STRESS ANALYSIS OF TYPICAL FLAWS IN AEROSPACE STRUCTURAL COMPONENTS USING
3-D HYBRID DISPLACEMENT FINITE ELEMENT METHOD

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Abstract

An assumed hybrid displacement finite element method for the solution of modes I, II and III stress intensity factors which vary along an arbitrary curved three-dimensional crack front in structural components was developed for linearly elastic materials. The present method can be applied to three-dimensional mixed mode fracture problems with complex crack and structural geometries. Special crack front singular elements were developed where proper asymptotic solutions for displacements and stresses are embedded. The above finite element method is presently used to analyze some important and typical flaw (fracture) problems which are commonly encountered in aerospace structural component applications.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_m$</td>
<td>volume of $m^{th}$ finite element ($m = 1,2,3,...,N$)</td>
</tr>
<tr>
<td>$S_m$</td>
<td>entire boundary of $m^{th}$ element</td>
</tr>
<tr>
<td>$\delta V_m$</td>
<td>that portion of $\delta V$ where $T_i$ are specified</td>
</tr>
<tr>
<td>$Su_m$</td>
<td>that portion of $\delta V$ where $u_i$ are specified</td>
</tr>
<tr>
<td>$u_i,{u_i}$</td>
<td>independently assumed interior displacements for each element</td>
</tr>
<tr>
<td>$e_{ij}$</td>
<td>$(u_{ij}+u_{ij})/2$ within each element $V_m$</td>
</tr>
<tr>
<td>$v_i,{v_i}$</td>
<td>independently assumed inter-element boundary displacements which are inherently compatible</td>
</tr>
<tr>
<td>$T_{Li},{T_{Li}}$</td>
<td>Lagrange multiplier terms which are physically the independently assumed arbitrary inter-element boundary tractions</td>
</tr>
<tr>
<td>$F_i$</td>
<td>body forces</td>
</tr>
<tr>
<td>$T_i$</td>
<td>specified surface tractions on $S_m$</td>
</tr>
<tr>
<td>$u_i$</td>
<td>specified boundary displacements on $S_u$</td>
</tr>
<tr>
<td>$E_{ijkL}$</td>
<td>elasticity tensor</td>
</tr>
<tr>
<td>$dV$</td>
<td>elemental volume</td>
</tr>
<tr>
<td>$dS$</td>
<td>elemental surface area</td>
</tr>
<tr>
<td>$[U_R]$</td>
<td>regular polynomial basis functions, which do not include any rigid body modes, for interior displacements</td>
</tr>
<tr>
<td>$[U_{RB}]$</td>
<td>regular polynomial basis functions which contain only rigid body modes for interior displacements</td>
</tr>
<tr>
<td>$[U_s]$</td>
<td>known asymptotic solution for displacements (which produce singular strains and stresses)</td>
</tr>
<tr>
<td>$[B_i],[B_i^*]$,</td>
<td>unknown independent parameters</td>
</tr>
<tr>
<td>$[\sigma]$</td>
<td>vector representing modes I,II and III stress intensity factors</td>
</tr>
<tr>
<td>$[k_s]$</td>
<td>assumed interpolation functions for boundary displacements</td>
</tr>
<tr>
<td>$[L_s]$</td>
<td>assumed boundary tractions, which contain regular as well as singular modes</td>
</tr>
<tr>
<td>$[R_s]$</td>
<td>element nodal displacements</td>
</tr>
<tr>
<td>$[Q_s]$, $[Q^*]$</td>
<td>generalized nodal displacements</td>
</tr>
<tr>
<td>$[Q_1],[Q_2]$</td>
<td>global matrices after the assembly of element matrices</td>
</tr>
<tr>
<td>$x,y,z$</td>
<td>displacements in $x,y,z$ directions respectively</td>
</tr>
<tr>
<td>$v$</td>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>$r,\theta,\phi$</td>
<td>coordinates as defined in Ref. 27</td>
</tr>
<tr>
<td>$X_1,X_2,X_3$</td>
<td>outward normal vector at the boundary</td>
</tr>
<tr>
<td>$E(k)$</td>
<td>complete elliptic integral of the second kind</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>elliptical angle</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>(edge)crack length and dimension of the ellipse (crack)</td>
</tr>
<tr>
<td>$c$</td>
<td>dimension of the ellipse (crack)</td>
</tr>
<tr>
<td>$R$</td>
<td>radius of the hole</td>
</tr>
<tr>
<td>$W,H$</td>
<td>dimensions of the plate</td>
</tr>
<tr>
<td>$t$</td>
<td>thickness of plate in fastener hole problems and surface flaw problems and thickness of adhesive in bonded plate problem</td>
</tr>
<tr>
<td>$\mu_1,\mu_2$</td>
<td>shear moduli of adherents</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus of adhesive</td>
</tr>
<tr>
<td>$G/tt_i$</td>
<td>$i=1,2$</td>
</tr>
<tr>
<td>$S$</td>
<td>tensile load applied to the plate</td>
</tr>
<tr>
<td>$\gamma_1,\gamma_2$</td>
<td>tension stress factors</td>
</tr>
<tr>
<td>$k_i,k_2,k_3$</td>
<td>modes I, II, and III stress intensity factors</td>
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I. Introduction

The importance of determining the stress intensity factors for the investigation of fracture of structural components need not be overemphasized, especially so in the case of aerospace structural components where accurate estimates of stresses and...
Several approximate methods have been formulated and developed to solve two-dimensional fracture problems accurately in recent years. As summarized by Rice, a recent survey on the available methods of solution for fracture problems was done by Pian. Even though a considerable amount of research work has been done on two-dimensional fracture problems, the work on three-dimensional fracture problems is very limited. Currently, for analyzing problems of practical significance in aerospace structural applications such as corner cracks originating from fastener holes, bonded plate problems with cracks and surface flaw problems in plates, etc., only highly approximate solutions which involve various ad hoc "correction factors" are available. Tracey extended his two-dimensional approach and solved three-dimensional problems of buried, surface and corner (circular) cracks. Three-dimensional stress analysis of a finite slab containing a transverse central crack for mode I problems was carried out by Sin et al. Recently, Pian and Moriya developed special crack elements for three-dimensional fracture analysis using a hybrid-stress finite element method. Several circular and elliptical crack problems were solved by Cruse by using boundary-integral equation method. Bergan et al. used the concept of strain energy release rate to compute the stress intensity factors for three-dimensional problems. Special quadratic isoparametric elements which possess the property of having a singularity of 1/3 in strains with the midside nodes of the faces, which are perpendicular to the crack front, placed at the quarter points were developed by Barsoum for three-dimensional linear fracture problems. Such elements have been developed and used for fracture analysis of plates, shells and two-dimensional problems. Hsu and Liu investigated the variation of stress intensity factors along the crack front for problems of corner cracks emanating from holes in finite plates. Shah also studied the stress intensity factors for through and part through cracks originating at fastener holes. A parametric type of study on the quarter-elliptical cracks emanating from holes in plates was done by Ganong using Schwarz-Neumann alternating technique. Experiments on quarter-elliptical corner cracks originating from holes in plates were conducted by Hall and Finger for aluminum and titanium alloy specimens. Experimental investigation of fracture and fatigue crack growth behavior of surface flaw and flaws originating from fastener holes for steel specimens was done by Hall et al. A stress freezing photoelasticity technique was used by McGowan and Smith to study the stress intensity factors for deep corner cracks emanating from a hole in a plate.

Considerable attention is being paid, recently, to the problem of surface flaws in plates in tension and bending due to the fact that in the literature solutions of stress intensity factors vary widely when the plate thickness is smaller when compared with other dimensions of the specimen and crack penetrates deeper in the thickness direction. Analytical solutions being unavailable, several investigators studied this problem using approximate solution procedures. Kobayashi studied semi-elliptical and semi-circular surface flaw problems subjected to tensile and bending loads using Schwarz-Neumann alternating technique. Schroedl and Smith used three-dimensional photoelasticity and extensively studied the surface flaw problems. Approximate solutions using alternating technique for surface flaws in bending were obtained by Shah and Kobayashi. Kobayashi is improving his solution by considering more arcs on the boundary where the stresses are erased in the alternating technique. Adhesively bonded structures containing metallic laminates possess several well-known features which make them highly attractive from a standpoint of fracture control. Thus, from an analysis point of view there has been a considerable interest in assessing the influence of material, geometrical and other structural variables on the stress-intensity factors for cracks in such structures. In earlier work Erdogan and Arin considered a sandwich plate consisting of an isotropic sheet and an orthotropic sheet held together by an adhesive layer, and the sandwich plate was assumed to have a part-through and a debonding crack. More recently, Keer, Lin and Mura considered the problem of two elastic, isotropic sheets, one of which is cracked, which are bonded together by an adhesive of finite thickness. They assumed in their analysis that no debonding will occur around the crack. It should be noted here that the above works on adhesively bonded plate with crack are effectively two-dimensional analyses in nature and linear shear spring elements representing the adhesive layer were used to join the front and back sheets in these analyses. The above brief review indicates that a more thorough and a rigorous three-dimensional analysis of practically significant problems in aerospace structural applications like quarter-elliptical corner cracks emanating from holes in plates and surface flaws in plates under uniaxial tension and adhesively bonded plates with crack in one plate subjected to tensile loading is needed and such a task is carried out using the developed and tested three-dimensional assumed hybrid displacement finite element method.

II. Formulation

The present finite element method is based on a modified variational principle of potential energy with relaxed continuity requirements for displacements at the inter-element boundary. The variational principle is a three field principle, with arbitrary...
interior displacements for the element, inter-element boundary displacements and element boundary tractions as variables. The variational principle is the stationary condition of a modified variational principle of potential energy for the functional to be varied is

\[ I_{HD}(u_1, v_1, T_{II}) = \frac{1}{2} \sum_{m=1}^{N} \left( \frac{1}{6} \int_{\Gamma} \sum_{i,j=1}^{4} E_{ij} k_{ij} \varepsilon_{ij} d\Gamma - \int_{\Gamma} T_{II} (u_1 - v_1) d\Gamma \right) dV - \int_{\Gamma} \tilde{T}_I v_1 d\Gamma - \int_{\Gamma} \sum_{i=1}^{4} T_{II} (u_1 - v_1) d\Gamma \]  

(1)

The interior displacements \( u \), assumed within \( v \) need not be continuous across \( \partial O \) and the inter-element boundary displacement \( v \), assumed independently on \( \partial O \), is continuous across \( \partial O \) and subject to the condition \( v = u \) on \( \partial O \). The first variation of the modified total potential energy, \( I_{HD} \), would yield the following Euler equation and natural boundary conditions: a) The interior displacements \( u \) in \( v \) generate stresses which satisfy local equilibrium; b) The interior displacements \( u \), coinciding with boundary displacements \( v \) on the element boundary \( \partial O \); c) The tractions, \( T_1 \), generated by the interior displacements \( u \), coincide with the independently assumed boundary tractions \( T_1 \) on the element boundary \( \partial O \). The stationary condition of the functional \( I_{HD} \) for arbitrary admissible variations of \( u \), \( v \), and \( T_1 \) also yield the essential boundary condition that the independently assumed boundary tractions \( T_1 \) coincide with the prescribed boundary tractions \( T_1 \) on the boundary \( \partial O \).

In analyzing the three-dimensional elastic fracture problems by the hybrid displacement model, the domain of the problem is divided into two distinct regions: 1) A small finite region near the crack front where the dominant singular behavior of strains and stresses is present and the elements in this region are referred to as "singular elements"; 2) the region away from the crack front where the effects of singularity are no longer felt and the behavior of stresses and strains may be approximated by regular polynomial basis functions. The elements in this region are referred to as "regular elements."

Essentially, in constructing the hybrid displacement finite element model for the singular elements the following three field functions are assumed: 1) interior displacement field within each element which need not satisfy the compatibility condition at the inter-element boundary a priori and includes the proper asymptotic displacement solutions; 2) boundary displacement field on the boundary of each element which is inherently compatible with the boundary displacement field of neighboring elements at the inter-element boundary; 3) the Lagrange multipliers, which are physically the independently assumed boundary tractions, assumed on the boundary of each element to match the independently assumed interior displacement field and the boundary displacement field on the boundary of each element. The above three field variables are assumed as shown below.

\[
\begin{align*}
{u}_I &= [R_I] [\beta] + [U_R] [\beta'] + [U_S] [k] \\
{v}_I &= [L_I] [q] \\
{T}_{II} &= [R_S] [\alpha]
\end{align*}
\]

By assuming the three field functions as shown above, the proper form of singularity in stresses and strains are embedded in the finite element procedure. New, Ref. 24 and 25 are substituted in Eq. (1) to obtain the total potential energy in terms of the nodal displacements \( \{q\} \), stress-intensity factor vector \( \{k\} \), and the undetermined parameters \( \{\alpha\} \) and \( \{\beta\} \). It can be easily shown that, for no body force, the contribution by the rigid body modes to the total potential energy (terms corresponding to \( \{\beta\} \)) would become zero. As the undetermined parameters \( \{\alpha\} \) and \( \{\beta\} \) are independent for each of the elements, they need to be eliminated from the expression of total potential energy and expressed only in terms of \( \{q\} \) and \( \{k\} \), which are associated with the entire system of elements and segments along the crack front, respectively. This can be done as follows. The first variation of the total potential energy functional with respect to the parameters \( \{\alpha\} \) and \( \{\beta\} \) is taken and equated to zero to obtain two equations. From these equations \( \{\alpha\} \) and \( \{\beta\} \) can be found in terms of \( \{q\} \) and \( \{k\} \). Then these are substituted in the equation of the total potential energy to obtain the total potential energy only in terms of \( \{q\} \) and \( \{k\} \). Using the variational principle the following final algebraic system of equations are obtained where the unknowns are the nodal displacements and the three elastic stress intensity factors along the crack front.

\[
\begin{align*}
[K_1] [q^*] + [K_2] [k^*] &= [Q_1] \\
[K_2] [q^*] + [K_3] [k^*] &= [Q_2]
\end{align*}
\]

(5)

(6)

Further details of the theoretical and numerical details of the formulations are not provided here for want of space and can be found in Ref. 24 and 25. In the following, a brief description of the three independently assumed field functions namely the interior displacements, the boundary displacements and the boundary tractions is given. These field functions are the modified and improved version found in Ref. 26.

A. Assumed Interior Displacements for Singular Elements

The form of the assumed interior displacement for singular elements is given in equation (2).

The field function \( [R_I] \) which contains pure straining modes is assumed in terms of isoparametric coordinate system and cartesian coordinate system. The geometry of the basic element considered is a 20 node isoparametric brick element with 60 degrees of freedom (see Fig. 1).
Fig. 1 Three Dimensional Mapping of Singular Element

[\{U\}_B] is then written in the cartesian component form as:

\[
\begin{align*}
\mathbf{u}_R &= \mathbf{\beta}_1 x + \mathbf{\beta}_2 y + \mathbf{\beta}_3 z + \mathbf{\beta}_4 \mathbf{\xi}_1 + \mathbf{\beta}_5 \mathbf{\xi}_2 + \mathbf{\beta}_6 \mathbf{\xi}_3 + \mathbf{\beta}_7 \mathbf{\xi}_4 \\
+ \mathbf{\beta}_8 \xi_1^2 + \mathbf{\beta}_9 \xi_2^2 + \mathbf{\beta}_{10} \xi_3^2 + \mathbf{\beta}_{11} \xi_4^2 + \mathbf{\beta}_{12} \xi_5^2 + \mathbf{\beta}_{13} \xi_6^2 \\
+ \mathbf{\beta}_{14} \xi_7^3 + \mathbf{\beta}_{15} \xi_8^3 + \mathbf{\beta}_{16} \xi_9^2 + \mathbf{\beta}_{17} \xi_{10}^2 + \mathbf{\beta}_{18} \xi_{11}^3 \\
+ \mathbf{\beta}_{19} \xi_{12}^3 + \mathbf{\beta}_{20} \xi_{13}^2 \\
\mathbf{v}_R &= \mathbf{\beta}_2 x + \mathbf{\beta}_2 y + \mathbf{\beta}_2 z + \mathbf{\beta}_3 \mathbf{\xi}_4 + \mathbf{\beta}_4 \mathbf{\xi}_5 + \mathbf{\beta}_5 \mathbf{\xi}_6 \\
+ \mathbf{\beta}_{25} \xi_1^2 + \mathbf{\beta}_{26} \xi_2^2 + \mathbf{\beta}_{27} \xi_3^2 + \mathbf{\beta}_{28} \xi_4^2 + \mathbf{\beta}_{29} \xi_5^2 + \mathbf{\beta}_{30} \xi_6^2 \\
+ \mathbf{\beta}_{31} \xi_7^2 + \mathbf{\beta}_{32} \xi_8^2 + \mathbf{\beta}_{33} \xi_9^2 + \mathbf{\beta}_{34} \xi_{10}^2 + \mathbf{\beta}_{35} \xi_{11}^2 \\
+ \mathbf{\beta}_{36} \xi_3^3 + \mathbf{\beta}_{37} \xi_4^3 \\
\mathbf{w}_R &= \mathbf{\beta}_3 x + \mathbf{\beta}_2 y + \mathbf{\beta}_3 z + \mathbf{\beta}_4 \mathbf{\xi}_4 + \mathbf{\beta}_5 \mathbf{\xi}_6 + \mathbf{\beta}_6 \mathbf{\xi}_7 \\
+ \mathbf{\beta}_{42} \xi_2^2 + \mathbf{\beta}_{43} \xi_3^2 + \mathbf{\beta}_{44} \xi_4^2 + \mathbf{\beta}_{45} \xi_5^2 + \mathbf{\beta}_{46} \xi_6^2 + \mathbf{\beta}_{47} \xi_7^2 \\
+ \mathbf{\beta}_{48} \xi_8^2 + \mathbf{\beta}_{49} \xi_9^2 + \mathbf{\beta}_{50} \xi_{10}^2 + \mathbf{\beta}_{51} \xi_{11}^2 + \mathbf{\beta}_{52} \xi_{12}^2 \\
+ \mathbf{\beta}_{53} \xi_3^3 + \mathbf{\beta}_{54} \xi_4^3 \\
\end{align*}
\]

(7)

The restriction that [\{U\}_B] should contain only pure straining modes forces the terms corresponding to \(\mathbf{\beta}_1, \mathbf{\beta}_2, \mathbf{\beta}_3, \mathbf{\beta}_4, \mathbf{\beta}_5, \mathbf{\beta}_6, \mathbf{\beta}_7, \mathbf{\beta}_8\) to be assumed in terms of physical coordinate system. Similarly, [\{U\}_B] which contains the pure rigid body terms can be written in component form as:

\[
\begin{align*}
\mathbf{u}_{RB} &= \mathbf{\beta}_5 x - \mathbf{\beta}_6 y + \mathbf{\beta}_7 z \\
\mathbf{v}_{RB} &= \mathbf{\beta}_8 x + \mathbf{\beta}_9 y - \mathbf{\beta}_{10} z \\
\mathbf{w}_{RB} &= -\mathbf{\beta}_{11} x + \mathbf{\beta}_{12} y + \mathbf{\beta}_{13} z \\
\end{align*}
\]

(8)

The singular displacement field function [\{U\}_B], which plays the predominant role in the singular element, has the identical form as the asymptotically correct near field displacement solution with the stress intensity factors being used directly as undetermined parameters [\(\mathbf{k}\_s\)].

B. Assumed Boundary Displacements for Singular Element

As mentioned earlier, the boundary displacement field \(\mathbf{u}_B\), on the boundary of each element, is assumed in such a way that it is inherently compatible with the boundary displacement field of neighboring elements at the inter-element boundaries. This is accomplished by using the same and unique interpolation functions on the inter-element boundary. This assures the inter-element displacement compatibility since the nodal displacements are common for elements that share the common boundary. Referring to Fig. 1, with AE being the crack front, the boundary displacement can be assumed as follows on six faces of the element.

Face ABFE (\(\zeta = -1\))

\[
\begin{align*}
\mathbf{u} &= a_1 + a_2 \xi + a_3 \eta + a_4 \xi^2 + a_5 \eta^2 + a_6 \xi \eta + a_7 \xi^2 \\
+ a_8 \eta^2
\end{align*}
\]

(9)

Face CDHG (\(\zeta = +1\))

\[
\begin{align*}
\mathbf{u} &= q_5 (1+\xi) (1-\eta)(-\xi-\eta-1)/4 + q_6 (1-\xi^2) (1-\eta)/2 \\
+ q_7 (1-\xi)(1-\eta) (-\xi-\eta-1)/4 + q_{11} (1+\xi)(1-\eta)/2 \\
+ q_{12} (1-\xi)(1-\eta)^2/2 + q_{17} (1+\xi)(1-\eta) (-\xi-\eta-1)/4 \\
+ q_{18} (1-\xi^2)(1-\eta)/2 + q_{19} (1+\xi)(1-\eta) (-\xi-\eta-1)/4
\end{align*}
\]

(10)

Faces ABCD (\(\zeta = -1\)) and EFHG (\(\zeta = +1\))

\[
\begin{align*}
\mathbf{u} &= a_1 + a_2 \xi + a_3 \eta + a_4 \xi - a_5 \eta \cos \frac{\theta}{2} (1-2\eta^2 \cos^2 \frac{\theta}{2} \\
+ a_6 \xi^2 \cos \frac{\theta}{2} (1-2\eta \sin^2 \frac{\theta}{2}) + a_7 \eta \cos \frac{\theta}{2} (1-2\eta) \\
+ \sin^2 \frac{\theta}{2} + a_8 \eta \sin \frac{\theta}{2} (1-2\eta + \cos^2 \frac{\theta}{2})
\end{align*}
\]

(11)

Face ADHE (\(\xi = -1\))

\[
\begin{align*}
\mathbf{u} &= a_1 + a_2 \xi + a_3 \eta + a_4 \xi - a_5 \eta \sin \frac{\theta}{2} (1-2\eta^2 \cos^2 \frac{\theta}{2} \\
+ a_6 \xi^2 \sin \frac{\theta}{2} (1-2\eta \sin^2 \frac{\theta}{2}) + a_7 \eta \sin \frac{\theta}{2} (1-2\eta) \\
+ \sin^2 \frac{\theta}{2} + a_8 \eta \sin \frac{\theta}{2} (1-2\eta + \cos^2 \frac{\theta}{2})
\end{align*}
\]

(12)

Face BCGF (\(\xi = +1\))

\[
\begin{align*}
\mathbf{u} &= q_3 (1-\xi)(1-\zeta)(-\zeta-\xi-1)/4 + q_4 (1-\eta)(1-\xi^2)/2 \\
+ q_5 (1-\xi)(1-\xi)(-\zeta-\xi-1)/4 + q_{10} (1-\eta^2)(1-\zeta)/2 \\
+ q_{11} (1-\xi^2)(1-\zeta)/2 + q_{15} (1+\eta)(1-\xi)(\xi-\zeta-1)/4 \\
+ q_{16} (1+\eta^2)(1-\xi)/2 + q_{17} (1+\eta^2)(1-\eta)(\eta-\zeta-1)/4
\end{align*}
\]

(13)

The parameters \((a_1, a_2, \ldots, a_8)\) in Eqs. (9), (11), and (12) are determined uniquely in terms of the relevant \(u\)-displacement coordinates on faces ABFE, ABCD & EFHG, and ADHE, respectively. For example, \((a_1, a_2, \ldots, a_8)\) in Eq. (9) for face ABFE are uniquely found in terms of the relevant nodal displacements \(q_1, q_2, q_3, q_4, q_9, q_{10}, q_{11}, \) and \(q_{12}\). It should be noted here that similar interpolates are used for the other two displacements \(v\) and \(w\) corresponding to all six faces of the element and the other three elements which contain the crack front \((\eta/2 \leq \eta \leq \eta; -\eta/2 \leq \eta \leq \eta; -\eta \leq \eta \leq \eta/2)\).

C. Assumed Boundary Tractions for Singular Element

The boundary tractions [\(T_{UL}\)] mathematically
interpreted as Lagrange multipliers in the assumed displacement hybrid finite element method, as such can be assumed in any convenient form. However, one of the relevant natural conditions of the present variational principle is that the boundary tractions generated by the assumed element interior displacements \( \{u_1\} \) must match the independently assumed boundary tractions \( \{T_{on}\} \) at the inter-element boundary. Also, the element interior displacement field \( \{u_1\} \) is forced to satisfy the equilibrium equation through the variational principle. Thus for better numerical accuracy of the formulation the assumed boundary tractions \( \{T_{on}\} \) are obtained from an equilibrated stress field. Moreover, since the assumed interior displacement field contains \( a/r \) variation in displacements, the assumed tractions \( \{T_{on}\} \) must also contain a \( 1/r \) variation in tractions for the singular element. Thus the assumed boundary tractions are essentially assumed in two parts namely regular and singular parts. For the regular part, the equilibrated stress field can be obtained from three-dimensional Maxwell's stress functions. The relationships between Maxwell's stress functions and the stresses are given as follows:

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 \chi_3}{\partial y^2} + \frac{\partial^2 \chi_2}{\partial z^2} ; \quad \sigma_{yz} = -\frac{\partial^2 \chi_1}{\partial y \partial z} \\
\sigma_{yy} &= \frac{\partial^2 \chi_1}{\partial z^2} + \frac{\partial^2 \chi_3}{\partial x^2} ; \quad \sigma_{zx} = -\frac{\partial^2 \chi_2}{\partial z \partial x} \\
\sigma_{zz} &= \frac{\partial^2 \chi_2}{\partial x^2} + \frac{\partial^2 \chi_1}{\partial y^2} ; \quad \sigma_{xy} = -\frac{\partial^2 \chi_3}{\partial x \partial y}
\end{align*}
\] (14)

The singular part of the stress field can be assumed from the asymptotic near field solution for stresses. The assumed stress functions which produce the needed stress modes are given below.

\[
\begin{align*}
\chi_1 &= \alpha_1 \eta^2 + \alpha_2 \xi \eta + \alpha_3 \xi^2 + \alpha_4 \xi^2 \eta + \alpha_5 \xi \eta^2 + \alpha_6 \xi^2 \\
&+ \alpha_7 \xi \eta^2 + \alpha_8 \xi^3 \eta + \alpha_9 \xi \eta^3 + \alpha_{10} \xi^2 \eta^2 + \alpha_{11} \xi \eta^2 + \alpha_{12} \xi^2 \eta^3 + \alpha_{13} \xi \eta^3 + \alpha_{14} \xi \eta^4 + \alpha_{15} \xi \eta^4 + \alpha_{16} \xi^2 \eta^3 + \\
&+ \alpha_{17} \xi \eta^4 \\
\chi_2 &= \alpha_{18} \xi^2 + \alpha_{19} \xi \eta + \alpha_{20} \eta^2 + \alpha_{21} \xi^2 \eta + \alpha_{22} \xi \eta^2 \\
&+ \alpha_{23} \eta^2 + \alpha_{24} \xi \eta + \alpha_{25} \xi^2 \eta^2 + \alpha_{26} \xi \eta^2 + \alpha_{27} \xi \eta^3 + \alpha_{28} \xi \eta^4 + \alpha_{29} \xi^2 \eta + \alpha_{30} \xi \eta^2 + \alpha_{31} \xi \eta^2 + \alpha_{32} \xi \eta^2 + \\
&+ \alpha_{33} \xi \eta^2 + \alpha_{34} \xi \eta^4 \\
\chi_3 &= \alpha_{35} \xi^2 + \alpha_{36} \xi \eta + \alpha_{37} \eta^2 + \alpha_{38} \xi^2 \eta + \alpha_{39} \xi \eta^2 + \alpha_{40} \xi \eta^2 + \alpha_{41} \xi \eta^3 + \alpha_{42} \xi \eta^2 + \alpha_{43} \xi \eta^2 + \alpha_{44} \xi \eta^2
\end{align*}
\] (15)

Once the stresses are calculated by using Eq. (14), the tractions are obtained as \( \{T_{on}\} = \{\sigma_{on}\} \). In the traction modes, assumed from the singular stress field, the stress intensity factors \( k_1, k_2, k_3 \) are replaced by the unknown parameters \( \alpha_{52}, \alpha_{53} \) and \( \alpha_{54} \).

### III. Problems

The above hybrid displacement finite element method was used to solve for the solution of stress intensity factors along the crack front for three problems namely quarter-elliptical corner cracks emanating from fastener holes subjected to tension, plates with surface flaws subjected to tension and bending, and bonded plates with holes and crack in one plate under tensile load.

#### A. Corner Cracks Near Fastener Holes

For simplicity of modelling, and the knowledge that the stress intensity values for single and double (symmetric) corner cracks originating from holes differ only by about ten percent, the problem of double (symmetric) quarter-elliptical corner cracks originating from fastener holes was considered for the analysis by the present three-dimensional hybrid displacement finite element model. Fig. 2 illustrates the nomenclature of the double corner cracked hole problem. The leading

Fig. 2 Corner Cracks Emanating from Holes in Plates

for this problem would be uniaxial tension at \( z = H \). Two different geometries have been considered and they are given in the following table, along with corresponding Poisson's ratios.

The dimensions \( W \) and \( H \) are given by the following equations

\[
2W = 6(2R + c)
\]
Due to symmetry of the problem, only a quarter of the structure needs to be considered for the finite element breakdown. The finite element breakdown of quarter of the problem is given in Fig. 3; the total number of finite elements and the total number of degrees of freedom for this breakdown are 156 and 2670 respectively. The results and discussions for this problem are presented in the next section.

<table>
<thead>
<tr>
<th>a/c</th>
<th>c/R</th>
<th>a/t</th>
<th>ν</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.99142</td>
<td>0.48</td>
<td>0.25</td>
</tr>
<tr>
<td>1.5</td>
<td>1.0</td>
<td>0.75</td>
<td>0.30</td>
</tr>
</tbody>
</table>

![Fig. 3 Finite Element Breakdown of Quarter of the Corner Cracked Hole Problem](image)

### B. Plates with Surface Flaws

The problems of semi-circular and semi-elliptical surface flaws in thin plates with different crack depth to thickness ratios under bending and tensile loads were also considered for the analysis by the present finite element model. The problem is schematically shown in Fig. 4 and Fig. 5 illustrates the finite element breakdown of quarter of structure with 156 total number of finite elements and 2670 total number of degrees of freedom. In Fig. 5, the thickness direction (A0 or A1) is shown elongated to show the finite element breakdown clearly. The geometric parameters of the problems considered are given below.

![Fig. 4 Surface Flaws in Plates in Tension and Bending](image)

<table>
<thead>
<tr>
<th>a/c</th>
<th>a/t</th>
<th>W/c</th>
<th>H/c</th>
<th>Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.6</td>
<td>4.0</td>
<td>4.0</td>
<td>Tension &amp; Bending</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8</td>
<td>4.0</td>
<td>4.0</td>
<td>Tension &amp; Bending</td>
</tr>
<tr>
<td>0.2</td>
<td>0.6</td>
<td>4.0</td>
<td>4.0</td>
<td>Bending</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>4.0</td>
<td>4.0</td>
<td>Bending</td>
</tr>
</tbody>
</table>

![Fig. 5 Finite Element Breakdown of Quarter of the Surface Flaw Problem in Plates](image)

The Poisson's ratio is assumed to be 1/3 for the surface flaw problems in plates. The results and discussions for these problems are also given in the next section.

In the solution procedure for surface flaw problems and corner cracks near fastener hole problems, in order to eliminate the translational and rotational (rigid body) movements of the plates, at least one displacement in each direction was sup-
pressed. The symmetry conditions of the problems were obtained by property suppressing the displacement in the directions perpendicular to the plane of symmetry. The quarter-ellipse (or circle) was divided into six segments for these problems to obtain the variation of the stress intensity factor along the crack front.

C. Cracks in Adhesively Bonded Plates with Holes

The present method was also used to solve the problem of two isotropic adhesively bonded plates containing a through-the-thickness hole, and a single crack near the hole in one of the plates. The schematic diagram of the problem is given in Figs. 6 and 7 illustrates the finite element breakdown of the problem with 216 total number of finite elements and 3822 total number of degrees of freedom. The Young's Moduli of the plates and adhesive were assumed to be 1.03 x 10^6 psi and 2.8 x 10^5 psi respectively. The Poisson's ratios of the plates and the adhesive were assumed to be 0.33 and 0.4, respectively. The ratio t_a/t_h was assumed as 1.0. The ratio of adherent thickness to adhesive thickness was assumed to be 7.0. The loading is the uniaxial tension which is uniform on both the plates and the adhesive. The non-dimensional parameter (W/πh)^2 was taken to be 10.0. Then the problem was solved for crack length to hole radius ratios of 0.35, 1.14, 2.1 and 3.5. The results of the stress intensity factor variation are given in the next section.

IV. Results and Discussions

A. Corner Cracks Near Fastener Holes

The solutions of the stress intensity factors for the symmetric corner cracks emanating from holes in finite plates under tension for the two different geometries are given in Figs. 8 and 9. These stress...
intensity factors are normalized with respect to the theoretical value of the stress intensity factor at the point where the crack front and the minor axis intersect when the complete elliptical crack is embedded in an infinite medium and is subjected to remote uniform tension $\sigma_o$. The normalizing factor is given by $\sigma_M/\sigma_o$, where $b$ is the semi-minor axis (minimum of $a$ and $c$). As a result, Figs. 8 and 9 depict the actual variation of the stress intensity factors, but for a constant. Solutions of stress intensity factors for the problem of corner cracks emanating from hole with geometric parameters $a/c = 1.1$, $c/b = 0.99162$ and $a/t = 0.48$ by the present method and by stress freezing photoelasticity method by McGowan and Smith are presented in Fig. 8. Present solution by the finite element model for the problem with geometric parameters $a/c = 1.5$, $c/b = 1.0$ and $a/t = 0.75$, along with Ganong's solution for single corner crack, is given in Fig. 9. Though the present solution cannot be compared directly with the solution of Ganong in Fig. 9 because of the difference in the problem, the present solution seems to differ slightly from Ref. 14. The solution of the first problem in Fig. 8 differs from that of the results obtained from the experiments by McGowan and Smith; where only the end values of the stress intensity factors were available for comparison. The reasons for the differences and discrepancies of these results can be enumerated as follows.

Ganong estimates that his results are about 20 percent higher than the previous estimates. Due to inadequate experimental results, it was not possible to verify the present results with them. The end values of stress intensity factors obtained by McGowan and Smith through experiments in Fig. 8 show that they agree at the front surface and not at the hole surface. Theoretically one would expect the stress intensity factor to be higher at the hole surface due to the stress concentration effect along the hole surface. Further comments on the reasons for discrepancies are given later in this section.

B. Plates With Surface Flaws

The solution of stress intensity factors along the crack front for semi-circular surface flaws in plates subjected to bending as well as tension for crack depth to thickness ratios 0.6 and 0.8 are in Figs. 10 and 11. Fig. 12 give the solution for the stress intensity factors along the crack front for the semi-elliptical surface flaws in plates subjected to bending for crack depth to thickness ratios 0.6 and 0.8. They are normalized with respect to the value of stress intensity factor of either a completely embedded circle or at the minor axis of the ellipse, when the circle or ellipse is completely embedded in an infinite solid and subjected to remote tension. Thus, Figs. 10, 11 and 12 represent the actual variation of stress intensity factors but for a constant. The corresponding solutions obtained by Kobayashi et al. and Raju et al. for semi-circular surface crack problems are also given in Figs. 10 and 11. For the semi-circular surface problems in Figs. 10 and 11, it can be seen that the solution of stress intensity factors for bending loads by the present procedure agrees well with the results estimated by Kobayashi except for a 15 percent difference at the intersection of the crack front and the front free surface. However, there is a large difference in the solution for tensile load between the present and Kobayashi's procedure. A recent solution by Raju and Newman given in Figs. 10 and 11 for tensile loads compares with the present solution better than that of Kobayashi. It should be noted here that that degrees of freedom used by Raju and Newman for the semi-circular surface flaw problems are 4317. As the solutions corresponding to the problem of semi-elliptical surface flaws in plates are being revised by the author of Ref. 18 (private communication), we choose not to compare the present solution at the present time.
In the alternating technique used by Kobayashi for semi-circular and semi-elliptical surface flaw problems, one of the steps involved is the removal of tractions on the boundary of the structure. These tractions are removed only over a small area near the crack in Ref. 18. As mentioned before, the solutions are being revised and improved i.e., the tractions are now being removed over a larger area near the crack which would result in better convergence of the solution. This may be one of the reasons for the discrepancies in solution of stress intensity factors by the present and Kobayashi's procedures. In analyzing the above problems, the previous investigators have used highly approximate solution procedures or two-dimensional analogs and various ad-hoc correction factors. But the present hybrid displacement finite element model is a rigorous three-dimensional finite element procedure where all the boundary conditions, such as free surfaces, are automatically taken into account and satisfied in an integral average sense.

C. Cracks in Adhesively Bonded Plates with Holes

In solving the crack problems in adhesively bonded plates with holes, the finite element grid was arrived at after a careful study of the expected stress concentration near the hole and after verifying that the stresses near the hole without the crack are accurate enough so that the obtained results for stress intensity factors for cracks near the hole may be meaningful. Before solving the problem of cracks in bonded plates with holes, the same finite element grid was used to solve a simple problem of a plate with a hole in tension. This was carried out to make sure that the finite element grid was fine enough to generate a stress concentration factor of about 3.0 at the hole. The solution of the stress concentration factor for the above problem at mid-thickness of the plate is given in Fig. 13. As can be seen from the figure, the stress concentration at the hole is 3.213. Also, the stress concentration is found to vary along the thickness direction of the plate. The value of the stress concentration decreased from 3.213 at the mid-thickness to 3.134 at the front and back surfaces of the plate. After making sure that the stress concentration effects are well accounted for, the problems of cracks in bonded plates with holes were solved.

The solution of the stress intensity factors for various crack length to hole radius ratios for the problem of two isotropic adhesively bonded plates containing a through-the-thickness hole and a single crack near the hole in one of the plates is given in Fig. 14. The stress intensity factors are normalized by using the value of the stress intensity factors of a single infinite plate with the crack. In analyzing these problems, the adhesive was considered as a continuum and was assumed to debond with the crack along the plane of the crack until it reaches the plate with no crack. Special crack elements were used only in the cracked plate and all the elements for the adhesive were modeled using regular isoparametric brick elements. In Fig. 14, three different sets of stress intensity factors are presented. The middle curve represents the varia-
tation of stress intensity factors computed directly from the uxyz-displacement finite element method in which the stress intensity factors are assumed as unknowns and computed directly. These values may be considered as the average values of the stress intensity factors over the length of the crack front. The upper and lower curves represent the variation of stress intensity factors at the outer and inner tips of the crack front. The outer tip corresponds to the crack tip at the free surface of the cracked plate and the inner tip corresponds to the crack tip where the cracked plate join the adhesive. The stress intensity factors for the outer and inner tips are computed from the displacement solution of the finite element method using crack opening displacement (COD) method.3

Acknowledgments

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