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A Finite-Element Program for Fracture Mechanics Analysis of Composite Material


ABSTRACT: An assumed displacement hybrid finite-element procedure developed for treating a general class of problems involving mixed-mode behavior of cracks is used to solve some two-dimensional, fracture mechanics problems involving rectilinear-anisotropic materials. This finite-element program uses four "singular" elements which surround the crack tip and "regular" elements which occupy the remaining region. The singular element has a built-in displacement field of the $\sqrt{r}$ type with the two modes of stress intensity factors, $K_1$ and $K_2$, as unknowns. Displacement compatibility between singular and regular elements is also maintained. Isoparametric transformations are used to derive the stiffness matrix of quadrilateral curved elements. Rectilinear anisotropic, nonhomogeneous, but linear elastic, material properties are considered. The program was checked out by analyzing a bimaterial tension plate with an eccentric crack and a centrally-cracked orthotropic tension plate. The results thus obtained agreed well with those by Erdogan and Biricikoglou, and Bowie and Freese, respectively. The program was then used to analyze two fracture test specimens for which analytical solutions do not exist. The first specimen was the doubly edge-notched tension plate with material principal directions oriented $0^\circ$-$90^\circ$ or $\pm 45^\circ$ to the geometric axes of symmetry and with varying crack length. The second specimen was the three-point bend specimen with material principal directions oriented $0^\circ$-$90^\circ$ to the geometric axes of symmetry. Finally, an orthotropic tension plate with an oblique center crack was analyzed. Finite-element solutions of most of these problems do not seem to have appeared in prior literature.

KEY WORDS: fracture properties, stresses, composite materials, finite-element procedure, crack propagation, fatigue (materials)

Fracture studies of composite materials are much more complex than those of homogeneous, but brittle, engineering materials. In general, the

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singular state of stress surrounding the crack tip is not that of $1/\sqrt{r}$, and crack extension could be nonsymmetric despite the geometric symmetry of the problem. In addition, the possible presence of formation and coalescence of voids as well as craze formation in the singular region surrounding the crack tip distinguish the composite fracture from that of structural metals. Despite the complexity of the actual fracture process, the composite fracture should be dependent, to some extent, on the severity of the local singular state of stress which is governed by the strengths of singularity, $K_1$ and $K_2$. These two quantities, which are the well-known stress intensity factors in linear fracture mechanics, should provide means of estimating the fracture load of a cracked composite material regardless of the failure criterion used. The purpose of this paper is to present a computational procedure by which the various physical quantities related to such a fracture criterion can be determined. For an up-to-date review of specific fracture criterion for composite materials as well as a summary of some of the known results published to date, the readers are referred to an excellent survey presented in Ref. 1.

The analytical difficulties in determining stress intensity factors in cracked homogeneous structures are compounded in composite materials because of the anisotropy and inhomogeneity of the materials involved. For composite materials, the limited analytical solutions of stress intensity factors for isotropic materials [2,3] are drastically reduced to a few problems with straight cracks in rectangular composite plates subjected to uniform loading conditions. The lack of even a handful of stress intensity factor solutions for different geometries and loading conditions makes it impossible to estimate stress intensity factors of real cracks in composite materials. It thus becomes imperative, in analyzing cracked composite material, to utilize the method of finite element analysis which has been used extensively to determine Mode I stress intensity factors as well as some mixed Modes I and II stress intensity factors in two-dimensional problems [4-9]. The simple computational procedure of using the crack opening displacement (COD) [4] or the computationally convenient procedure of using strain energy release rate [5,7] to estimate stress intensity factors, however, cannot be used to analyze cracked composite materials since the mixed modes of crack-tip deformation are more likely to co-exist in these problems. The finite-element procedure used in composite material analysis should therefore contain the proper singular state with stress intensity factors as unknowns which are to be determined together with the nodal forces and displacements.

In a recent paper, Tong and Pian [10] have shown that, for problems with singularities, the convergence rates of the finite-element method are

$K_1$ and $K_2$ are defined as the stress intensity factors for a general singular state of stress $a_0 = K/r^a$ where $a$ is not necessarily equal to 1/2.

The italic numbers in brackets refer to the list of references appended to this paper.
dominated by the singular nature of the solution near the crack tip. Thus, the order and format of stress singularities must be known a priori so that these singular stresses can be properly incorporated in the assumed functions in order to improve the rate of convergence. The special elements by Tracey [11] and Byskov [12] do not satisfy the inter-element boundary compatibility criteria and again, convergence of such a solution cannot be guaranteed, as has been proven by Tong and Pian [13].

To overcome these convergence problems, Pian, Tong, and Luk [9] used a special crack-tip finite element for isotropic materials with the correct singularities, of the $1/r^2$ type for elastic analysis, in the assumed stresses of the hybrid finite-element model originated by Pian [14]. Parameters for the singular terms, together with the nodal displacements, are then solved from the final set of matrix equations. The magnitudes of singular terms, which are in fact the stress intensity factors $K_i$ and $K_{ii}$, are solved directly in this procedure. The results of Ref 9 have shown that the stress intensity factors for isotropic elastic material can be calculated accurately by using a much smaller number of degrees of freedom than by using only conventional elements.

Tong, Pian, and Lasry [15] have recently improved upon the analysis of Ref 9 by combining the hybrid element concept with those of the complex variable techniques developed by Bowie in his modified boundary collocation method for crack problems [16]. The proper $1/r$ stress singularity was incorporated in the region surrounding the crack tip and led to very efficient programming. However, for anisotropic materials the complex variable formulation becomes somewhat complicated.

In this paper, application of a hybrid displacement model to two-dimensional problems in cracked composite material is presented. The discussion will be limited to the linear elastic, but arbitrary, anisotropic materials. The relative merits of the hybrid displacement method are presented.

**Method of Approach**

The mathematical formulation of the hybrid displacement model for analyzing cracked structures is described in detail in Refs 17 and 18 and will not be repeated here. Basically the variational principle which governs the assumed displacement hybrid model [19,20] is a modified principle of minimum potential energy. In a solution of an elasticity problem, the displacements $u_i$ in the interior element, the inter-element boundary displacement, $v_i$, and the element surface traction, $T_{ij}$, are treated as unknown variables. The vanishing of the variation $\delta \Pi$ in the modified minimum potential energy principle, for arbitrary variations in displacements, $\delta u_i$, in the element, for arbitrary $\delta T_{ij}$ on inter-element boundaries, and for
admissible $\delta v$, on inter-element boundaries, results in the respective Euler equations:

(a) satisfaction of the local equilibrium equations in the element, $V_m$;

(b) values of interior displacements, $u$, at inter-element boundaries co-
    incide with the inter-element boundary displacements $v$, which are treated
    as independent unknowns; and

(c) finally, the boundary tractions $T_l$, which are treated as independent
    unknowns in the present problem, coincide with the tractions $\frac{1}{2}E_k(u_{k,l} +
    v_j)$ generated at inter-element boundaries by the functions $u$, where $v_j$
    is the direction cosine of the element boundary, $\partial V_m$. Thus, it can be seen
    that the hybrid finite-element model enables one to choose element dis-
    placement functions that are completely arbitrary and need not satisfy the
    inter-element compatibility condition. This inter-element compatibility
    is enforced a posteriori by introducing the inter-element boundary dis-
    placement $v$, as an independent variable and enforcing the constraint
    condition $v = u$, on $\partial V_m$. The boundary tractions $T_l$, which
    are treated as independent variables in the present problem, can also be viewed
    mathematically as Lagrangean multipliers to enforce the constraint $v = u$.

In applying this model to cracked composite plates, the plate is divided
into two regions: (a) a small but fixed region near the crack tip where the
singular, near-field solution is predominant; and (b) a region away from
the crack tip where the effect of the singularity is not felt. Assuming that
the nature of this singularity is known, appropriate displacements cor-
responding to the singular stresses (of $1/r^a$ type, where $a$ is not necessarily
equal to $1/2$) are incorporated in the assumed approximate functions for
$u$ in the elements surrounding the crack tip as follows:

$$\{u\}_S = [U_S] \{\beta\} + [U_S] \{K_1\}_S = [U_S] \{\beta\} + [U_S] \{K_3\} \quad (1)$$

where $U_k$ are simple polynomials, $\beta$ are unknown independent parameters,
$K_1$ and $K_2$ are stress intensity factors for Mode I and Mode II crack
deformation, and $U_S$ are known displacement functions for plane problems
which yield the correct singular behavior for stresses and strains in linear
elastic analysis. In the remainder of this paper, the stress singularity will
be assumed to be of the $1/\sqrt{r}$ type although all discussions are applicable
to other singularities of $1/r$ type.

An independent element boundary displacement, $v$, which is uniquely
interpolated by displacements at the nodes along the boundary, is then
assumed. Since these nodes are common for elements that share the com-
mon boundary, inter-element boundary displacement compatibility is thus
ensured as mentioned previously. For the elements adjacent to the crack
tip, a $\sqrt{r}$ type displacement behavior is built into the boundary displace-
ment along element boundaries which pass through the crack tip. Details
of this arrangement are given in Refs 17 and 18.

Finally, element boundary tractions (which can also be identified as
Lagrangean multipliers in the present formulation) are assumed for a
singular element. Since the assumed displacement, $u_h$, for the singular ele-
ment contains a $\sqrt{r}$ type behavior, these displacements will generate
singular boundary tractions of the $1/\sqrt{r}$ type along the boundary. Thus, for
numerical accuracy in the present formulation, the assumed tractions,
$T_{li}$, for a singular element must also contain a $1/\sqrt{r}$ type behavior to be
compatible with those generated by $u_h$. Again, further details of the
actual construction of the singular and regular elements can be found
in Refs 17 and 18.

Advantages of the Hybrid Displacement Model

1. The present formulation leads to matrix equations with nodal dis-
placements and stress intensity factors as unknowns. Since inter-element
displacement compatibility is satisfied, it is easy to establish mathematical
convergence of the solution for nodal displacements as well as stress in-
tensity factors. Details of numerical convergence studies have already
been presented in Ref 18.

2. Since the displacement along the boundary of a quadrilateral element
in the present hybrid formulation is quadratic in nature, these quadrilateral
elements are automatically "compatible" with similar quadrilateral ele-
ments with quadratic boundary displacements, but derived through the
common (compatible displacement) finite-element model. Thus, the stiff-
ness matrix for the four singular elements derived from the present pro-
cedure can be first assembled and then merged into the stiffness matrix
for the remainder of the structure obtained through the more common
displacement finite-element method. Since inter-element compatibility is
still maintained, earlier arguments about convergence still apply.

3. Since boundary tractions for each element are introduced as inde-
pendent variables in the present formulation, it is easy to satisfy any
stress-free conditions, especially on the crack surface. This is done by
simply assuming zero tractions in the formulation, on boundary segment
of the element where such conditions should exist. Thus, stress-free con-
ditions are satisfied a priori in the present formulation, as opposed to
these conditions becoming the natural boundary conditions of the vari-
tional principle in the usual displacement formulation. In the hybrid stress
formulation of Ref 9, such stress-free boundary conditions are also en-
forced a priori by properly choosing the stress field within each element
such that it generates zero tractions on any segment of the boundary. Such
a procedure becomes tedious for arbitrary quadrilateral elements that may
have curved boundaries. An accurate satisfaction of the stress-free conditions is shown to be mandatory for the numerical performance of the hybrid stress model in Ref 9.

4. Since element boundary displacements are assumed independently, it is an easy task to assume boundary displacements along boundaries of elements that share the crack tip as a node, such that they have the correct variation with respect to radial distance ($\sqrt{r}$ in the present analysis).

5. In evaluating the strain energy density integrals for singular elements, where in the assumed displacement field involves a combination of regular polynomial and $\sqrt{r}$ types of behavior, it can be seen that one encounters three types of integrands: (a) products of regular type tensors, (b) products of singular stress tensor ($1/\sqrt{r}$ type) and singular strain tensor of $1/\sqrt{r}$ type, and (c) product of regular stress tensor and singular strain tensor. It can be shown easily that, since the asymptotic solution for the singular stress tensor near the crack tip is self-equilibrated, by employing the divergence theorem and proper variable transformations, integrals of types (b) and (c) can be converted to boundary integrals free from singularities. Thus, the problem of having to numerically integrate singular integrals is avoided.

6. Available solutions for plastic yielding near the crack tip suggest that, in the limit of perfect plasticity, there may be singularities in strains but only finite stresses. Thus, construction of the hybrid displacement element near the crack tip that has proper strain singularities, but finite stress, appears attractive. The hybrid stress formulation, on the other hand, appears more complicated for such plasticity problems, and has not been attempted so far.

Applications

Center Crack in a Bimaterial Tension Plate

The legend in Fig. 1 shows the geometry, loading conditions, and finite element breakdown for the case of an eccentric crack in a bimaterial tension plate. Each of two materials was considered as homogeneous and isotropic. The theoretical solution for the corresponding infinite plate problem was given by Erdogan and Biricikoglu [21]. The problem was considered as one of plane strain and the load intensities $\sigma_1$ and $\sigma_2$ (Fig. 1) were considered to be related as $(1 - \nu_1) \sigma_i/E_i = (1 - \nu_2) \sigma_i/E_i$. There are three singular points in the present problem: the two crack tips and the interface point. The nature of stress singularities, as seen from the referenced theoretical solution, are of $1/\sqrt{r}$ type near the crack tips and $1/r^2$ type (where $\sigma$ depends on material properties) near the interface point. The latter built-in singularity at the interface point is $1/r^{0.273692}$.
in stresses. Because of symmetry, only one half of the plate is used in the present finite-element analysis. A total of 60 quadrilateral elements with 215 nodes were employed. In the present analysis, a total of six singular elements are used: two each at the above-mentioned singular points. The calculated values of the four stress intensity factors are given in the following:

<table>
<thead>
<tr>
<th></th>
<th>$K_1$ (Material 1)</th>
<th>$K_1$ (Material 2)</th>
<th>$K_0$ (Interface)</th>
<th>$K_t$ (Interface)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present Erdogan and Biricikoglu [21]</td>
<td>1.459 $\sigma_1$</td>
<td>2.890 $\sigma_1$</td>
<td>$-0.116 \sigma_1$</td>
<td>0.0093 $\sigma_1$</td>
</tr>
<tr>
<td></td>
<td>1.375 $\sigma_1$</td>
<td>2.767 $\sigma_1$</td>
<td>$-0.119 \sigma_1$</td>
<td>0.0437 $\sigma_1$</td>
</tr>
</tbody>
</table>

Thus, there is an error of about 6 percent in $K_1$, and an error of 4.5 percent in $K_2$. However, the stress-intensity factors at the interface are in errors of over one-hundred percent. Attempts were not made at the refinement of the finite element grid to study to rate of convergence of the present solution to that given in the referenced report.
Center Crack Orthotropic Tension Plate

Accuracy of the developed procedure was assessed by analyzing a centrally-cracked orthotropic tension plate and by comparing the obtained results with the analytical solution by Bowie and Freese [22]. Stress intensity factors in this and subsequent figures (except Fig. 6) are presented in non-dimensional form. The presented results can thus be scaled provided the material properties remain identical. Principal directions of the orthotropic material were aligned in the two lines of symmetry and therefore only one quadrant of the plate was considered in this analysis. The legend in Fig. 2 shows the 24-element and 93-node quadrant used in this analysis. The material properties being considered are: ratios of modulus of elasticity, \( E_x/E_y = 0.3, 0.7, 1.0, 1.5, \) and \( 4.5 \), and the computed variations in stress intensity factors are shown in terms of the ratios of modulus of elasticity. An excellent agreement between the results of finite-element analysis and that of Bowie and Freese is noted.

Double Edge-Cracked Tension Plate

The legend in Fig. 3 shows element and nodal breakdown of two orthotropic double-edge-cracked tension specimens with two aspect ratios of 4.
and 1, respectively. The material properties\(^1\) were held constant in this case and the crack depth was varied from 0.2 to 0.8 of half-specimen width. Figure 3 shows the variation in stress intensity factor with variation in crack depth for two orientations of the principal directions of orthotropic material properties. Noticeable differences in the stress intensity factor with difference in aspect ratios are noted for the material which is very rigid in longitudinal direction. Also noted are the stress intensity factors for shorter crack length which are significantly lower than the theoretical values of 1.12 for isotropic double-edge-notched tension plate with short cracks.

The legend in Fig. 4 shows the element and nodal breakdown of the same double-edge-cracked tension plate but with the principal direction oriented \(\pm 45^\circ\) to the geometric axis of symmetry. Due to the antisymmetry of the problem, the entire plate had to be used in finite-element analysis.

\(^1\)The material properties shown in Figs. 4 and 5 relate to carbon epoxy laminate with parallel ply at 0 and 90 deg and angle ply at \(\pm 45\) deg.
thus increasing the number of elements and nodes to 80 and 103, respectively. Obviously, the same nodal breakdown can be used for arbitrary orientation of the principal axes of the material. Figure 4 shows the variation in stress intensity factor with variation in crack length for $E_{+45}/E_{-45} = 14.285$. These stress intensity factors do not differ substantially from those shown in Fig. 3. The large values of $K_{II}$, however, indicate the influence of the orientation of the reinforcing fibers in the material.

**Three-Point Bend Specimen**

The legend in Fig. 5 shows element and nodal breakdowns of a three-point bend specimen composed of orthotropic materials. Again, the material properties were held constant and the stress intensity factors were computed for crack depth from 0.4 to 0.8 of specimen width for two

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Footnote: The material properties shown in Figs. 4 and 5 relate to carbon-epoxy laminate with parallel ply at 0 and 90 deg and angle ply at $\pm 45$ deg.
orientations of principal directions of the material properties. Figure 6 also shows a comparison of the computed stress intensity factor and that by Srawley et al [23] for an isotropic material.

Orthotropic Tension Plate with Slanted Crack

The legend in Fig. 6 shows element nodal breakdown of an orthotropic tension plate with a crack inclined at 45° to the direction of loading (y-axis). The principal directions of orthotropy are the x and y-axes. Because of geometrical asymmetry, the entire plate was analyzed, which involved 96 elements and 260 nodes. The finite-element solutions for the stress intensity factors were obtained as: $K_1 = 1.0195$ and $K_{II} = 1.0759$. The available solution for an orthotropic infinite plate by Sih, Paris, and Irwin [24] suggests that $K_1 = K_{II} = 1.0539$. Thus, the calculated value for $K_1$ for the present finite plate is about 3 percent lower than the theoretical value for the infinite plate, whereas the computed $K_{II}$ is about 2 percent higher than the corresponding theoretical value for the infinite plate. It is interesting to note, however, that for a similar case of an isotropic plate with a slanted crack, Wilson [25] has found that $K_1$ is higher and $K_{II}$ is lower than the corresponding theoretical solutions.
FIG. 6—Finite-element breakdown in an orthotropic tension plate with a slanted crack.

Conclusion

The validity of an assumed displacement hybrid finite-element procedure, in calculating stress intensity factors for two-dimensional fracture problems involving composite materials, is established. Several new results for cracks running through dissimilar media, double-edge cracks in orthotropic tension specimens for different orientations of principal axes of orthotropy, different crack depths, and different crack orientations have been presented.

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