NON-LINEAR VIBRATIONS OF A FLAT PLATE
WITH INITIAL STRESSES

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(Received 16 December 1974, and in revised form 25 April 1975)

Non-linear free vibrations of a simply supported rectangular elastic plate are examined, by using stress equations of free flexural motions of plates with moderately large amplitudes derived by Herrmann. A modal expansion is used for the normal displacement that satisfies the boundary conditions exactly, but the in-plane displacements are satisfied approximately by an averaging technique. Galerkin technique is used to reduce the problem to a system of coupled non-linear ordinary differential equations for the modal amplitudes. These non-linear differential equations are solved for arbitrary initial conditions by using the multiple-time-scaling technique. Explicit values of the coefficients that appear in the forementioned Galerkin system of equations are given, in terms of non-dimensional parameters characterizing the plate geometry and material properties, for a four-mode case, for which results for specific initial conditions are presented. A comparison of the results with those obtained in previous studies of the problem is presented. In addition, effects of prescribed edge loadings are examined for the four-mode case.

1. INTRODUCTION

Non-linear effects have long been recognized to play an important role in determining the stability and response of thin plates and shells. Chu and Herrmann [1] first presented an analysis for flat plates, where they also demonstrated the consistency of neglecting in-plane inertia terms in the study of non-linear vibrations of a plate. The approach used by Chu and Herrmann was to convert the stress equations of motion first derived by Herrmann [2] to displacement equations of motion, neglect the in-plane inertias, choose single mode shapes for the displacements w (normal) and for u and v (in-plane) which satisfy the boundary conditions and the u and v equations of motion identically, and solve the remaining equation, neglecting the effects of all but the lowest modes, for period as a function of amplitude.

The purpose of this paper is to present a solution to the stress equations of motion derived by Herrmann [2] with in-plane inertias neglected for a general n-mode case for w, and also to study the effects of prescribed stress resultants along the edges. In this solution, a stress function is assumed which identically satisfies the u and v equations of motion, but which cannot satisfy exactly the boundary conditions on u and v. Therefore these boundary conditions are satisfied in an average sense, which has been shown to be a satisfactory approximation for a plate by Bolotin [3], Fralich [4], and Dowell [5].

2. BASIC EQUATIONS AND PROBLEM FORMULATION

The stress equations of free flexural motions of rectangular elastic plates with moderately
large amplitudes are shown by Herrmann [2] to be (a list of notation is given in Appendix II)

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2},
\]

\[
\frac{\partial N_y}{\partial y} + \frac{\partial N_{yx}}{\partial x} = \rho \frac{\partial^2 v}{\partial t^2},
\]

\[
\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial}{\partial x} \left( N_x \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( N_y \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial x} \left( N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{yx} \frac{\partial w}{\partial x} \right) = \rho h \frac{\partial^2 w}{\partial t^2}.
\]

(1)

If the inertia in the plane of the plate is neglected, the first two of equations (1) can be satisfied identically by the definition of an in-plane stress function \( F \), such that

\[
N_x = \frac{\partial^2 F}{\partial y^2}, \quad N_y = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}.
\]

(2)

Substituting expressions (2) into the third of equations (1) and writing the moments \( M_x, M_y, \) and \( M_{xy} \) in terms of displacements, for an elastic, isotropic plate, one has:

\[
-D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \frac{\partial^4 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x^2} = \rho h \frac{\partial^2 w}{\partial t^2}.
\]

(3)

The second equation which needs to be satisfied is obtained from compatibility of in-plane strains. Together with equation (3), modified to include applied edge loadings, this provides two equations in the two unknowns, \( w \) and \( F \):

\[
-D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \left( \frac{\partial^4 F}{\partial y^2} + N_x^2 \right) \frac{\partial^2 w}{\partial x^2} + \left( \frac{\partial^4 F}{\partial y^2} + N_y^2 \right) \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x^2} = \rho h \frac{\partial^2 w}{\partial t^2},
\]

(4a)

\[
\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = Eh \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right].
\]

(4b)

It is assumed that the plate is simply supported, and thus, the boundary conditions on the normal displacements are,

\[
\begin{align*}
&at x = 0, a, \quad w = \frac{\partial^2 w}{\partial x^2} = 0, \\
&at y = 0, b, \quad w = \frac{\partial^2 w}{\partial y^2} = 0.
\end{align*}
\]

(5)

Following Bolotin [3], Fralich [4], and Dowell [5], one can satisfy the in-plane displacements on the average:

\[
\int_0^b \int_0^a \frac{\partial u}{\partial y} \, dx \, dy = \int_0^b \left[ u(a, y) - u(0, y) \right] \, dy = 0,
\]

(6a)

\[
\int_0^b \int_0^a \frac{\partial v}{\partial y} \, dy \, dx = \int_0^a \left[ v(x, b) - v(x, 0) \right] \, dx = 0,
\]

(6b)

\[
\int_0^b N_{xy} \, dy \, dx = 0.
\]

(6c)

The first two of these state that the in-plane displacements "on the average" are zero at \( x = 0, a \) and \( y = 0, b \) respectively and the last that the average shear is zero.
By using the non-dimensional variables,
\[ \xi = x/a, \quad \eta = y/a, \quad \tilde{w} = w/h, \quad \delta = h/a, \quad r = a/b, \quad F = F/D. \]
\[ N_\xi^2 = (a^2/D) N_\xi^2, \quad N_\eta^2 = (a^2/D) N_\eta^2, \quad N_\xi_\eta = (a^2/D) N_\xi_\eta, \quad \tau = t[D/\rho a^4]^{1/2}, \]
the equations (4) and (5) can be written in non-dimensional form as
\[ \frac{\partial^4 \tilde{w}}{\partial \xi^4} + 2 \frac{\partial^4 \tilde{w}}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 \tilde{w}}{\partial \eta^4} + \frac{\partial^2 \tilde{w}}{\partial \xi^2} = \frac{\partial^2 F}{\partial \xi^2} \frac{\partial^2 \tilde{w}}{\partial \eta^2} + 2 \frac{\partial^2 F}{\partial \xi \partial \eta} \frac{\partial^2 \tilde{w}}{\partial \eta^2} + \frac{\partial^2 \tilde{w}}{\partial \xi \partial \eta} \frac{\partial^2 \tilde{w}}{\partial \eta^2} + N_\xi^2 \frac{\partial^2 \tilde{w}}{\partial \xi^2} + N_\eta^2 \frac{\partial^2 \tilde{w}}{\partial \eta^2} - 2 N_\xi_\eta \frac{\partial^2 \tilde{w}}{\partial \xi \partial \eta}, \]
\[ \frac{\partial^2 F}{\partial \xi^2} + 2 \frac{\partial^2 F}{\partial \xi^2 \partial \eta^2} + \frac{\partial^2 F}{\partial \eta^4} = 12(1 - \nu^2) \left[ \left( \frac{\partial^2 \tilde{w}}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 \tilde{w}}{\partial \xi^2} \frac{\partial^2 \tilde{w}}{\partial \eta^2} \right], \]
with the boundary conditions and additional definitions,
\[ \text{at } \xi = 0, 1, \quad \tilde{w} = \partial^2 \tilde{w}/\partial \xi^2 = 0 \]
\[ \text{at } \eta = 0, 1, \quad \tilde{w} = \partial^2 \tilde{w}/\partial \eta^2 = 0, \]
\[ \int_0^{1/r} \int_0^{1/r} \frac{\partial \tilde{u}}{\partial \xi} \, d\xi \, d\eta = 0, \]
\[ \int_0^{1/r} \int_0^{1/r} \frac{\partial \tilde{v}}{\partial \eta} \, d\eta \, d\xi = 0, \]
\[ \int_0^{1/r} \int_0^{1/r} N_\xi_\eta \, d\eta \, d\xi = 0, \]
\[ \frac{\partial^2 F}{\partial \xi^2} = (a^2/D) N_\xi, \]
\[ \frac{\partial^2 F}{\partial \eta^2} = (a^2/D) N_\eta, \]
\[ \frac{\partial^2 F}{\partial \xi \partial \eta} = (a^2/D) N_\xi_\eta. \]
A modal expansion for \( \tilde{w} \) can be assumed as
\[ \tilde{w}(\xi, \eta, \tau) = \sum \sum A_{mn}(\tau) \sin m\pi \xi \sin n\pi \eta. \]
By substituting equation (10) into the right-hand side of equation (8b), one can solve for \( F \) completely as,
\[ F = F_h + F_p, \]
where subscripts \( h \) and \( p \) denote "homogeneous" and "particular", respectively.
It can be shown easily that,
\[ F_p = \sum I \sum \sum A_{ij} A_{mn} [a_{ijmn} \cos (m - i) \pi \xi \cos (n + j) \pi \eta + b_{ijmn} \cos (m + i) \pi \xi \cos (n + j) \pi \eta + c_{ijmn} \cos (m - i) \pi \xi \cos (n + j) \pi \eta + d_{ijmn} \cos (m + i) \pi \xi \cos (n - j) \pi \eta], \]
where
\[ a_{mn} = \begin{cases} 
\frac{3(1-v^2)r^2(mnij - m^2j^2)}{((m-i)^2 + r^2(n-j)^2)^2}, & \text{if } m \neq i, \text{ or } n \neq j, \\
0, & \text{if } m = i \text{ and } n = j, 
\end{cases} \]
\[ b_{mn} = \frac{3(1-v^2)r^2(mnij - m^2j^2)}{((m+i)^2 + r^2(n+j)^2)^2}, \]
\[ c_{mn} = \frac{3(1-v^2)r^2(mnij + m^2j^2)}{((m-i)^2 + r^2(n-j)^2)^2}, \]
\[ d_{mn} = \frac{3(1-v^2)r^2(mnij + m^2j^2)}{((m+i)^2 + r^2(n-j)^2)^2}. \]

For the homogeneous part, keeping in mind the “average” in-plane displacement boundary conditions, one can assume a simple function, as
\[ F_h = \frac{1}{2}(\tilde{N}_x \zeta^2 + \tilde{N}_v \eta^2 - 2 \tilde{N}_{xy} \zeta \eta), \] (12b)
where \( \tilde{N}_x, \tilde{N}_v \) and \( \tilde{N}_{xy} \) are physically the in-plane restraint stresses generated at the edges of the plate, due to the prevention of the in-plane displacements on the “average”. By substituting for \( F \) from equation (11) in equation (9f) and using boundary conditions (9c-e), it easily can be shown that
\[ R_x = \frac{5}{2} \pi^2 \sum_{m,n} A_{mn}^2 (m^2 + vn^2 r^2), \] (13a)
\[ R_v = \frac{3}{2} \pi^2 \sum_{m,n} A_{mn}^2 (n^2 r^2 + vm^2), \] (13b)
\[ R_{xy} = \sum_{m,n} S_{mn} A_{mn} A_{nm}, \] (13c)
where
\[ S_{mn} = \begin{cases} 
4r[a_{mn} + b_{mn} + c_{mn} + d_{mn}] & \text{if } (m+i) \text{ and } (n+j) \text{ are odd}, \\
0 & \text{if } (m+i) \text{ or } (n+j) \text{ is even.} 
\end{cases} \]

Once the functions \( F_h \) and \( F_p \) are solved for, as in equations (12a) and (12b)–(13c), respectively, one can substitute for the total \( F \) and for the assumed displacement function \( \tilde{w} \) as given in equation (10), into equation (8a), to obtain a single non-linear differential equation in time for the variables \( A_{mn} \). From this equation, by using the Galerkin technique and weighting in turn by each of the functions \( \sin n \pi \zeta \sin m \pi \eta \), where \( n = 1, 2 \ldots N, m = 1, 2 \ldots M \), and integrating over the plate midsurface, a system of \( M \times N \) ordinary, coupled non-linear differential equations in time can be obtained for the unknowns, \( A_{mn} \):
\[ \frac{d^2 q_i}{dt^2} + \sum_j K_{ij} q_j + \sum_m \sum_n L_{ijmn} A_{jm} A_{nm} = 0 \quad \text{for } i = 1, 2 \ldots M \times N, \] (14a)
where, for added convenience, the variables have been redefined as
\[ \begin{bmatrix} A_{11} \\ A_{12} \\
\vdots \\
A_{1N} \\
A_{21} \\
\vdots \\
A_{MN} \end{bmatrix} = \{q_i\}. \] (14b)
Explicit values of the coefficients $K_{ij}$ and $L_{ijkl}$ are given in Appendix I in terms of the non-dimensional parameters defined as equations (7). It can be shown that equation (14a) is correct to order $(q)^5$ in the principal non-linear terms, because of inherent assumptions in the plate equations (1).

3. SOLUTION OF THE PROBLEM

3.1. APPLICATION OF SOLUTION TECHNIQUE

One can rewrite equation (14a) in matrix form as

$$\left(\frac{d^2 q_i}{d\tau^2}\right) + \{K_{ij}\}\{q_j\} + \{L_{ijkl}\}\{q_j q_k q_l\} = 0.$$  \hspace{1cm} (15)

By finding the eigenvalues and eigenvectors of the linear part of equation (15), equation (15) can be written in terms of normal co-ordinates such that the linear part of equation (15) is uncoupled, as

$$\left(\frac{d^2 \tilde{q}_i}{d\tau^2}\right) + \{\tilde{K}_{ij}\}\{\tilde{q}_j\} + \{L_{ijkl}\}\{\tilde{q}_j \tilde{q}_k \tilde{q}_l\} = 0,$$ \hspace{1cm} (16)

where $\tilde{q}_i$ are the normal co-ordinates and are related linearly to $q_i$ as

$$\{\tilde{q}_i\} = [M_{ij}]{q}_j,$$ \hspace{1cm} (16b)

and similarly $\tilde{K}_{ij}$ is related to $K_{ij}$ as

$$[\tilde{K}_{ij}] = [K_{ij}][M_{ij}].$$

and $L_{ijkl}$ is related to $L_{ijkl}$ by substitution of co-ordinates from equation (16b) into the third term of equation (16) and suitable rearrangement of products. Equation (16) now can be rewritten in the form,

$$d^2 \tilde{q}_i/d\tau^2 + \Omega_i^2 \tilde{q}_i + \sum_j \sum_k \sum_l L_{ijkl} \tilde{q}_j \tilde{q}_k \tilde{q}_l = 0 \quad \text{for } i = 1, 2 \ldots M \times N.$$  \hspace{1cm} (17)

The system of equations (17) now can be solved for any given initial conditions of the type

$$\tilde{q}_i = e^{1/2} \alpha_i, \quad \frac{d\tilde{q}_i}{d\tau} = e^{1/2} \beta_i \quad \text{at } \tau = 0,$$ \hspace{1cm} (18)

where $\varepsilon$ is an arbitrarily small parameter that defines the amplitudes of the initial conditions. Correspondingly, one can define, for the solution of equation (17), that

$$\tilde{q}_i(\tau) = e^{1/2} \tilde{q}_i(\tau).$$  \hspace{1cm} (19)

Upon substituting equation (19) into it, equation (17) can be reduced to

$$d^2 \tilde{q}_i/d\tau^2 + \Omega_i^2 \tilde{q}_i + \varepsilon \sum_j \sum_k \sum_l L_{ijkl} \tilde{q}_j \tilde{q}_k \tilde{q}_l = 0.$$  \hspace{1cm} (20)

Equations of the previous type can be solved by using the perturbation method of multiple-time-scales which is discussed for single-degree-of-freedom systems by Kevorkian [6] and Nayfeh [7].

One can define multiple-time-scales $\tau_0, \tau_1, \ldots$ etc., accordingly as

$$\tau_m = (\varepsilon)^m \tau.$$ \hspace{1cm} (21)

One now can assume that there exists a uniformly valid asymptotic solution for $\tilde{q}_i$ of the form

$$\tilde{q}_i = \sum_{m=0}^{M} \varepsilon^m \tilde{q}_{im}(\tau_0, \tau_1, \ldots \tau_m) + O[\varepsilon^{M+1}],$$ \hspace{1cm} (22)

where $q_{im}$ are now functions of the independent time scales $\tau_0, \tau_1, \ldots$ etc. From equation (21) it can be seen that

$$d/d\tau = \sum_m \varepsilon^m \partial/\partial \tau_m, \quad d^2/d\tau^2 = \sum_m \sum_n \varepsilon^m \varepsilon^n \partial^2/\partial \tau_m \partial \tau_n.$$ \hspace{1cm} (23)
Substituting for $\frac{d^2}{dr^2}$ and $\tilde{q}_i$ from equations (23) and (22), respectively, into equation (20), and identifying terms multiplied by equal powers of $e$, one can obtain the following system of equations:

\begin{align*}
text{terms with } e^0: & \quad \frac{\partial^2}{\partial t^2} \tilde{q}_{i0}/\partial \tau_0^2 + \Omega_i^2 \tilde{q}_{i0} = 0; \\
text{terms with } e^1: & \quad \frac{\partial^2}{\partial t^2} \tilde{q}_{i1}/\partial \tau_1^2 + \Omega_i^2 \tilde{q}_{i1} + 2 \frac{\partial^2}{\partial t^2} \tilde{q}_{i0}/\partial \tau_0 \partial \tau_1 + \sum_{I} L_{iI} \tilde{q}_{j0} \tilde{q}_{k0} \tilde{q}_{l0} = 0.
\end{align*}

The boundary conditions, equation (18), likewise can be transformed as
\begin{align*}
\tilde{q}_{i0} = & \alpha_i; \quad \tilde{q}_{i1} = \tilde{q}_{i2} = \cdots = 0 \quad \text{for } \tau_m = 0, \\
\frac{\partial^2 \tilde{q}_{i0}}{\partial \tau_0^2} = & \beta_i, \quad \frac{\partial^2 \tilde{q}_{i0}}{\partial \tau_1} + \frac{\partial \tilde{q}_{i1}}{\partial \tau_0} = 0, \quad \frac{\partial^2 \tilde{q}_{i1}}{\partial \tau_2} + \frac{\partial \tilde{q}_{i2}}{\partial \tau_0} = 0, \text{ etc.}, \quad \text{for } \tau_m = 0.
\end{align*}

3.2. SOLUTION FOR THE ZEROTH-ORDER SYSTEM

Equation (24) has the general solution
\begin{equation}
\tilde{q}_{i0} = A_i(\tau_1, \tau_2, \ldots) e^{i\Omega_i \tau_0} + A_i^\dagger(\tau_1, \tau_2, \ldots) e^{-i\Omega_i \tau_0},
\end{equation}
where $\lambda = \sqrt{-1}$, $A_i$ is a complex quantity that is a function of the time scales $\tau_1, \tau_2, \ldots$ and $A_i^\dagger$ is the complex conjugate of $A_i$. $A_i$ and $A_i^\dagger$ can be determined from the initial conditions (26) and (27).

3.3. SOLUTION FOR THE FIRST-ORDER SYSTEM

After substituting for $\tilde{q}_{i0}$ from equation (28), the first-order system, equation (25) can now be written as
\begin{equation}
\frac{\partial^2 \tilde{q}_{i1}}{\partial \tau_0^2} + \Omega_i^2 \tilde{q}_{i1} = -2\lambda \Omega_i \frac{\partial A_i}{\partial \tau_1} e^{i\Omega_i \tau_0} + 2\lambda \Omega_i \frac{\partial A_i^\dagger}{\partial \tau_1} e^{-i\Omega_i \tau_0} - \\
- \sum_{j} \sum_{k} \sum_{I} L_{iI} \left[ A_j A_k A_i e^{i(\Omega_j + \Omega_k + \Omega_i) \tau_0} + A_i^\dagger A_k^\dagger A_i e^{-i(\Omega_j + \Omega_k + \Omega_i) \tau_0} + \\
+ A_j^\dagger A_k A_i e^{i(\Omega_j - \Omega_k + \Omega_i) \tau_0} + A_j A_k^\dagger A_i e^{-i(\Omega_j - \Omega_k + \Omega_i) \tau_0} + \\
+ A_j A_k^\dagger A_i e^{i(\Omega_j - \Omega_k - \Omega_i) \tau_0} + A_j^\dagger A_k A_i e^{-i(\Omega_j - \Omega_k - \Omega_i) \tau_0} + \\
+ A_j^\dagger A_k^\dagger A_i e^{i(\Omega_j - \Omega_k - \Omega_i) \tau_0} + A_j A_k A_i e^{-i(\Omega_j - \Omega_k - \Omega_i) \tau_0} \right].
\end{equation}

In solving equation (29), the terms on the right-hand side which vary with frequency $\Omega_i$ must be suppressed; otherwise, these would lead to spurious resonances in the solution and hence would destroy its uniformity. Thus, by grouping terms on the right-hand side which vary with frequency $\Omega_i$, equation (29) can be written as
\begin{equation}
\frac{\partial^2 \tilde{q}_{i1}}{\partial \tau_0^2} + \Omega_i^2 \tilde{q}_{i1} = \left( 2\lambda \Omega_i \frac{\partial A_i}{\partial \tau_1} + A_i \sum_n \gamma_n A_n A_i^\dagger \right) e^{i\Omega_i \tau_0} + \\
\left( 2\lambda \Omega_i \frac{\partial A_i^\dagger}{\partial \tau_1} - A_i^\dagger \sum_n \gamma_n A_n A_i \right) e^{-i\Omega_i \tau_0} + \sum_r P_r e^{i\Omega_r \tau_0} + \sum_r P_r^\dagger e^{-i\Omega_r \tau_0}.
\end{equation}

In the foregoing, $\Omega_r$ stands for the combinations
\begin{equation}
\Omega_r = \pm \Omega_j \pm \Omega_k \pm \Omega_i
\end{equation}
such that $\Omega_r \neq \Omega_i$. The explicit expressions for $P_r$ are lengthy and hence are not recorded here. After some manipulation, the coefficients $\gamma_n$ associated with terms varying with frequency $\Omega_i$ in equation (30) can be shown to be
\begin{equation}
\gamma_n = 3L_{i11i1} \quad \text{for } n = i, \quad \gamma_n = 2(L_{i1in} + L_{in1i} + L_{i1ni}) \quad \text{for } n \neq i.
\end{equation}
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Spurious resonance in the solution of equation (30) can be eliminated by setting

\[ 2\lambda\Omega I \partial A / \partial \tau + A \sum_n \gamma_n A_n N = 0. \]

The system of equations (33) can be solved easily by letting

\[ A_n = \psi_i e^{\lambda_i \tau}, \]

where \( \psi_i \) is a real quantity. Substituting equation (34) into equation (33) and separating the real and imaginary parts, one obtains

\[ \partial \psi_i / \partial \tau = 0 \quad \text{and} \quad \partial \psi_i / \partial \tau = (1/2\Omega I) \sum_n \gamma_n \psi_i^2. \]

From the first of equations (35) one can write

\[ \psi_i = \tilde{\psi}_i(\tau_2, \tau_3, \ldots \tau_m) \]

and hence, from the second of equations (35),

\[ \theta_i = (1/2\Omega I) \left[ \sum_n \gamma_n \psi_i^2 \right] \tau_1 + \theta_{i0}(\tau_2, \tau_3, \ldots \tau_m). \]

By using equations (36) and (37), equation (34) can be written as

\[ A_i(\tau_1, \tau_2, \ldots \tau_m) = \tilde{A}_i \exp \left[ -\left( \frac{\lambda}{2\Omega I} \sum_n \gamma_n \psi_i^2 \right) \tau_1 \right] \]

where

\[ \tilde{A}_i(\tau_2, \tau_3, \ldots \tau_m) = \tilde{\psi}_i(\tau_2, \tau_3, \ldots \tau_m) \exp \left[ \lambda \theta_{i0}(\tau_2, \tau_3, \ldots \tau_m) \right]. \]

Finally, one now can solve equation (30) as

\[ q_i = B_i(\tau_1, \tau_2, \ldots \tau_m) e^{i\lambda \Omega I \tau} + B_i^*(\tau_1, \tau_2, \ldots \tau_m) e^{-i\lambda \Omega I \tau} + \sum_r (P_r e^{i\lambda \Omega I \tau} + P_r^* e^{-i\lambda \Omega I \tau})/(\Omega_r^2 - \Omega_i^2), \]

where \( B_i \) and \( B_i^* \) are determined from the initial conditions

\[ \bar{q}_{i} = 0, \quad \partial \bar{q}_{i0} / \partial \tau = -\partial \bar{q}_{i0} / \partial \tau \quad \text{for} \quad \tau_m = 0. \]

By using equations (19), (21), (22), (28), (38) and (40), the solution for \( \bar{q}_i(\tau) \), correct to the order \( \varepsilon^{3/2} \), can be written as

\[ \bar{q}_i(\tau) = e^{1/2} \bar{q}_{i0} + e^{3/2} \bar{q}_{i1} + O(\varepsilon^{5/2}) \]

\[ = e^{1/2} \left[ \tilde{A}_i e^{i\lambda \Omega I \tau} + \tilde{A}_i^* e^{-i\lambda \Omega I \tau} \right] + e^{3/2} \left[ B_i e^{i\lambda \Omega I \tau} + B_i^* e^{-i\lambda \Omega I \tau} \right] + \]

\[ + (P_r e^{i\lambda \Omega I \tau} + P_r^* e^{-i\lambda \Omega I \tau})/(\Omega_r^2 - \Omega_i^2) \] + \( O(\varepsilon^{5/2}) \),

where \( \tilde{A}_i(\tau_2, \tau_3, \ldots \tau_m) \) and \( B_i(\tau_1, \tau_2, \ldots \tau_m) \) are determined from initial conditions (26), (27) and (41), and where

\[ \bar{q}_i = \Omega_i[1 + (\varepsilon/2\Omega_i^2) \sum_n \gamma_n \tilde{A}_n A_n^*]. \]

Equation (42) indicates that, because of the non-linear coupling between the modes, even though the initial conditions correspond only to a single mode, other modes also would be excited. Further, it can be seen that if the principal response of the excited mode is of order \( O(\varepsilon^{1/2}) \), the other modes are excited to order \( O(\varepsilon^{3/2}) \). Equation (43), on the other hand, indicates the effect of coupling between modes on the frequencies of natural oscillations of each mode. The nature of the non-linearity (whether hard-spring type or soft-spring type) depends on whether the quantity \( \sum_n \gamma_n \tilde{A}_n A_n^* \) is positive or negative. From equation (32), it can be seen that \( \gamma_n \) depends on the non-linearity coefficients \( \tilde{L}_{ukt} \). From Appendix I, it can be seen that each of these coefficients depends on the non-dimensional properties of the plate as well as the mode numbers under consideration. Specific examples to illustrate this follow.
4. EXAMPLES AND DISCUSSION

For purposes of illustration we will retain only the first four modes for \( \ddot{w} \): i.e., \( \ddot{w} = \sum_{m=1}^{2} \sum_{n=1}^{2} q_{m} \sin mn \xi \sin n \eta \), where

\[
q_{k} = \begin{pmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{pmatrix} = A_{mn}.
\]

For comparison with other authors, we will look at the case of no applied edge loadings first.

Consider the plate excited in the first mode with the initial conditions,

\[
\ddot{w}(\xi, \eta, \tau) = 0, \quad \partial \ddot{w} / \partial \tau = e^{1/2} \beta_{1} \Omega_{1} \sin \pi \xi \sin \pi \eta \quad \text{at} \quad \tau = 0,
\]

where \( \Omega_{1} \) is the linear natural frequency of the first mode. The corresponding initial conditions for the generalized co-ordinates are

\[
\ddot{q}_{k} = 0 \quad (i = 1, 2, \ldots), \quad \partial \ddot{q}_{k} / \partial \tau = \Omega_{1} \beta_{1}, \quad \partial \ddot{q}_{k} / \partial \tau = 0 \quad (i = 2, 3, \ldots) \quad \text{at} \quad \tau = 0. \tag{45}
\]

From equations (28) and (38), the zeroth-order solution, with initial conditions (45), can be written as

\[
\ddot{q}_{10} = \beta_{1} \sin \tilde{\Omega}_{1} \tau, \quad \ddot{q}_{k0} = 0, \quad k > 0, \tag{46}
\]

where

\[
\tilde{\Omega}_{1} = \Omega_{1}[1 + e\beta_{1}^{2} \gamma_{1}/8 \Omega_{1}^{2}] \tag{47}
\]

From Appendix I and equation (32), it can be seen that

\[
\Omega_{1} = \pi^{2}(1 + r^{2}) \tag{48}
\]

and

\[
\gamma_{1} = 3 L_{1111} = 3 \pi^{4}[\frac{1}{2}(1 - \nu^{2})(1 + r^{4}) + \frac{3}{2}(1 + 2 \nu r^{2} + r^{4})]. \tag{49}
\]

From equation (49) it is evident that \( \gamma_{1} \) is always positive and hence the non-linear frequency always increases with amplitude, as indicated by equation (47). This result agrees with that

![Figure 1. First mode. Comparison of present results with the results obtained by Chu and Herrmann in reference [1]. ——, Present results; ——, Chu and Herrmann; \( N_{I} = N_{S} = 0 \).](image-url)
found by Chu and Herrmann [1], and is plotted in Figure 1 for \( \delta = 0.01, \nu = 0.3 \). Also, the first-order solution \( q_{11} \) as given by equation (40), and initial conditions (41), can be found as

\[
q_{11} = (L_{i1}/\Omega_1)(\varepsilon \beta_1 - 3 \beta_1^3 L_{1111}/4(\Omega_1^2 - 9\Omega_1^2)) \sin \Omega_1 \tau + [\beta_1^3 L_{1111}/4(\Omega_1^2 - 9\Omega_1^2)] \sin 3\Omega_1 \tau
\]  

(50a)

and

\[
q_{ii} = (3 \beta_1^3 L_{i1}/4\Omega_1^3)(L_{1111}/(\Omega_1^2 - \Omega_i^2) - L_{1111}/(\Omega_1^2 - 9\Omega_1^2))(\sin \Omega_i \tau - [3 \beta_1^3 L_{1111}/4(\Omega_1^2 - 9\Omega_1^2)] \sin 3\Omega_1 \tau)
\]

\times \sin \Omega_i \tau + [\beta_1^3 L_{1111}/4(\Omega_1^2 - 9\Omega_1^2)] \sin 3\Omega_i \tau \quad \text{for} \ i \neq 1.

(50b)

Thus, from equations (42), (46), (50a) and (50b), the response of the plate subjected to initial conditions (44) is given by

\[
\ddot{w}(\xi, \eta, \tau) = e^{1/2} \beta_1 \sin \Omega_1 \tau \sin \pi \xi \sin \pi \eta + e^{3/2} \sum_{m=1}^{M} \sum_{n=1}^{N} q_{ii} \sin m \pi \xi \sin n \pi \eta,
\]

(51)

where \( q_i \) is defined in terms of \( A_{mn} \) in equation (14b).

It can be seen from equations (50a, b) that corresponding to an initial condition in the first mode of order \( e^{1/2} \), the response consists of the first mode with amplitude \( O(\varepsilon^{1/2}) \) the frequency \( \Omega_i \) and the other modes excited with amplitudes of order \( O(\varepsilon^{3/2}) \) because of the third-order non-linear terms present in the equation of motion. Equations (50a, b) also indicate the presence of superharmonics of order 3 in the response.

As a second example, consider the plate excited initially in the second mode: i.e.,

\[
\ddot{w}(\xi, \eta, \tau) = 0, \quad \partial \ddot{w}/\partial \tau = e^{1/2} \beta_2 \sin \pi \xi \sin 2\pi \eta \quad \text{at} \ \tau = 0.
\]

(52)

By following the same procedure, the zeroth-order solution can be written as

\[
q_{2i} = \beta_2 \sin \Omega_2 \tau \quad \text{and} \quad q_{10} = 0, \quad i \neq 2,
\]

(53)

where

\[
\Omega_2 = \Omega_2[1 + e^{1/2} \gamma_2/\Omega_2^2]
\]

(54)

and

\[
\gamma_2 = 3L_{2222}.
\]

(55)

It can be shown from Appendix I that

\[
\gamma_2 = 3\pi^4[3(1 - \nu^2)(1 + 16\nu^4)].
\]

(56)

The ratio of \( \tilde{\Omega}_2/\Omega_2 \) is plotted in Figure 2(a).

As a third example, consider the plate excited in the fourth mode: i.e.,

\[
\ddot{w}(\xi, \eta, \tau) = 0, \quad \partial \ddot{w}/\partial \tau = e^{1/2} \beta_4 \sin 2\pi \xi \sin 2\pi \eta \quad \text{at} \ \tau = 0.
\]

(58)
Again by following the same procedure, the zeroth-order solution can be written as
\[ q_{a0} = \tilde{\beta}_a \sin \tilde{\Omega}_a \tau \quad \text{and} \quad \tilde{\Omega}_{a0} = 0, \quad i \neq 4, \] (59)
where
\[ \tilde{\Omega}_a = \Omega_a [1 + \varepsilon \beta_a^2 \gamma_a / 8 \Omega_a^2] \] (60)
with
\[ \Omega_a = 4\pi^2(1 + r^2) \] (61)
and
\[ \gamma_a = 3L_{44+4} = 18\pi^4(1 - v^2)(1 + r^4). \] (62)
The ratio of \( \tilde{\Omega}_a / \Omega_a \) is plotted in Figure 2(b).

5. EFFECTS OF APPLIED EDGE LOADINGS

The effects of applied edge loadings on the non-linear frequency response of the plate are now examined for the same three examples given in previous paragraphs by equations (44), (52) and (58). For simplicity, \( N_{a0} \) is taken to be zero, while \( N_{a}^* \) and \( N_{a}^* \) are varied as fractions of their respective unidirectional buckling loads.

From Appendix I and equation (17), it can be seen that the inclusion of non-zero applied edge loadings \( N_{a}^* \) and \( N_{a}^* \) affects only \( \tilde{\Omega}_a \). It easily can be shown then that the solutions of the examples with non-zero applied edge loadings \( N_{a}^* \) and \( N_{a}^* \) will be identical to those with zero applied edge loadings, as given in the previous section in equations (46)–(47), (49)–(51), (53)–(54), (56), (60), and (62), except for the following changes in \( \tilde{\Omega}_a \):
\[ \tilde{\Omega}_a = \left[ \pi^4(1 + r^2)^2 + \pi^2(N_{a}^* + 4r^2 N_{a}^*) \right]^{1/2}, \] (63)
\[ \tilde{\Omega}_3 = \left[ \pi^4(1 + 4r^2)^2 + \pi^2(N_{a}^* + 4r^2 N_{a}^*) \right]^{1/2}, \] (64)
\[ \tilde{\Omega}_4 = \left[ \pi^4(4 + 4r^2)^2 + \pi^2(4N_{a}^* + 4r^2 N_{a}^*) \right]^{1/2}. \] (65)
The non-linear frequencies for the first mode are plotted in Figures 3(a) and (b). It can be seen that negative (compressive) edge loadings produce a softening effect on the non-linear frequency, as one expects, for amplitude ratios less than one. However, as the amplitude ratio becomes greater than one, Figures 3(a) and (b) show that negative edge loadings begin to produce a hardening effect on the non-linear frequency, a result not consistent with experiment. This result shows that the present analysis, which includes only third-order non-linearities, is not valid for amplitude ratios larger than one. Thus, to accurately study the non-linear frequency for amplitude ratios larger than one, fifth-order non-linearities must be included in the analysis.

Figure 3. First mode. Effects of applied edge loadings. (a) \( r = 1.0; N_{a}^* = 0; ---, N_{a}^* = 0; --, N_{a}^* = -0.5 \ (4\pi^2); ---, N_{a}^* = +0.5 \ (4\pi^2) \). (b) \( r = 0.5; N_{a}^* = 0; ---, N_{a}^* = 0; --, N_{a}^* = -0.5 \ (\pi^2(1.25)^2); --, N_{a}^* = +0.5 \ (\pi^2(1.25)^2) \).
6. SUMMARY AND CONCLUSIONS

A general n-mode solution for moderately large amplitude vibrations of flat plates with initial stresses has been presented. In-plane intertias have been neglected and boundary conditions on the in-plane displacements have been satisfied in an average sense.

With no initial stresses, good agreement was shown with the results obtained by Chu and Herrmann [1] for the first mode. As in the first mode case, non-linearities have a hardening effect on the higher modes: i.e., frequency increases with amplitude.

Finally, it has been shown that for amplitude ratios less than one, the inclusion of negative applied edge loadings produces a softening effect on the non-linear frequencies and the inclusion of positive applied edge loadings produces a hardening effect, a result consistent with experiment. For amplitude ratios greater than one, however, the effects begin to reverse themselves, indicating that fifth-order non-linearities must be included in the analysis to accurately study the non-linear frequencies for amplitude ratios greater than one.

REFERENCES


APPENDIX I

The coefficients of the Galerkin system of equations (14a) are given here in terms of the non-dimensional properties of the beam, as defined in the text. It should be noted that in the following, i is related to m and n by equation (14b); similarly, j, k, and l are related to a and b, c and d, and e and f, respectively, in the same manner.

\[ K_{ij} = (m^2 + n^2 r^2) \pi^4 + (m^2 N_1^2 + n^2 r^2 N_2^2) \pi^2 + 2N_3^2 \tilde{C}_{abmn} \] (66)

where

\[ \tilde{C}_{abmn} = \begin{cases} 16abmn & \text{when } a + m \text{ and } b + n \text{ are odd,} \\ 0 & \text{otherwise,} \end{cases} \]

\[ L_{ijkl} = 2\tilde{C}_{abefmn} - \tilde{C}_{abdefmn} + \delta_{ac} \delta_{bd} \delta_{em} \delta_{fn} 3\pi^4 [m^2(a^2 + b^2 r^2) + n^2 r^2(b^2 r^2 + a^2)] - 2S_{abdefmn}, \] (67)

where

\[ \tilde{a}_{abcdefmn} = \pi^4 r^3 e^2 \left \{ \int_{0}^{1/r} \int_{0}^{1/r} [a_{abcd}(d-b)^2 \cos (c-a) \pi \xi \cos (d-b) \pi \eta + \right. \]

\[ + b_{abcd}(d+b)^2 \cos (c+a) \pi \xi \cos (d+b) \pi \eta + \right. \]

\[ + c_{abcd}(d+b)^2 \cos (c-a) \pi \xi \cos (d+b) \pi \eta + \right. \]

\[ + d_{abcd}(d-b)^2 \cos (c+a) \pi \xi \cos (d-b) \pi \eta] \times \]

\[ \times \sin e \pi \xi \sin f \pi \eta \sin m \pi \xi \sin n \pi \eta \, dc \, dn, \]
\[ b_{abcdefmn} = \pi^4 r^3 f^2 \int_0^{\pi/2} \int_0^{\pi/2} [a_{abcd}(c - a)^2 \cos (c - a) \pi \xi \cos (d - b) \pi \eta + \\
+ b_{abcd}(c + a)^2 \cos (c + a) \pi \xi \cos (d + b) \pi \eta + \\
+ c_{abcd}(c - a)^2 \cos (c - a) \pi \xi \cos (d + b) \pi \eta + \\
+ d_{abcd}(c + a)^2 \cos (c + a) \pi \xi \cos (d - b) \pi \eta] \times \\
\times \sin \pi \xi \sin \pi \eta \sin m \pi \xi \sin \pi \eta \, d\xi \, d\eta, \]

\[ c_{abcdefmn} = \pi^4 r^3 f^2 \int_0^{\pi/2} \int_0^{\pi/2} [a_{abcdefmn}(c - a)(d - b) \sin (c - a) \pi \xi \sin (d - b) \pi \eta + \\
+ b_{abcdefmn}(c + a)(d + b) \sin (c + a) \pi \xi \sin (d + b) \pi \eta + \\
+ c_{abcdefmn}(c - a)(d + b) \sin (c - a) \pi \xi \sin (d + b) \pi \eta + \\
+ d_{abcdefmn}(c + a)(d - b) \sin (c + a) \pi \xi \sin (d - b) \pi \eta] \times \\
\times \cos \pi \xi \cos \pi \eta \sin m \pi \xi \sin \pi \eta \, d\xi \, d\eta, \]

\[ S_{abcdefmn} = -S_{abcd} c_{efmn}. \]

with

\[ a_{abcd} = \begin{cases} \\
3(1 - v^2) r^2 (abcd - b^2 c^2) & \text{when } a \neq c \text{ or } b \neq d, \\
0 & \text{when } a = c \text{ and } b = d. \\
\end{cases} \]

\[ b_{abcd} = \frac{3(1 - v^2) r^2 (abcd - b^2 c^2)}{((c + a)^2 + r^2(d + b)^2)^2}, \]

\[ c_{abcd} = \frac{3(1 - v^2) r^2 (abcd + b^2 c^2)}{((c - a)^2 + r^2(d + b)^2)^2}, \]

\[ d_{abcd} = \frac{3(1 - v^2) r^2 (abcd + b^2 c^2)}{((c + a)^2 + r^2(d - b)^2)^2}, \]

\[ S_{abcd} = \begin{cases} \\
4r(a_{abcd} + b_{abcd} + c_{abcd} + d_{abcd}) & \text{when } a + c \text{ and } d + b \text{ are odd,} \\
0 & \text{otherwise}. \\
\end{cases} \]

**APPENDIX II: NOTATION**

- \( x, y \) in-plane co-ordinates
- \( z \) normal co-ordinate
- \( u, v \) in-plane displacement
- \( w \) normal displacement
- \( h \) thickness of plate
- \( a, b \) length and width of plate, respectively
- \( w/h \)
- \( u/a \)
- \( v/a \)
- \( w/a \)
- \( \xi x/a \)
- \( \eta y/a \)
- \( \delta h/a \)
- \( r a/b \)
- \( E, v \) Young's modulus, Poisson's ratio
- \( \rho \) density
- \( D \) \( E h^3/12 (1 - v^2) \)
- \( \tau \) \( t(D/\rho a^4)^{1/2} \) non-dimensional time
$F$ stress function
$F' = F/D$

$N_{x}, N_{y}, N_{xy}$ in-plane stress resultants
$N_{x}', N_{y}', N_{xy}' = (N_{x}, N_{y}, N_{xy})(a^2/D)$

$N_{x}, N_{y}, N_{xy}$ applied stress resultants
$N_{x}' = N_{x}(a^2/D)$
$N_{y}' = N_{y}(a^2/D)$
$N_{xy}' = N_{xy}(a^2/D)$

$A_{1j}$ non-dimensional modal amplitudes, also redefined as $q_1, q_2, q_3,$ etc.

$\Omega_{l}$ linear frequency

$\varepsilon$ a small parameter that characterizes the amplitudes of initial conditions on $q_i$

$\tau_{m}$ multiple time scales

$q_{im}$ component of $q_i$ multiplied by $\varepsilon$