Theoretical Formulation
of Finite-Element Methods
in Linear-Elastic Analysis
of General Shells*

SATYANADHAM ATLURI** and THEODORE H. H. PIAN

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASS.

ABSTRACT

A systematic classification of the variational functionals whose stationarity conditions (Euler equations) can be used alternately to solve for the various unknowns in a boundary-value problem in linear-shell theory is made. The application of these alternate variational principles to a finite-element assembly of a shell and thus, the development of the properties of an individual discrete element are studied in detail. A classification of the finite-element methods, formulated from the variational principles by systematically relaxing the continuity requirements at the interelement boundaries of adjoining discrete elements is made.

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**Presently with the Department of Aeronautics and Astronautics, University of Washington, Seattle.
I. INTRODUCTION

With the widespread use of thin shells as load-carrying members in various structures, the need for a reliable analysis has been greatly felt over the past decade. Exact analytical solutions in linear-shell theory have been possible only in very limited cases of shells with simple geometry (such as cylinders, cones, and spheres), shells whose boundaries coincide with the coordinate curves used to define the shell surface, and shells without discontinuities (cutouts), or without any attachments (stringers, etc.). To overcome these shortcomings and the additional complexities caused by variable thickness and variable anisotropic properties, the use of an approximate method of analysis becomes necessary.

One such approximate method is the “finite-element method.” A review and a systematic classification of the finite-element methods as formulated from the variational principles in solid mechanics are given by Pian and Tong [1], who also discussed the application of such methods in plate-bending analysis. The problem of a shell is more complicated because, in the general bending behavior, the variables defining the displacement field are, in general, coupled.

In the finite-element analysis of shells, Green, Strome, and Weikel [2] used an assemblage of quadrilateral or triangular flat membrane elements. Johnson [3] compared several schemes of flat plate elements in the solution of cylindrical panels. In this approach, in addition to the structural idealization errors inherent in the finite-element solution, geometrical idealization errors are also present. For shells of revolution, the use of conical frusta as finite elements has been attempted by Grafton and Strome [4]. The extension of the conical element for unsymmetric deformations by means of Fourier series was made by Percy, Pian, Klein, and Navaratna [5]. For thick shells of revolution, a ring element of triangular cross section was used, for symmetrical deformations, by Clough and Rashid [6], and later extended for the asymmetrical case by Wilson [7]. All the abovementioned finite elements were derived from a “compatible displacement model” which uses displacements that are compatible both in the interior of the element as well as along the interelement boundaries.

For curved shells with cutouts, sections of shells, and curved shells that are not surfaces of revolution, there is a necessity for using doubly curved quadrilateral or triangular elements. Quadrilateral circular cylindrical thin shell elements were developed by Cantin and Clough [8] who use five rigid body modes and 15 straining modes of displacements for each element, by Bogner,
Fox, and Schmit [9] who use 48 straining modes of displacement for each element, and by Olson and Lindberg [10] who use 28 modes of displacement for each element. All these authors used a Kirchhoff deformation hypothesis for shell deformation which reduces the displacement variables to three, viz., the two displacements parallel and the one displacement normal to the reference surface of the shell. Key [11] developed a quadrilateral doubly curved shell element which incorporates the effect of transverse shear deformation in the shell. All the above elements were based on a "displacement model" (to be discussed later in this paper). As has been discussed by Atluri [12], however, all the above elements violate, to various degrees, the criteria for the convergence of the displacement solution, as discussed by Tong and Pian [13], viz., 1) compatibility in displacements and slopes at interelement boundaries, 2) choice of displacements such that constant strain states are represented, and 3) the inclusion of a complete set of rigid-body modes in the element interior in the assumed displacement field.

The purpose of this paper is to systematically consider the various variational functionals whose stationary conditions can be used alternately to solve for various unknowns in linear-shell theory. In applying these variational principles to the finite-element assembly of a shell, various "modified variational principles" can be generated by systematically relaxing the continuity (compatibility) requirements along the interelement boundary. Several finite-element schemes can be generated using these modified variational principles, the unknowns in the final algebraic set of equations being different in each case. The properties of a discrete element in each of the different finite-element schemes are discussed. Presented are 1) a "compatible displacement model," 2) two "hybrid displacement models," 3) two "equilibrium stress models," 4) a "hybrid stress model" and several "mixed models." An attempt is made to keep the formulations within the framework of a non-Kirchhoff-type linear-shell theory, as proposed by Reissner [14], which includes both transverse shear stresses and couple-stress stress couples. Specializations to a Kirchhoff-type shell theory are also indicated.

II. GENERAL THEORY OF THIN SHELLS WITH COUPLE STRESSES

Referring to Fig. 1, the geometry of the midsurface of a general shell is represented by a radius vector \( \mathbf{R} \) which is a function of two arbitrary curvilinear coordinates \( \xi^1 \) and \( \xi^2 \). The base vectors \( \mathbf{a}_i \) on the midsurface and the unit vector \( \mathbf{n} \), normal to the midsurface are given by
Fig. 1  Nomenclature for shell midsurface.

\[ a_a = \frac{\partial \mathbf{R}}{\partial \xi^a} \text{ and } \mathbf{n} = a_1 \times a_2 / \sqrt{a} \]  

(1)

where

\[ a = \det |a_{a\beta}| \]

and the metric tensors are given by

\[ a_{a\beta} = a_a \cdot a_\beta \]  

(2)

The contravariant metric tensors \( a^{a\beta} \) and the contravariant base vectors \( \mathbf{a}^a \) are defined by

\[ a^{a\beta} a_{\beta\gamma} = \delta^a_{\gamma} \]

where

\[ \delta^a_\beta = \begin{cases} 0 \ (\alpha \neq \beta) \\ 1 \ (\alpha = \beta) \end{cases} \]  

(3)

and

\[ \mathbf{a}^a = a^{a\beta} a_\beta \]
For further use, the Gauss–Weingarten relations for a surface are represented by

\[ \frac{\partial a_{\xi}}{\partial \xi} = \Gamma_{\xi\xi}^\gamma a_{\gamma} + h_{\xi\gamma} n \quad (4) \]

where

\[ \Gamma_{\xi\xi}^\gamma = \frac{1}{2} \alpha^\gamma \left( \frac{\partial a_{\xi\xi}}{\partial \xi} + \frac{\partial a_{\xi\gamma}}{\partial \gamma} - \frac{\partial a_{\gamma\gamma}}{\partial \xi} \right) \]

\[ h_{\xi\gamma} = n \cdot \frac{\partial a_{\xi}}{\partial \xi} \]

and

\[ \frac{\partial a}{\partial \xi} = -a_{\gamma\xi} \frac{\partial a_{\gamma}}{\partial \xi} \quad (5) \]

As shown in Fig. 2a, the vector of stress resultants \( N^\gamma \), and of stress couples \( M^\gamma \), acting on a side \( \xi^\gamma = \text{const} \) can be written as

\[ N^\gamma = \sqrt{\alpha} (N^\gamma a_{\gamma} + q^\gamma n) \quad (6) \]

and

\[ M^\gamma = \sqrt{\alpha} (n \times (M^\gamma a_{\gamma}) + P^\gamma n) \quad (7) \]

![Fig. 2a Stress resultants and stress couples acting on a shell element.](image)
where $N^{a2}$ are the "membrane stress resultants," $q^a$ are the transverse shear resultants, $M^{a2}$ are the stress couples, and $P^a$ are the couple-stress stress couples which were first introduced by Gunther [15].

The vectors of externally applied forces and moments per unit midsurface area can be written, respectively, as

\[ F = F^a a_x + F^3 n \]  
\[ m = n \times (m^a a_x) + m^3 n \]

The two vector equations of static equilibrium for the shell element depicted in Fig. 2b become

\[ \frac{\partial N^a}{\partial \xi^2} + F \sqrt{\dot{a}} = 0 \]  
\[ \frac{\partial M^a}{\partial \xi^2} + a_x \times N^a + m \sqrt{\dot{a}} = 0 \]

which are equivalent to six scalar equations that relate the 12 scalar components of $N^a$ and $M^a$.

Consistent with the state of description of stress, the strain state of the
shell is represented by membrane strains \( \varepsilon_m \), transverse shear strains \( \gamma_\theta \), curvature strains \( \kappa_\theta \), and couple-strain strain couples \( \lambda_\sigma \). These strains will, in turn, depend on an appropriately chosen description of the state of deformation of the shell midsurface. Thus, the midsurface strain vector \( \varepsilon_s \) and the curvature strain vector \( \kappa_s \) can be defined as

\[
\varepsilon_s = (1/\sqrt{\alpha})(\varepsilon_m a^\theta + \gamma_\theta n)
\]

(12)

and

\[
\kappa_s = (1/\sqrt{\alpha})[n \times (\kappa_\theta a^\theta) + \lambda_\sigma m]
\]

(13)

The internal energy of the shell per unit midsurface area can be then expressed consistently as

\[
\delta U = N^s \cdot \delta \varepsilon_s + M^s \cdot \delta \kappa_s
\]

(14)

The translational displacement vector \( u \) and the rotation vector \( \phi \) are then introduced in order to be able to write the virtual work per unit midsurface area for the external surface loads and moments as

\[
\delta W = F \cdot \delta u + m \cdot \delta \phi
\]

(15)

Making use of Eqs. (10) and (11), one can show following Reissner [14], that for the virtual work principle to be satisfied, the following relations must hold:

\[
\delta \varepsilon_s = \frac{1}{\sqrt{\alpha}} \left( \frac{\partial^2 u}{\partial \xi^2} + a_\theta \times \delta \phi \right)
\]

(16)

and

\[
\delta \kappa_s = \frac{1}{\sqrt{\alpha}} \frac{\partial \delta \phi}{\partial \xi^2}
\]

(17)

Hence, within the realm of linear theory,

\[
\varepsilon_s = \frac{1}{\sqrt{\alpha}} \left( \frac{\partial u}{\partial \xi^2} + a_\theta \times \phi \right)
\]

(18)

and

\[
\kappa_s = \frac{1}{\sqrt{\alpha}} \frac{\partial \phi}{\partial \xi^2}
\]

(19)
The individual components of strain are derived from

\[ e_{ab} = \sqrt{a}(a_a \cdot a_b); \gamma_a = \sqrt{a}(a_a \cdot n) \]  

(20)

and

\[ \kappa_{ab} = \sqrt{a}[\kappa_a \cdot (n \times a_b)]; \lambda_a = (\kappa_a \cdot n)\sqrt{a} \]  

(21)

Since the four strain vectors \( e_a \) and \( \kappa_a \) are derived from only two vectors \( u \) and \( \phi \), the two "compatibility relations" for strain measures can be immediately observed to be

\[ \delta^{ab} \frac{\partial \kappa_a}{\partial \xi^b} = 0 \]  

(22)

and

\[ \delta^{ab} \left( \frac{\partial e_a}{\partial \xi^b} + a_a \times \kappa_b \right) = 0 \]  

(23)

where

\[ \delta^{12} = -\delta^{21} = 1; \delta^{11} = \delta^{22} = 0 \]  

(24)

Noting the form of Eqs. (10) and (11) and Eq. (22) and (23), it can be observed that if the mathematical analogy

\[ N^a \equiv \delta^{ab} \kappa_b \]  

and \[ M^a = \delta^{ab} e_b \]  

(25)

is observed, the equilibrium equations in the absence of external loads and moments (10) and (11) and the compatibility conditions (22) and (23) can be derived from one another. This analogy is generally referred to as the "static geometric analogy."

Just as the strains were derived from two deformation vectors \( u \) and \( \phi \), the stresses can be derived from analogous stress functions \( F \) and \( H \) such that

\[ N^a = \delta^{ab} \frac{\partial F}{\partial \xi^b} \]  

(26)

and

\[ M^a = \delta^{ab} \left( \frac{\partial H}{\partial \xi^b} + a_a \times F \right) \]  

(27)
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where the further analogy

\[
\phi = F \text{ and } u = H
\]  

(28)

is identified. Hence, one can write

\[
u = u' a_x + w n
\]

(29)

\[
\phi = n \times (\phi' a_x) + \Omega n
\]

(30)

\[
H = H' a_x + K n
\]

(31)

\[
F = F'(a_x \times n) + J n
\]

(32)

The expressions for \( \epsilon_{\alpha \beta} \) etc., in terms of \( u \) and \( \phi \) and for \( N^\alpha \) etc., in terms of \( F \) and \( H \) are given in the Appendix.

In a Kirchhoff-type shell theory the transverse shear strains \( \gamma_x \) and the stress couples \( P^\alpha \) are assumed to be zero, which leads to the assumptions

\[
\gamma_x = \epsilon_x \cdot n \equiv \frac{\partial u}{\partial \xi} \cdot n - \phi \cdot (a_x \times n) = 0
\]

(33)

and

\[
P^\alpha = M^\alpha \cdot n \equiv \delta^\alpha \left[ \frac{\partial H}{\partial \xi} \cdot n - F \cdot (a_x \times n) \right] = 0
\]

(34)

From Eqs. (33) and (34) one obtains relations between \( \phi \) and \( u \), and \( F \) and \( H \), respectively. Thus \( u \) and \( H \) become the only independent displacement and stress function vectors, respectively. The relations between \( \epsilon_{\alpha \beta} \) and \( u \), and \( N^\alpha \) and \( H \), respectively, for a Kirchhoff-type shell theory are listed in the Appendix where the usual definitions \( \epsilon_{12} = \epsilon_{21} = \frac{1}{2} (\epsilon_{12} + \epsilon_{21}) \) and \( N^{12} = N^{21} = \frac{1}{2} (N^{12} + N^{21}) \) are employed.

III. A GENERAL VARIATIONAL PRINCIPLE IN SHELL THEORY

Neither the two vector equilibrium equations for the four vector stress resultants nor the two vector compatibility conditions for the four vector strain resultants can be solved independently. These equations need to be
supplemented by introducing the constitutive relations. In order to accomplish this, it is convenient to postulate the existence of the stress potential \( A \) and the strain potential \( B \) such that

\[
A = A_M + A^m + A^s; \quad B = B_M + B^m + B^s
\]

(35)

where the subscript \( M \) refers to "membrane energy," \( B \) to "bending energy," and the superscripts \( m \) and \( s \) refer to the parts of bending energy contributed by moments and transverse shear, respectively. The stress and strain potentials are defined by

\[
N^{ab} = \frac{\partial A_M}{\partial \varepsilon_{ab}}; \quad M^{ab} = \frac{\partial A^m}{\partial \kappa_{ab}}; \quad q^a = \frac{\partial A^s}{\partial \gamma_a}; \quad p^a = \frac{\partial A^s}{\partial \lambda_a}
\]

(36)

and

\[
e_{ab} = \frac{\partial B_M}{\partial N^{ab}}; \quad \kappa_{ab} = \frac{\partial B^m}{\partial M^{ab}}; \quad \gamma_a = \frac{\partial B^s}{\partial q^a}; \quad \lambda_a = \frac{\partial B^s}{\partial p^a}
\]

(37)

For convenience in bookkeeping, generalized gradients are defined by

\[
e_a = \nabla_{N^a} B \equiv \frac{\partial B}{\partial N^a}; \quad \kappa_a = \nabla_{M^a} = \frac{\partial B}{\partial M^a}
\]

(38)

and

\[
N^a = \nabla_{e_a} A \equiv \frac{\partial A}{\partial N^a}; \quad M^a = \nabla_{\kappa_a} = \frac{\partial A}{\partial \kappa_a}
\]

(39)

where

\[
\nabla_{\kappa_a} \equiv a_2 \left( \frac{\partial}{\partial \kappa_{a2}} \right) + a_1 \left( \frac{\partial}{\partial \kappa_{a1}} \right)
\]

(40)

\[
\nabla_{e_a} \equiv a_2 \left( \frac{\partial}{\partial e_{a2}} \right) + a_1 \left( \frac{\partial}{\partial e_{a1}} \right)
\]

(41)

\[
\nabla_{M^a} \equiv -a_2 \left( \frac{\partial}{\partial M^{a2}} \right) + a_1 \left( \frac{\partial}{\partial M^{a1}} \right) + n \left( \frac{\partial}{\partial p^a} \right)
\]

(42)

and

\[
\nabla_{e_a} \equiv -a_2 \left( \frac{\partial}{\partial \kappa_{a2}} \right) + a_1 \left( \frac{\partial}{\partial \kappa_{a1}} \right) + n \left( \frac{\partial}{\partial \lambda_a} \right)
\]

(43)
With these definitions, one can now write the shell theory analog of the most general functional $\pi_F$ as was done for three-dimensional elasticity by Washizu [16, 18], Hu [17], and Reissner [19], where

$$\pi_F = \int\int \left[ N^p \left\{ \frac{1}{\sqrt{a}} \left( \frac{\partial u}{\partial \xi} + a_2 \times \phi \right) - e_2 \right\} + M^s \left\{ \frac{1}{\sqrt{a}} \frac{\partial \phi}{\partial \xi} - \kappa \right\} ight] \left( F \cdot u + m \cdot \phi \right) + A_M - A_M^s + A_B \right] \sqrt{a} d\xi^1 d\xi^2$$

$$- \int_{C_\partial} \mathbf{N} \cdot \mathbf{u} + \mathbf{M} \cdot \phi ds - \int_{C_\partial} \left[ (\mathbf{u} - \bar{\mathbf{u}}) \cdot N + (\phi - \bar{\phi}) \cdot M \right] ds \quad (44)$$

where $C_S$ is the midsurface edge curve on which stress vectors $\mathbf{N}$ and $\mathbf{M}$ are prescribed, and $C_\partial$ is the midsurface edge curve where deformation vectors $\mathbf{u}$ and $\phi$ are prescribed. By setting the variation $\delta \pi_F$ with respect to each of the quantities $N^p, M^s, e_2, \kappa, u,$ and $\phi$ equal to zero, one obtains as Euler differential equations: 1) the equilibrium equations, viz., Eqs. (10) and (11), 2) the constitutive relations, viz., Eq. (44), 3) the strain-displacement relations, viz., Eqs. (20) and (21), and 4) the boundary conditions

$$N = \mathbf{N}; M = \mathbf{M} \text{ on } C_S \text{ and } u = \bar{u}; \phi = \bar{\phi} \text{ on } C_\partial \quad (45)$$

### IV. THE STATIC GEOMETRIC ANALOG OF THE GENERAL VARIATIONAL PRINCIPLE

From the static geometric analogy, described earlier, a variational functional that uses the equations for stresses in terms of stress functions, and the complementary strain energy density $B$ can be written immediately as

$$\pi_F^* = \int\int \left[ e^g \left( \frac{\partial F}{\partial \xi} + N^p - N^g \right) \right] d\xi^1 d\xi^2$$

$$+ \kappa \left( \mathbf{u} - \mathbf{a} \times \mathbf{F} \right) + M^s - M^g + B_M + B_M^s + B_B^g \right] ds$$

$$- \int_{C_\partial} (\mathbf{a} \cdot \mathbf{F} + \kappa \cdot \mathbf{H}) ds - \int_{C_S} (\mathbf{H} - \mathbf{H} \cdot \kappa + (\mathbf{F} - \mathbf{F}) \cdot \varepsilon) ds \quad (46)$$

In the above functional, $e^g, \kappa, \mathbf{F}, \mathbf{H}, N^s,$ and $M^s$ all can be varied independently; however, $N^p$ and $M^p$ are the particular parts of the solution corre-
sponding to given external loads and hence cannot be subjected to variation. Thus, setting the variation $\delta \pi_F^*$ equal to zero leads in addition to the Euler equations: 1) the compatibility conditions (22) and (23), 2) the constitutive relations (38), 3) the stress-stress function relations (26) and (27), and 4) the boundary conditions

$$\epsilon = \epsilon, \kappa = \kappa \text{ on } C_D \text{ and } H = \overline{H}; F = F \text{ on } C_S$$

(47)

In using the variational functionals $\pi_F$ and $\pi_F^*$ in connection with a Kirchhoff-type shell theory, the constraint relations between $\phi$ and $u$ (33), the relation between $F$ and $H$ (34), respectively, should be observed.

In the following, several subclasses of the general variational principles $\delta \pi_F = 0$ and $\delta \pi_F^* = 0$, and their applications to finite-element theory are indicated.

V. THE FINITE-ELEMENT VARIATIONAL TECHNIQUE

A. The General Approach

It should be noted that in the functional $\pi_F$ given by (44), the quantities $u$, $\phi$, $\epsilon$, $\kappa$, $N^a$, and $M^a$ can be varied independently, and that in the functional $\pi_F^*$ given by (46), the quantities $F$, $H$, $N^a$, $M^a$, $\epsilon$, and $\kappa$ can be varied independently. In practice, for a given material of a shell, the constitutive relations are known; hence, one can find $N^a$ and $M^a$ if $\epsilon$ and $\kappa$ are known, and vice versa. Several subclasses of the two general variational functionals can be constructed by assuming \textit{a priori} some of the shell theory relations that would otherwise follow from the general functionals. Thus, for instance, by assuming \textit{a priori} the stress-strain relations and the strain displacement relations, the functional $\pi_F$ can be used to find, through a variational technique, the displacement field generated in the shell in equilibrium under the action of applied loads. By making proper assumptions \textit{a priori} in the two functionals $\pi_F$ and $\pi_F^*$, the solution of any boundary-value problem of a shell can be reduced to finding, from the stationary conditions of the specialized functionals, any vector pair of independent variables out of the possible pairs: $(u; \phi)$, $(\epsilon; \kappa)$, $(N^a, M^a)$, $(F, H)$, $(u^a, M^a)$, and $(\phi, N^a)$ subject to the given boundary conditions. However, a special form of the variational principle as indicated later, can be used to solve for the pairs $(u, \phi)$ and $(N^a, M^a)$ simultaneously. Once these independent variables are solved for, the
other shell variables can be deduced from the relations that were considered a priori.

In the finite-element procedure, the shell domain is divided into a finite number of regions called discrete elements; which are, in practice, well-conditioned triangles or quadrilaterals in shape for general shells or meridionally curved frusta for shells of revolution. Mathematically, the finite-element technique consists of solving for the alternate sets of independent variables as indicated earlier, at a finite number of control points called "nodes" of each discrete element; these nodes are generally located at the boundaries of each element, but may also be in the interior of each element. The values of the above mentioned basic unknown variables in the interior of each element are interpolated, in a predetermined fashion, in terms of their respective nodal values. However, in choosing these interpolation functions in the interior of each element, there are several criteria regarding the distribution of a given unknown quantity at the interelement boundary so as to maintain "compatibility" (in a broad sense) with the neighboring element, so that the respective variational functional can be defined for the finite-element assembly of the shell as the sum (extending over the number of elements) of the value of the functional for each individual element. However, such interelement "compatibility" or "continuity" conditions can be relaxed by "modifying" the variational functionals by using a Lagrangian multiplier technique as discussed by Pian and Tong [1] and to be discussed further in this paper. Additional criteria to insure the monotonic convergence of the solution, as the size of the discrete element is progressively reduced, have to be taken into account.

In the following subsections, several alternate schemes of finding the basic unknown variables, as indicated earlier, at the nodes of a finite-element assembly of a shell are examined.

**B. Displacement Models**

Here, the finite-element models that can be constructed through the initial assumption of a continuous displacement field in the interior of each element are studied. Since the description of the state of deformation in a Kirchhoff-type theory differs from that in a Reissner-type theory, the basic differences arising, therefore, in the finite-element technique using the above theories are pointed out.

If in the functional $\pi_F$ of Eq. (44), the strain-displacement relations Eqs. (18) and (19) and the stress-strain relations Eq. (39) are assumed a priori, then the functional $\pi_F$ reduces to
\[
\pi_F = \int_{C_1} [A_M + A_B^m + A_B^n - F \cdot u - m \cdot \phi] ds \\
- \int_{C_2} [\bar{N} \cdot u + \bar{M} \cdot \phi] ds - \int_{C_3} [(u - \bar{u}) \cdot N + (\phi - \bar{\phi}) \cdot M] ds 
\]

In the above, \( C_1 \) and \( C_2 \) are the edge curves of the shell where the stresses and displacements are actually prescribed. However, it should be remarked that in extending the function \( \pi_F \) to an assemblage of finite elements such as \( ABCD \) (see Fig. 3) \( u \) can be recognized as the displacement field in the interior of each element, \( C_3 \) refers to the interelement boundary, i.e., line \( ABCDA \), \( \bar{u} \) is the interelement boundary displacement, and the quantities \( \bar{N} \) and \( \bar{M} \) are the line loads that are present along \( ABCDA \), especially when \( ABCDA \) or a part of it is aligned with the edge of a shell.

Fig. 3 An isolated discrete element \( ABCD \).

1. Compatible Displacement Model (CDM). If the displacement field in the interior is assumed in such a way that interelement boundary displacements are compatible, the line integral on \( C_3 \) in Eq. (48) vanishes for each element. Such a model is termed a "compatible displacement model."

Since the deformation state in a Reissner-type theory is described by two independent vectors \( u \) (translation) and \( \phi \) (rotation), and since the deformation state in a Kirchhoff-type theory is defined by \( u \) alone, the appropriate generalized coordinates at each node \( A, B, C, \) and \( D \) (see Fig. 3) are:
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Reissner-type theory

\[ u = u^a a^a + w n \]
\[ \phi = n \times (\phi^a a^a) + \Omega n \] (49)

Kirchhoff-type theory

\[ u = u^a a^a + w n \]
\[ \phi_1 = -\frac{1}{\alpha} \frac{\partial w}{\partial \xi} + \frac{u^1}{R_{11}} + \frac{u^2}{R_{12}} \]
\[ \phi_2 = -\frac{1}{\alpha} \frac{\partial w}{\partial \eta} + \frac{u^1}{R_{12}} + \frac{u^2}{R_{22}} \] (50)

Thus, in a Reissner-type theory, the interpolation function that is used to describe the interior displacement field \((u, \phi)\) is such that \(u\) and \(\phi\) are linear along any edge, say, \(BC\) in Fig. 3, such that they could be determined uniquely from their respective values at nodes \(B\) and \(C\). This automatically insures interelement displacement field continuity, whereas, in a Kirchhoff-type theory, the displacement field should be so chosen that not only \(u^a\) and \(w\) are continuous across the interelement boundary, but also the first derivatives of \(w\) should be continuous so that the rotations \(\phi_2\) would be continuous across the interelement boundary.

If the displacement fields are so chosen, the total potential energy for the finite-element assemblage of the shell, using discrete elements of the CDM type, is denoted by \((\pi_p)_{\text{CDM}}\) and can be written as

\[
(\pi_p)_{\text{CDM}} = \sum_m \left\{ \int_{a_m} \left[ A_m + A_m^w + A_m^\phi \right] - (F \cdot u + m \cdot \phi) \right\} da
- \int_{C_{S_m}} (\bar{N} \cdot u + \bar{M} \cdot \phi) \, ds \right\} \] (51)

where \(a_m\) refers to the area of the \(m\)th discrete element, \(C_{S_m}\) refers to that portion of the edge curve of the \(m\)th element where stresses are prescribed, and the summation extends over the \(N\) elements of the system.

Henceforth, let \(v\) represent the displacement field in a general sense, to mean either \(u, \phi\) in a Reissner-type theory or \(u\) in a Kirchhoff-type theory. Let the matrix of the assumed displacement field for each element be

\[ v = [U_1] \{ \beta \} \] (52)
where \([U_i]\) is a matrix of appropriately assumed functions in coordinates \(\xi,\\eta\) and \(\{\beta\}\) are undetermined parameters. As pointed out by Melosh [20], and Tong and Pian [13], although it is not necessary from a convergence point of view, it is desirable to include a complete set of rigid-body displacement modes in the set of functions \(U_i\). However, in general, it is extremely difficult to include these rigid-body modes and yet satisfy the interelement compatibility. The inclusion of such rigid-body modes will be considered in the subsequent discussion on the hybrid displacement models.

The corresponding strain distribution for each elements is

\[
e = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} = [W] \{\beta\}
\]

Using Eqs. (52) and (53) in Eq. (51), one obtains

\[
(\pi_P)^{CDM} = \sum_m \{4[\beta][H]_m\{\beta\} - [\beta]\{F_m\} - [\beta]\{S_m\})
\]

where

\[
[H_m] = \int_{\alpha_m} \int_{\beta_m} [W]^T [C] [W] da
\]

\[
[\beta]\{F_m\} = \int_{\alpha_m} \int_{\beta_m} (F \cdot u + m \cdot \phi) da
\]

\[
[\beta]\{S_m\} = \int_{\alpha_m} \int_{\beta_m} (N \cdot u + \bar{M} \cdot \phi) dS
\]

and \([C]\) is the elastic constant matrix.

The values of the nodal generalized displacements defined in terms of the local coordinate system for each element can be expressed as

\[
\{q\} = [B]\{\beta\}
\]

If one takes the same number of displacement parameters as the nodal coordinates, the transformation matrix \([B]\) is square. Hence,

\[
\{\beta\} = [B]^{-1}\{q\} = [U]\{q\}
\]

Using Eq. (59) in Eq. (54), one can write

\[
(\pi_P)^{CDM} = \sum_m \{4[q][K_m]\{q\} - [q]\{Q_m\})
\]
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where, from usual definitions,

\[ [K_m] = \text{stiffness matrix for the } m\text{th element} \]
\[ = [J]^T [H] [J] \tag{61} \]

and

\[ \{Q_m\} = \text{element nodal load matrix for the } m\text{th element} \]
\[ = [J]^T \{F_1 + S_1\} \tag{62} \]

The element nodal displacements \( \{q\} \) for different elements, which are not independent, can be related to a column of independent global displacements \( \{q^*\} \) thus:

\[ \{q_1, q_2, q_3, \ldots, q_n\} = [J^*] \{q^*\} \tag{63} \]

where \( [J^*] \) includes the effect of transforming from local coordinates for individual elements to the global coordinates for the element assembly. Thus, Eq. (60) can be written as

\[ (\pi p)_{CDM} = \frac{1}{2} [q^*] [K] [q^*] - [q^*] [F^*] \tag{64} \]

where

\[ [K] = [J^*]^T \begin{bmatrix} K_1 \\ & K_2 \\ & & \ddots \\ & & & K_m \end{bmatrix} [J^*] \tag{65} \]

and

\[ \{F^*\} = [J^*]^T \{Q_1 \cdots Q_m\} \tag{66} \]

If some of the global coordinates are prescribed, Eq. (64) can be rewritten as

\[ (\pi p)_{CDM} = \frac{1}{2} [q^* q^*_2] \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{bmatrix} q^*_1 \\ q^*_2 \end{bmatrix} - [q^*_1 q^*_2] [F^*_2] \tag{67} \]

where \( \{q^*_1\} \) is unknown, \( \{q^*_2\} \) is prescribed, \( \{F^*_2\} \) is given, and \( \{F^*_2\} \) is unknown. Then the stationarity condition of the functional in Eq. (67) leads to

\[ [K_{11}] [q^*_1] = \{Q^*_1\} \tag{68} \]
which can be solved for \( \{q^*_i\} \). Then one obtains

\[
\{F^*_i\} = [K_{12}]{q^*_i}
\]

A doubly curved quadrilateral element for a general shell of revolution based on the above model was developed by Atluri [12] and successfully tested in the program STACUSS by Kotanchik [21].

2. Hybrid Displacement Model-I (HDM-I). The condition of interelement compatibility on the choice of displacement field for each element can be relaxed by choosing an arbitrary displacement field in the interior of each element and including the Lagrangian multiplier terms

\[
\begin{aligned}
&- \int_{c_{D_m}} [(u_e - \vec{u}) \cdot N_L + (\phi_e - \vec{\phi}) \cdot M_L] ds \\
&= \int_{c_{D_m}} [\mathbf{N} \cdot (u_e - \vec{u}) + \mathbf{M} \cdot (\phi_e - \vec{\phi})] ds
\end{aligned}
\]

for each element in the functional of Eq. (51); where \( u_e(\phi_e) \) denotes the displacement (rotation) on the boundary which results from the displacement (rotation) field assumed for the interior of the element, \( \vec{u}(\vec{\phi}) \) is the interelement displacement (rotation) which can be prescribed independently of \( u_e(\phi_e) \) such that it is determinable uniquely in terms of the generalized nodal displacements and the stress resultants \( N_L \) and \( M_L \) along the interelement boundary of element \( m \) playing the role of Lagrange multipliers. Adding the term in Eq. (70) to Eq. (51), the modified functional can be written as

\[
\begin{aligned}
(\pi_F)_{HDM-I} &= \sum_m \left[ \int_{a_m} \left( A_M + A_N - F \cdot u - m \cdot \phi \right) ds \\
&- \int_{c_{D_m}} [\mathbf{N} \cdot (u_e - \vec{u}) + \mathbf{M} \cdot (\phi_e - \vec{\phi})] ds \\
&- \int_{c_{D_m}} [(u_e - \vec{u}) \cdot N_L + (\phi_e - \vec{\phi}) \cdot M_L] ds
\end{aligned}
\]

It should be noted in the above equation that the independent variables in each element are \( u, \phi, N_L, \) and \( M_L \), whereas the prescribed interelement displacement fields \( \vec{u} \) and \( \vec{\phi} \) are not independent for each element.

Since the variations \( \delta u, \delta \phi, \delta N_L, \) and \( \delta M_L \) are arbitrary and independent for each individual element, it can be shown that the Euler equations for the functional in Eq. (71) are

\[
\frac{\partial N_L^i}{\partial \vec{u}} + F \cdot \vec{u} = 0 \quad \text{on} \quad a_m
\]
Equations (72) and (73) state that the stress developed by the assumed interior displacement field satisfy the static equilibrium for each element. The condition that the interelement displacements generated by the assumed interior displacement field match the prescribed interelement element displacements along C_D is represented by Eq. (74). The fact that the interelement stresses generated by the interior assumed displacement field match the prescribed Lagrangian stresses is represented by Eq. (75). Finally, for edges C_s along which there are prescribed (actually present) stress resultants, Eq. (76) states that these prescribed stresses are matched by the edge tractions which result from the assumed internal displacement field. The fact that the Lagrangian stresses are assumed independently for each element indicates that stress discontinuities are still present across the interelement boundaries in this formulation.

Since there is considerable latitude in the selection of a displacement field in the interior, one selects: 1) a six-mode rigid-body displacement field and 2) a general linearly independent n mode straining displacement field. Thus

$$\{\nu\} = \begin{bmatrix} \{\phi\} = [U_1]{\beta} + [U_2]\{\beta\}$$

where $$[U_1]$$ and $$[U_2]$$ represent the straining and the six rigid-body modes, respectively. The strains can be represented as

$$\{\varepsilon\} = [W]\{\beta\}$$

since the rigid-body modes generate zero strains. Likewise the Lagrangian multiplier terms at the boundary of each element can be chosen in two parts: a set corresponding to the homogeneous solution, and a set corresponding to the particular solution, respectively, of the equilibrium equation. Thus, for each element, one can write

$$\begin{bmatrix} \{N\} \\ \{M\} \end{bmatrix} = [R_1]\{\alpha\} + [R_2]\{\delta\}$$
where \([R_1]\) and \([R_2]\) correspond to homogeneous and particular solutions, respectively. As will be shown later, it is better to have the dimensions of \(\{a\}\) and \(\{\delta\}\) in Eq. (79) equal to those of \(\{\beta\}\) and \(\{\hat{\beta}\}\), respectively, as in Eq. (77).

Further, the prescribed edge displacement field is represented in terms of a finite number of nodal displacements \(\{q\}\) as

\[
\begin{bmatrix}
\{\delta_x\} \\
\{\delta_y\}
\end{bmatrix} \equiv [L] \{q\}
\]  

Substituting Eqs. (77), (79), and (80) in Eq. (71), one can show that

\[
(p_F)_{HDM-1} = \sum_n \left\{ \beta \right\} H_1 \left\{ \beta \right\} - \beta \left\{ F_1 \right\} - \beta \left\{ F_2 \right\} - \beta \left\{ P_1 \right\} \{a\}
- \beta \left\{ P_2 \right\} \{\delta\} - \beta \left\{ P_3 \right\} \{\bar{a}\} + \{G_1\} \{q\} + \{G_2\} \{q\}
\]  

where

\[
[H_1] = \int_{\partial e} \int [W]^T[C][W]
\]  

\[
\int_{\partial e} \left( F \cdot u + m \cdot \phi \right) ds + \int_{\partial e} \left( N \cdot u + \overline{M} \cdot \phi \right) = \left\{ \beta \right\} \left\{ F_1 \right\} + \left\{ \beta \right\} \left\{ F_2 \right\}
\]  

\[
\int_{\partial e} \left( N_L \cdot u + M_L \cdot \phi \right) ds = \left\{ \beta \right\} \left\{ P_1 \right\} \{a\} + \left\{ \beta \right\} \left\{ P_2 \right\} \{\delta\} + \left\{ \beta \right\} \left\{ P_3 \right\} \{\bar{a}\}
\]  

and

\[
\int_{\partial e} \left( N_L \cdot \bar{u} + M_L \cdot \bar{\phi} \right) ds = \{a\} \left\{ G_1 \right\} \{q\} + \{\bar{a}\} \left\{ G_2 \right\} \{q\}
\]  

In the above \([C]\) is the elastic constant matrix, and the matrices \([P_1]\) and \([P_3]\) are square because of the equality of the dimensions of \(\{a\}\) and \(\{\beta\}\), and \(\{\delta\}\) and \(\{\hat{\beta}\}\), respectively. Since \(\{a\}\), \(\{\beta\}\), \(\{\delta\}\), and \(\{\hat{\beta}\}\) are independent for each element, taking the variations with respect to them leads to the equations

\[
\left\{ H_1 \right\} \{\beta\} - \{F_1\} - \left\{ P_1 \right\} \{a\} - \left\{ P_3 \right\} \{\bar{a}\} = 0
\]  

\[
\{F_2\} + \left\{ P_3 \right\} \{\delta\} = 0
\]  

\[
-\left\{ P_1 \right\}^T \{\beta\} + \left\{ G_1 \right\} \{q\} = 0
\]  

\[
-\left\{ P_2 \right\}^T \{\beta\} - \left\{ P_3 \right\}^T \{\hat{\beta}\} + \left\{ G_2 \right\} \{q\} = 0
\]
Using Eqs. (86) through (89), \( \{\beta\}, \{\hat{\beta}\}, \{x\}, \) and \( \{\hat{x}\} \) can each be expressed in terms of \( \{q\} \) only. Thus,

\[
(\pi_P)_{HDM-1} = \sum_m \{q\}^T [G_i^T][P_i^{-1}] [H_i] [P_i^{-1}]^T [G_i] \{q\}
- \{q\}^T [G_i] [P_i^{-1}] \{F_i\} + \{G_i\}^T - [G_i^T][P_i^{-1}][P_i^{-1}] \{F_i\})
\]

(90)

Hence, one can identify the element stiffness and load matrices as

\[
[K_m] = [G_i] [P_i^{-1}] [H_i] [P_i^{-1}]^T [G_i] \quad \text{(91)}
\]

and

\[
\{Q_m\} = [G_i] [P_i^{-1}] \{F_i\} - \{G_i\}^T - [G_i^T][P_i^{-1}][P_i^{-1}] \{F_i\}
\]

(92)

Thus, the unknowns in the final set of matrix equations are the generalized nodal displacements.

The above model was used by Tong [22] to solve plate-bending problems. A doubly curved quadrilateral element for a shell of revolution was developed by Atluri [12] and successfully tested in the program STACUSS by Kotanchik [21].

3. Hybrid Displacement Model-2 (HDM-2). The requirement of inter-element displacement compatibility can also be relaxed, as suggested by Yamamoto [23], by introducing Lagrangian multiplier terms, such that the generalized displacement coordinates of the system are the integrated averages of the edge displacements rather than the nodal displacement as in the earlier two models. The functional for this model can be written as

\[
(\pi_P)_{HDM-2} = \sum_m \left\{ \int_{s_m} \left[ A - F \cdot u - m \cdot \phi \right] ds - \int_{c_{s_m}} (N \cdot u + M \cdot \phi) ds \right\}
- \int_{c_{D_m}} (N_L \cdot u + M_L \cdot \phi) ds
\]

(93)

where \( N_L \) and \( M_L \) are the Lagrangian multiplier terms, which can be interpreted physically as the edge tractions for each element, and are chosen for each discrete element as

\[
\begin{bmatrix} N_L \\ M_L \end{bmatrix} = \{T\} = [\phi] [\lambda]
\]

(94)

The parameters \( \{\lambda\} \) can be interpreted as the generalized nodal forces for
each element and the functions \( \phi \) are such that the edge traction \( \{ T \} \) of any particular kind for a particular element could be determined uniquely in terms of the relevant quantities \( \{ \lambda \} \) at the nodes pertaining to the edge of the element in question. The element displacement field which need not satisfy the interelement compatibility can be chosen arbitrarily for each element as

\[
\{ v \} = [U_1](\beta) + [U_2](\dot{\beta})
\]  

(95)

where \([U_1]\) and \([U_2]\) refer to straining and rigid-body modes, respectively. Neglecting, for the present, the possibility of any actual edge loadings \( N \) and \( M \) for each element, one can write Eq. (93) as

\[
(\pi_p)_{\text{HDM-2}} = \sum_m \{ \beta \} [H] \{ \beta \} + \{ \beta \} \{ F_1 \} + \{ \dot{\beta} \} \{ F_2 \} - \{ \beta \} [D_1] \{ \beta \} - \{ \dot{\beta} \} [D_2] \{ \lambda \}
\]  

(96)

where, as before,

\[
[H] = \int_a \int_a \{ W \} [C] \{ W \} \, da
\]

\[
\int_a \int_a (F \cdot u + m \cdot \phi) \, da = \{ \beta \} \{ F_1 \} + \{ \dot{\beta} \} \{ F_2 \}
\]

and

\[
\int_a (N_L \cdot u + M_L \cdot \phi) \, ds = \{ \beta \} \left[ \int_a \{ \phi \} \, ds \right] \{ \lambda \}
\]  

(97)

From Eq. (97), by analogy, the generalized nodal displacements corresponding to generalized nodal forces \( \{ \lambda \} \) can be defined as

\[
[q] = \begin{bmatrix} \beta \\ \dot{\beta} \end{bmatrix} [D_1]
\]  

(98)

and thus are the weighted integrals of the boundary displacements. Hence, maintaining the compatibility of these generalized displacements does not guarantee compatibility along the entire boundary.

Since \( \{ \beta \} \) and \( \{ \dot{\beta} \} \) are the only independent variables for each element, the stationarity conditions for the potential in Eq. (93) give

\[
[H] \{ \beta \} + \{ F_1 \} - [D_1] \{ \lambda \} = 0
\]  

(99)
and

\[ \{F_2\} - [D_2]\{\lambda\} = 0 \]  

(100)

It can be observed that Eq. (100) represents six equations of static equilibrium for the generalized forces around the element. Thus, Eq. (100) can be rewritten as

\[ \{F_2\} - \begin{bmatrix} D_{21} \\ D_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0 \]

and thus,

\[ \{\lambda_2\} = [D_{22}]^{-1}\{F_2\} - [D_{22}]^{-1}[D_{21}]\{\lambda_1\} \]  

(101)

Thus, the independent generalized nodal forces for each element are \( \{\lambda_1\} \) only. Using Eqs. (99) and (100) one can express Eq. (93) entirely in terms of the unknown forces \( \{\lambda_1\} \). Thus using the generalized forces \( \{\lambda^*_1\} \) for the finite-element assembly, and since these can be subjected to independent variation, one obtains the final set of equations of the form

\[ [E]\{\lambda^*_1\} = \{G\} \]  

(102)

Thus, even though the model has been constructed with an initial assumption of element displacements, the unknowns in the final set of equations are the redundant generalized nodal forces \( \{\lambda^*_1\} \). Since this is a matrix force method, considerable care should be exercised in the choice of the basic unknown forces \( \{\lambda^*_1\} \) such that the matrix equations are not ill-conditioned. It should be noted that in this model, even though the interelement displacement compatibility is satisfied only on an average, interelement stress continuity is maintained. To the authors' knowledge, no shell element has been developed so far based on this model.

C. Stress Models

In this subsection, finite-element models that could be constructed through the initial assumption of equilibrium stress field within each element are studied. First, a stress model that can be constructed by applying the static geometric analogy to the "compatible displacement model" is presented. In this approach, the unknowns to be solved from the final set of equations are the nodal values of the stress functions. Two other models, involving the
initial assumption of a stress field are also presented: in both of the models, the unknowns to be solved in the final set of equations obtained from the governing variational statement for the assembled structure are the generalized nodal displacements.

1. Equilibrium Stress Model-I (ESM-I). As discussed by Pian [24], this model can be constructed directly from the compatible displacement model through the static geometric analogy as explained in Eqs. (25) through (28), where it has already been shown that there exists a direct analogy between the displacement vector \( \mathbf{u} \) and the stress function vector \( \mathbf{H} \), between the rotation vector \( \mathbf{\phi} \) and the stress function vector \( \mathbf{F} \), between the stress-resultant vector \( \mathbf{N}' \) and the curvature strain vector \( \mathbf{K}_g \), and between the stress-couple vector \( \mathbf{M}'' \) and the stretching strain vector \( \mathbf{e}_u \). Thus, just as the CDM (where displacements are assumed) is used to solve for a displacement boundary-value problem in shells, an analogous model using stress functions can be used to solve a stress boundary-value problem. If the analogy between displacement variables and the stress function variables is observed, it then follows that for similar types of functions, as are used in the CDM, the stress functions would be continuous across the interelement boundary. Even though displacement compatibility is maintained in the CDM, strains, however, are not necessarily continuous across the interelement boundary. By direct analogy, it then follows that stresses will not be continuous, but nevertheless stress equilibrium conditions are satisfied at the interelement boundary in ESM-I, if the stress functions are analogous to the displacement functions as in CDM. Also, the unknowns in the finite-element formulation of ESM-I, are by analogy, the nodal values of stress functions. When once these are solved, the stress and strain distribution must be obtained from the derivatives of stress functions, while the displacements can only be calculated by integrating the stress displacement relations. In view of the approximate character of the stresses provided by the finite-element analysis, the deflection obtained by integration is, in general, dependent upon the chosen integration path; hence it is not a unique solution.

There are other drawbacks to this equilibrium model. In general, it is a difficult task to include the externally applied loads and moments on the shell surface in this formulation. Also it is often obscure to interpret physically the stress boundary conditions at the edges of the shell in terms of stress functions.

2. Equilibrium Stress Model (ESM)-2. If in the general variational functional of Eq. (44), the equilibrium equation for the shell, Eqs. (10) and (11), are assumed \( \text{a priori} \), the functional can be shown to reduce to
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\[-\pi_{FS} = \int \int_B da + \int_{S_m} [(\mathbf{N} - \mathbf{N}) \cdot \mathbf{u} + (\mathbf{M} - \mathbf{M}) \cdot \phi] \, ds
\]
\[\quad - \int_{C_{\partial m}} \overline{\mathbf{u}} \cdot \mathbf{N} + \overline{\phi} \cdot \mathbf{M} \, ds\]  \hspace{1cm} (103)

In the case of a finite-element assembly of a shell, Eq. (103) can be written as

\[-\pi_{FS} = \sum \int_B da + \int_{S_m} [(\mathbf{N} - \mathbf{N}) \cdot \mathbf{u} + (\mathbf{M} - \mathbf{M}) \cdot \phi] \, ds
\]
\[\quad - \int_{C_{\partial m}} (\overline{\mathbf{u}} \cdot \mathbf{N} + \overline{\phi} \cdot \mathbf{M}) \, ds\]  \hspace{1cm} (104)

In Eq. (104), \( \mathbf{N} \) can be interpreted as the stress at the interelement boundary and \( \overline{\mathbf{u}} \) as the interelement boundary displacement. If the stress field in the interior of each element is selected such that it generates interelement boundary tractions that are compatible with the neighboring element, the integral over \( C_{S_m} \) in Eq. (104) vanishes. Hence

\[-\pi_{FS} = \sum \int_B da - \int_{C_{\partial m}} (\overline{\mathbf{u}} \cdot \mathbf{N} + \overline{\phi} \cdot \mathbf{M}) \, ds\]  \hspace{1cm} (105)

Thus, the stress field that satisfies not only equilibrium, but also interelement compatibility can be chosen for each element as

\[\{\sigma\} = \begin{cases} \{\mathbf{N}\} \\ \{\mathbf{M}\} \end{cases} = [R_1](\alpha) + \{R_2\}\]  \hspace{1cm} (106)

where the set of functions \([R_1]\) can be derived from stress functions (to represent the homogeneous solution of equilibrium equations) with unknown parameters \(\alpha\) and \([R_2]\) represents any stress field that is statically equivalent to the applied loads and thus is considered as known. The corresponding edge tractions are

\[\{T\} = \begin{cases} \{\mathbf{N}\} \\ \{\mathbf{M}\} \end{cases} = [R_{1e}](\alpha) + \{R_{2e}\}\]  \hspace{1cm} (107)

For stress continuity along the interelement boundary, these edge tractions should be representable uniquely from a set of generalized loads at a finite number of control points along the boundary of the element in question.
Thus, \((T)\) should be expressible as

\[
\{T\} = [\phi]\{\lambda\}
\]

(108)

where \(\{\lambda\}\) are the generalized nodal loads. From Eqs. (107) and (108) there exists a unique relation

\[
\{\lambda\} = [\phi]^{-1}[R_1]\{z\} + [\phi]^{-1}[R_2]
\]

\[
= [G_1]\{z\} + \{G_2\}
\]

(109)

The matrix \(G_1\) obviously is a square matrix. The corresponding generalized edge displacements \(\{q\}\) are defined from the equation

\[
[\lambda](q) = \int_{c_n} (\bar{u} \cdot N + \bar{\phi} \cdot M) \, ds
\]

\[
= [\lambda] \int_{c_o} [\phi]^T \begin{bmatrix} \underline{u} \\ \underline{\phi} \end{bmatrix} \, ds
\]

(110)

and thus represent weighted integral averages of the boundary displacements. Using Eqs. (106) and (110), one can rewrite Eq. (105) as

\[
-\pi_{FS} = \sum_m \left[ \int_{c_m} \left\{ \frac{1}{2} [z][R_1^T][D][R_1]\{z\} + \frac{1}{2} [R_2][D][R_2] \\
+ \frac{1}{2} [z][R_1^T][D][R_1] \right\} \, ds - |\lambda|{\lambda}) \right]
\]

(111)

where \(D\) is the matrix of elastic constants. Carrying on the integration, one can then write

\[
-\pi_{FS} = \sum_m \left\{ \frac{1}{2} [z][J_{11}]\{z\} + [z][J_{22}] - [z][G_1^T]\{q\} - [G_2^T]\{q\} + C_m \right\}
\]

(112)

Taking variations with respect to \(\{z\}\), which are independent for each element, leads to an equation for \(\lambda\)'s in terms of \(q\)'s. Thus, the complementary energy of the entire can be expressed in terms of the system coordinates \(\{q^*\}\). Setting \(\delta \pi_{FS} = 0\) then leads to the usual matrix equations in \(\{q^*\}\).

To the authors' knowledge no shell element has been developed so far for this model.
3. Hybrid Stress Model-1 (HSM-1). This hybrid stress formulation is an analog of the hybrid displacement HDM-2 model. Pian [25] used this HSM-1 formulation first in plane stress and plate-bending problems.

In this method, an equilibrium stress field is assumed in the interior of each element and stress equilibrium along the interelement boundary need not be satisfied. Stress continuity along the interelement boundary is satisfied only in an average sense by introducing Lagrangian multipliers along the interelement boundary. These Lagrangian multipliers, which can be physically interpreted as the interelement boundary displacements, are chosen such that they are compatible with the neighboring elements. The functional for the HSM-1 model can be written as

$$-\pi_{PS} = \sum_{m} \left[ \int_{\Omega_m} B \, da - \int_{\partial\Omega_m} (u_L \cdot N + \phi_L \cdot M) \, ds \right]$$  \hspace{1cm} (113)

where $u_L, \phi_L$ (Lagrangian multipliers) can be chosen as

$$\begin{bmatrix} u_L \\ \phi_L \end{bmatrix} = [L] \{q\}$$  \hspace{1cm} (114)

where $[L]$ is a matrix of interpolation functions such that the displacement on any particular edge of the element is interpolated uniquely in terms of the nodal values $\{q\}$ along that edge; these generalized nodal displacements are to be regarded as global, a "master" in the sense that they are common to and apply to all elements which share any given node. For the assumed stress field in the interior of each element, one may write

$$\{\sigma\} = \begin{bmatrix} N^* \\ M^* \end{bmatrix} = [R_1]{\{\alpha\}} + {R_2}$$  \hspace{1cm} (115)

where $[R_1]$ represents the homogeneous solution with unknown parameters $\{\alpha\}$ and $\{R_2\}$ any particular solution. The boundary tractions generated by this stress field can be written as

$$\{T\} = [R_1]{\{\alpha\}} + {R_2e}$$  \hspace{1cm} (116)

The generalized nodal forces $\{\Gamma\}$ can be identified from the expression

$$\int_{\partial\Omega} (u_L \cdot N + \phi_L \cdot M) \, ds = \{q\} \int_{\partial\Omega} [L]^T ([R_1]{\{\alpha\}} + {R_2e}) \, ds$$

$$\equiv \{q\} [G_1]{\{\alpha\}} + \{G_2\}.$$

$$\equiv \{q\} \{\Gamma\}$$  \hspace{1cm} (117)
and thus are equal to the integrated averages of the edge fractions. Thus, the functional for the finite-element assemblage becomes

\[
\begin{align*}
\{ \Gamma \} &= \{ G_1 \} \{ x \} + \{ G_2 \} \\
\text{and thus are equal to the integrated averages of the edge fractions. Thus, the functional for the finite-element assemblage becomes}
\end{align*}
\]

\[
(- \pi_{FS})_{HSM-1} = \sum_{m} \left[ \int \int \left( \frac{1}{2} [x][R_1]^T [D][R_1](x) + [x][R_1][D][R_2] \right) da \\
\quad + [x][G_1](q) + [G_2](q) + C_m \right] \\
\]

which can be written as

\[
(- \pi_{FS})_{HSM-1} = - \sum_{m} \left\{ \frac{1}{2} [x][J_{11}](x) - [x][J_{22}] - [x][G_1](q) \\
\quad + [G_2](q) + C_m \right\} \\
\]

Taking variations with respect to \( \{ x \} \), one gets the equation

\[
[J_{11}](x) - [G_1](q) - [J_{22}] = 0 \\
\]

Solving for \( \{ x \} \) from Eq. (121) and substituting in Eq. (120) one obtains

\[
(- \pi_{FS})_{HSM-1} = - \sum_{m} \left[ \frac{1}{2} [q][K](q) - [q][Q] + C_m \right] \\
\]

where

\[
[K] = \text{stiffness matrix for the } m\text{th element} \\
\quad = [G_1][J_{11}]^T[G_1]^T \\
\]

and

\[
[Q] = \text{nodal loads for the } m\text{th element} \\
\quad = [G_1][J_{11}]^{-1}[J_{22}] + [G_{22}] \\
\]

Here, a comment about the particular solution \( (R_2) \) in Eq. (115) is necessary. Tong and Pian [26] have shown, for plate bending, that when stress modes are so chosen that the polynomials in \( (R_2) \) are of the same degree as in \( [R_1] \), then
the finite-element solutions will be independent of the particular solution.
Tanaka [27] developed a triangular shell element, which is part of a cone,
based on the above model, and which was tested successfully in the program
STACUSS by Kotanchik [21].

D. Mixed Models

Thus far, finite-element models that were based on either the assumption of
a displacement field or the assumption of a stress field within each element
have been considered. In shell theory, it is possible to construct finite-
element models based on a simultaneous assumption of a suitable combi-
nation of displacement and stress fields which are mutually independent.
These possibilities are explored in this subsection.

1. Models Based on Reissner's Variational Principle. If in the variational
functional given by Eq. (44), one assumes a priori that the stress-strain
relations (39) are fulfilled, one obtains the Reissner functional

$$
\pi_R = \int \left\{ \left[ N^* \cdot \frac{1}{\sqrt{a}} \left( \frac{\partial u}{\partial x^2} + n_\alpha \times \phi \right) \right] + M^* \cdot \frac{1}{\sqrt{a}} \frac{\partial \phi}{\partial x^2} - F \cdot u - m \cdot \phi - B \right\} \, da
- \int_{c_{s}} (N \cdot u + M \cdot \phi) \, ds - \int_{c_{b}} [(u - \bar{u}) \cdot N + (\phi - \bar{\phi}) \cdot M] \, ds
$$

(125)
in which \( u, \phi, N^*, \) and \( M^* \) can be varied independently. For reasons of clarity
and convenience, the Reissner functional is written in its expanded form for
Kirchhoff-type shell theory in orthogonal curvilinear coordinates as follows

$$
\pi_R = \int \left\{ N^{11} \left[ \frac{\partial u_1}{\partial x_2} \left( \frac{1}{x_1} \frac{\partial x_1}{\partial \xi_1} \right) + \frac{\partial u_2}{\partial x_1} \left( \frac{1}{x_2} \frac{\partial x_2}{\partial \xi_2} \right) + \frac{w}{R_{11}} \right] + N^{22} \left[ \frac{1}{x_2} \frac{\partial u_2}{\partial x_2} + \frac{\partial u_1}{\partial x_1} \left( \frac{1}{x_1} \frac{\partial x_1}{\partial \xi_1} \right) + \frac{w}{R_{22}} \right] \right.
+ N^{12} \left[ \frac{\partial x_1}{\partial x_2} \left( \frac{1}{x_2} \frac{\partial x_2}{\partial \xi_2} \right) + \frac{\partial x_2}{\partial x_1} \left( \frac{1}{x_1} \frac{\partial x_1}{\partial \xi_1} \right) \right] + M^{11} \left[ \frac{\partial w}{\partial x_1} \left( \frac{1}{x_1} \frac{\partial x_1}{\partial \xi_1} \right) - \frac{\partial w}{\partial x_2} + \frac{u_1}{R_{11}} \right]
+ \frac{1}{x_1} \frac{\partial x_2}{\partial x_1} \left( \frac{1}{x_2} \frac{\partial x_2}{\partial \xi_2} \right) + \frac{1}{x_1} \frac{\partial w}{\partial x_1} + \frac{u_1}{R_{11}} \right]
+ M^{12} \left[ \frac{\partial x_2}{\partial x_1} \left( \frac{1}{x_1} \frac{\partial x_1}{\partial \xi_1} \right) \right]
+ \frac{1}{x_2} \frac{\partial w}{\partial x_2} + \frac{u_1}{x_2} \right]
$$

(126)
Alluri and T. H. H. Pian

In the integrals on $C_1$ and $C_2$, the superscript (or subscript) $n$ is the direction normal to the local tangent of the boundary, and similarly the super (sub)script $s$ denotes the direction along the tangent to the boundary. Also, it should be noted that the Kirchhoff-reduced boundary tractions are used in the integral on $C$. In the finite-element analysis, stresses and displacements can be assumed separately for each individual element. Several different possibilities arise depending on the continuity requirements for 1) stresses and 2) displacements at the interelement boundaries so that the function $n$ can be defined for the finite-element assemblage. Pian and Tong [1] studied this type of problem in detail for plates.

Based on different requirements, Eq. (126) for $\pi_R$ can be written in different forms for finite-element analysis. Thus, one may write

$$
-\left(\frac{1}{R_{11}} - \frac{1}{R_{22}}\right)\left(\frac{\alpha_2}{\alpha_1} \frac{\partial}{\partial \xi_1} \left(\frac{u_2}{x_2} + \frac{\alpha_1}{\alpha_2} \frac{\partial}{\partial \xi_2} \left(\frac{u_1}{x_1}\right)\right) - B - F \cdot u - m \cdot \phi\right) da

- \int_{C_1} \left[ N^{n*} u_n + \left( \frac{N^{n*} + \frac{M^{n*}}{R_{n*}}}{\xi_1} \right) u_s + \left( \frac{\partial M^{n*}}{\partial \xi_1} \right) \phi + \frac{M^{n*} \phi_s}{\xi_1} \right] ds

- \int_{C_2} \left[ (u_n - \bar{u}_n) N^{n*} + (u_s - \bar{u}_s) N^{s*} + (\phi_n - \bar{\phi}_n) M^{n*} \right] ds

+ (\phi_s - \bar{\phi}_s) M^{s*} \right] ds

(126)

\]

In the integrals on $C_1$ and $C_2$, the superscript (or subscript) $n$ is the direction normal to the local tangent of the boundary, and similarly the super (sub)script $s$ denotes the direction along the tangent to the boundary. Also, it should be noted that the Kirchhoff-reduced boundary tractions are used in the integral on $C$. In the finite-element analysis, stresses and displacements can be assumed separately for each individual element. Several different possibilities arise depending on the continuity requirements for 1) stresses and 2) displacements at the interelement boundaries so that the function $n$ can be defined for the finite-element assemblage. Pian and Tong [1] studied this type of problem in detail for plates.

Based on different requirements, Eq. (126) for $\pi_R$ can be written in different forms for finite-element analysis. Thus, one may write

$$
\pi_R = \sum_m \left[ \int_{C_m} \left\{ N^{n*} \frac{1}{\sqrt{a}} \left( \frac{\partial u}{\partial \xi_1} + a_n \phi \right) + M^{s*} \frac{1}{\sqrt{a}} \frac{\partial \phi}{\partial \xi_2} - F \cdot u - m \cdot \phi - B \right\} da

- \int_{C_{m,s}} (N \cdot u + M \cdot \phi) ds - \int_{C_{m,s}} [(u - \bar{u}) \cdot N + (\phi - \bar{\phi}) \cdot M] ds - \lambda_m \right]

(127)

\]

where $\lambda_m$ is a constraint term of various types to allow for the various discontinuities of stresses and displacements at the interelement boundaries. For example, it can be seen from Eq. (126) that, if the displacements $(u_1, u_2,$ and $w)$ and the slope $\partial w/\partial n$ along the normal to the boundary of the element are discontinuous, the integral over the element area leads to a delta function. But the functional $\pi_R$ can be written in the form of Eq. (127), if the stress resultants $N^{n*}$ and $M^{s*}$ are continuous across the interface, by including a proper $\lambda_m$ term. On the other hand, if the displacements and normal slopes are continuous across the interface, the integral exists even if the stress resultants $N^{n*}$ and $M^{s*}$ are discontinuous.

One can verify in a similar manner that the sufficient conditions for the
functional $\pi_\mathbf{e}$ to be defined for the discrete element assemblage are given by each of the following sets of conditions, and the corresponding $\lambda_\mathbf{m}$ terms are indicated.

1) Continuity of $u_1$, $u_2$, $w$, and $\partial w/\partial n$ across the interelement boundary even though $N^{\alpha\beta}$ and $M^{\alpha\beta}$ may be discontinuous:

$$\lambda_\mathbf{m} = 0$$  \hspace{1cm} (128)

2) Continuity of $N^m$, $N^n$, $M^m$, $M^n$, and the normal derivative of $M^m$ across the interelement boundary, even though the quantities $u_1$, $u_2$, $w$, and $\partial w/\partial n$ are discontinuous:

$$\lambda_\mathbf{m} = \int [N^m u_1 + N^n u_2 + Q_n w + \partial M^m/\partial s - M^n \phi_n] \, ds$$  \hspace{1cm} (129)

3) Continuity of $u_1$, $u_2$, $w$, and $M^m$, even though $\partial w/\partial n$, $N^m$, $M^n$ are discontinuous:

$$\lambda_\mathbf{m} = \int \left( M^m \frac{\partial w}{\partial n} \right) \, ds$$  \hspace{1cm} (130)

4) Continuity of $u_1$, $u_2$, $M^m$, $M^n$, and $M^m/\partial n$, while $w$, $\partial w/\partial n$, and $N^{\alpha\beta}$ are discontinuous across the interelement boundary:

$$\lambda_\mathbf{m} = \int \left[ Q_n w + M^m \frac{\partial w}{\partial n} + M^n \frac{\partial w}{\partial s} \right] \, ds$$  \hspace{1cm} (131)

5) Continuity of $w$, $\partial w/\partial n$, $N^m$, and $N^n$, while $u_1$, $u_2$, $M^m$, and $M^n$ are discontinuous across the interelement boundary:

$$\lambda_\mathbf{m} = \int [N^m u_1 + N^n u_2] \, ds$$  \hspace{1cm} (132)

6) Continuity of $w$, $N^m$, $N^n$, and $M^m$, while $u_1$, $u_2$, and $\partial w/\partial n$ are discontinuous across the interelement boundary:

$$\lambda_\mathbf{m} = \int [N^m u_1 + N^n u_2 + M^m \phi_n] \, ds$$  \hspace{1cm} (133)

Several other combinations are possible, but they appear to be irrelevant to shell analysis. Again, it should be pointed out that several variations on the lines of hybrid models presented earlier are possible in all of the six cases presented here by relaxing the requirements on the continuity of the needed quantities in each case through a Lagrangian multiple technique.
The Reissner functional can be further modified in shell theory as shown in the next two subsections.

2. Mixed Model 1 (MM1). In the general functional \( \pi_F \) of Eq. (44) consider a priori 1) the moment equilibrium equation [Eq. (11)], 2) the membrane strain-displacement relations [Eq. (18)], and 3) the constitutive relations

\[
\gamma_a = \frac{\partial B^n_B}{\partial q^a}; \quad N^{ab} = \frac{\partial A_M}{\partial \varepsilon_{ab}}; \quad \kappa_a = \frac{\partial B^n_B}{\partial M^a}
\]  

(134)

It can be shown that if the usual definition \( N^{12} = N^{21} = \frac{1}{2}(N^{12} + N^{12}) \) is made, the moment equilibrium equations [Eq. (11)] contain only \( M^a \) and \( q^a \) terms. Likewise if the definition \( \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\varepsilon_{12} + \varepsilon_{21}) \) is made, the membrane strains \( \varepsilon_{ab} \) depend on \( u \) alone. Under the above a priori assumptions, the general functional \( \pi_F \) can be shown to reduce to

\[
\pi_{M} = \int \int \left\{ q^a \frac{1}{\sqrt{a}} \frac{\partial u}{\partial \xi^a} \cdot n - F \cdot u + A_M - B_B^n - B_r^n \right\} da
\]

\[
- \int_{C_a} \{ \dot{N} \cdot u - (\dot{M} - M) \cdot \phi \} ds - \int_{C_D} [(u - \ddot{u}) \cdot N - \ddot{\phi} \cdot M] ds
\]  

(135)

in which \( u \) and \( M^a \) can be subjected to variation and whose Euler equations are 1) the stress-resultant equilibrium equation [Eq. (10)], 2) the curvature strain-rotation relations [Eq. (19)], and 3) the boundary conditions \( u = \ddot{u}; \quad \phi = \ddot{\phi} \) on \( C_D \) and \( N = \dot{N}, \quad M = \dot{M} \) on \( C_r \). Thus, in the finite-element method, the displacement vector \( u \) and the moment resultants \( M^{ab} \) (and through them \( q^a \)) are assumed independently for each element. Note from the first two terms in the integral of \( \pi_{M} \) that it is defined if the displacements are discontinuous, even if the stress \( q^a \) are discontinuous across the interelement boundary. On the other hand, if \( w \) is discontinuous across the interelement boundary, the integral is defined if only the stresses are continuous. Thus, for a finite-element assemblage,

\[
\pi_{M} = \sum \left\{ \int_{\Omega_x} \left\{ q^a \frac{1}{\sqrt{a}} \frac{\partial u}{\partial \xi^a} \cdot n - F \cdot u + A_M - B_B^n - B_r^n \right\} da \right.
\]

\[
- \int_{C_{Dx}} \{ \dot{N} \cdot u - (\dot{M} - M) \cdot \phi \} ds - \int_{C_{Dx}} [(u - \ddot{u}) \cdot N - \ddot{\phi} \cdot M] ds + \lambda_x \left. \right\}
\]  

(136)

where \( \lambda_x = 0 \) if \( u \) is continuous across the interelement boundary, and
λₘ = \int q' wₜ ds if uₜ is discontinuous but q" is continuous across the interelement boundary.

Several variations of the method on the lines of the hybrid methods presented earlier are possible by relaxing the various continuity conditions through a Lagrange multiplier technique. A model for a triangular shallow shell element, with linear variations of u and Mₑ in each element, appears to be first presented by Prato [28].

3. Mixed Model 2 (MM2). If, in the general functional πₑ of Eq. (44) one assumes a priori 1) the stress-resultant equilibrium equation [Eq. (10)], 2) the curvature strain-displacement relation [Eq. (19)], and 3) the constitutive relations

\[ \begin{align*}
\pi_{Nφ} &= \int \left\{ (N^φ - N) \cdot u + (N^φ - \bar{N}) \cdot \bar{u} \right\} ds \\
&\quad - \int_{C_T} \left\{ (N^φ - N) \cdot M + (N^φ - \bar{N}) \cdot \bar{M} \right\} ds \\
&\quad - \int_{C_D} \left\{ (N^φ - N) \cdot q + (N^φ - \bar{N}) \cdot \bar{q} \right\} ds
\end{align*} \]

then the functional πₑ can be shown to reduce to

\[ \pi_{Nφ} = \int \left\{ (N^φ - N) \cdot u + (N^φ - \bar{N}) \cdot \bar{u} \right\} ds \]

in which φ and N are the independent variables, whose Euler equations are 1) the equilibrium equations for moments [Eq. (11)], 2) membrane strain-displacement relations [Eq. (18)], and 3) the relevant boundary conditions. If, further, the usual assumption N₁² = N²¹ is made, then one can find that

\[ \pi_{Nφ} = \int \left\{ (N^φ - N) \cdot u + (N^φ - \bar{N}) \cdot \bar{u} \right\} ds \]

In the finite-element analysis, the rotation vector φ and the stress resultant vector Nφ are assumed independently for each element. The stress resultants Nₑ and q" are chosen in two parts: one to satisfy the homogeneous equilibrium equations (using stress functions as indicated in the Appendix) and the other, any particular solution.

If a Reissner-type theory is used, the vector φ is independent of u, and since no differentiations of q" and φ are involved in the functional πₑ, the integral and hence the functional πₑ are defined even if q" or φ or both are
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discontinuous at the element interface. If a Kirchhoff-type theory is used, the first term in the integral in Eq. (139) is

\[ \int \int \{ q^1 \left( -\frac{1}{\alpha_1} \frac{\partial w}{\partial \xi^1} + \frac{u_1}{R_{11}} + \frac{u_2}{R_{12}} \right) + q^2 \left( -\frac{1}{\alpha_2} \frac{\partial w}{\partial \xi^2} + \frac{u_1}{R_{12}} + \frac{u_1}{R_{12}} \right) \} da \]  

(140)

Thus, as has been discussed before, the functional \( \pi_{\text{Kirchhoff}} \) using a Kirchhoff-type theory can be written as

\[ \pi_{\text{Kirchhoff}} = \sum_{\alpha_m} \left[ \int_{\Sigma_\alpha_m} \{ -\sqrt{a}(q^1 \phi_1 + q^2 \phi_2) - m \cdot \phi \sqrt{a} - B_M - B_n + A_n^2 \} \} da \right. 
- \int_{\Sigma_{\alpha_m}} \{ (N - N) \cdot u + M \cdot \phi \} ds - \int_{\Sigma_{\alpha_m}} \{ ((\phi - \bar{\phi}) \cdot M - \bar{M} \cdot N) ds + \lambda_m \} 
\]

(141)

where \( \lambda_m = 0 \) if \( w \) is continuous across the element interface and \( \lambda_m = \int q^1 w ds \) if \( w \) is discontinuous but \( q^1 \) is continuous across the interface. To the authors' knowledge no such element has been developed for shells.

**SUMMARY**

A general first-order shell theory including transverse shear strains and couple stress-strain couples, a general variational principle and its static geometric analog are discussed in tensor form. A systematic classification of the various possible finite-element models, depending on an initial assumption of the element displacement field (displacement models) or the stress field (stress models), or a proper combination of both (mixed models) is made, and is summarized in Table 1 for convenience.

**APPENDIX**

The expressions for individual physical components of strain in terms of the physical displacement components and for the individual physical components of stress and moment resultants in terms of physical stress function components are given for both Reissner- and Kirchhoff-type shell theories.
Orthogonal curvilinear coordinates (not necessarily line of curvature coordinates) are considered. In what follows, the notations

$$\alpha_i = \sqrt{a_{11}}; \quad \alpha_2 = \sqrt{a_{22}}; \quad \frac{1}{R_{\beta \beta}} = \frac{a_{\beta} \cdot n_{\beta} \beta}{\sqrt{a_{\alpha} a_{\beta \beta}}} \quad (A.1)$$

are employed.

A. Strain Displacement Relations of a Reissner-Type Theory

The relations given by Eqs. (18) and (19) can be expanded as

$$\varepsilon_{11} = \frac{1}{\alpha_1} \frac{\partial u_1}{\partial \xi^1} + \frac{u_2}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi^2} + \frac{w}{R_{11}}; \quad \varepsilon_{12} = \frac{1}{\alpha_2} \frac{\partial u_2}{\partial \xi^1} + \frac{u_1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi^1} + \frac{w}{R_{12}}$$

$$\varepsilon_{22} = \frac{1}{\alpha_1} \frac{\partial u_1}{\partial \xi^2} - \frac{u_1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi^1} + \frac{w}{R_{11}} - \Omega; \quad \gamma_{21} = \frac{1}{\alpha_2} \frac{\partial u_2}{\partial \xi^1} - \frac{u_2}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi^1} + \frac{w}{R_{12}} + \Omega$$

$$\gamma_1 = \varphi_1 + \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi^1} - \frac{u_1}{R_{11}} - \frac{u_2}{R_{12}}; \quad \gamma_2 = \varphi_2 + \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi^2} - \frac{u_2}{R_{12}} - \frac{u_1}{R_{11}}$$

$$\kappa_{11} = \frac{1}{\alpha_1} \frac{\partial \phi_1}{\partial \xi^1} + \frac{\phi_2}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi^1} + \frac{\Omega}{R_{11}}; \quad \kappa_{12} = \frac{1}{\alpha_2} \frac{\partial \phi_2}{\partial \xi^1} + \frac{\phi_1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi^1} + \frac{\Omega}{R_{12}}$$

$$\kappa_{22} = \frac{1}{\alpha_1} \frac{\partial \phi_1}{\partial \xi^2} - \frac{\phi_1}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi^2} - \frac{\Omega}{R_{11}}; \quad \kappa_{21} = \frac{1}{\alpha_2} \frac{\partial \phi_2}{\partial \xi^2} - \frac{\phi_2}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi^2} - \frac{\Omega}{R_{12}} \quad (A.2)$$

One could make the usual hypothesis $\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\varepsilon_{11} + \varepsilon_{22})$ and get

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2} \left[ \frac{1}{\alpha_1} \frac{\partial u_2}{\partial \xi^1} - \frac{1}{\alpha_2} \frac{\partial u_1}{\partial \xi^2} \right] + \frac{1}{2} \left[ \frac{\partial u_2}{\partial \xi^1} \frac{\partial \alpha_1}{\partial \xi^2} \frac{\partial \alpha_1}{\partial \xi^2} + \frac{\partial u_1}{\partial \xi^2} \frac{\partial \alpha_2}{\partial \xi^1} \frac{\partial \alpha_2}{\partial \xi^1} \right] + \frac{w}{R_{12}} \quad (A.3)$$

from which

$$\Omega = \frac{1}{2} \left[ \frac{1}{\alpha_1} \frac{\partial u_2}{\partial \xi^1} - \frac{1}{\alpha_2} \frac{\partial u_1}{\partial \xi^2} \right] - \frac{1}{2} \left[ \frac{\partial u_1}{\partial \xi^2} \frac{\partial \alpha_2}{\partial \xi^1} \frac{\partial \alpha_2}{\partial \xi^1} - \frac{\partial u_2}{\partial \xi^1} \frac{\partial \alpha_1}{\partial \xi^2} \frac{\partial \alpha_1}{\partial \xi^2} \right] \quad (A.4)$$

Likewise,

$$\kappa_{12} = \kappa_{21} = \frac{1}{2} (\kappa_{11} + \kappa_{22}) = \frac{1}{2} \left[ \frac{\partial \phi_2}{\partial \xi^1} \right] \frac{\partial \alpha_1}{\partial \xi^2} \frac{\partial \alpha_1}{\partial \xi^2} + \frac{1}{2} \left[ \frac{\partial \phi_1}{\partial \xi^2} \frac{\partial \alpha_2}{\partial \xi^1} \frac{\partial \alpha_2}{\partial \xi^1} \right]$$

$$- \Omega \left( \frac{1}{R_{11}} - \frac{1}{R_{22}} \right) \quad (A.5)$$
Thus, in the modified Reissner-type theory, when the strain tensor is made symmetric, the only independent displacement measures are $u^*$, $w$, $\phi_1$, and $\phi_2$, the third rotation component $\Omega$ being expressed in terms of $u^*$ and $w$.

### B. Strain Displacement Relations for Kirchhoff-Type Theory

Under the constraint condition the transverse shear strains $\gamma = 0$, it follows from Eq. (33) that

$$\phi_1 = -\frac{1}{\alpha_1} \frac{\partial w}{\partial \xi^1} + \frac{u_1}{R_{11}} + \frac{u_2}{R_{12}}; \quad \phi_2 = -\frac{1}{\alpha_2} \frac{\partial w}{\partial \xi^2} + \frac{u_2}{R_{22}} + \frac{u_1}{R_{12}}$$

(A.6)

Also, if the tensor $\varepsilon_{\theta \phi}$ is made symmetric as before,

$$\Omega = \frac{1}{2} \left[ \frac{\alpha_2}{\alpha_1} \frac{\partial}{\partial \xi^1} \left( \frac{u_2}{\alpha_2} \right) - \frac{\alpha_1}{\alpha_2} \frac{\partial}{\partial \xi^2} \left( \frac{u_1}{\alpha_1} \right) \right]$$

(A.7)

Hence the only independent displacement measures are $u^*$ and $w$. The strain displacement relations then become

$$\varepsilon_{11} = \frac{1}{\alpha_1} \frac{\partial u_1}{\partial \xi^1} + \frac{u_2}{\alpha_1 \alpha_2} \frac{\partial \alpha_1}{\partial \xi^2} + \frac{w}{R_{11}}; \quad \varepsilon_{22} = \frac{1}{\alpha_2} \frac{\partial u_2}{\partial \xi^2} + \frac{u_1}{\alpha_1 \alpha_2} \frac{\partial \alpha_2}{\partial \xi^1} + \frac{w}{R_{22}}$$

$$\varepsilon_{12} = \frac{1}{2 \alpha_1} \frac{\partial}{\partial \xi^1} \left( \frac{u_2}{\alpha_2} \right) + \frac{1}{2 \alpha_2} \frac{\partial}{\partial \xi^2} \left( \frac{u_1}{\alpha_1} \right) + \frac{w}{R_{12}}$$

$$\kappa_{11} = \frac{1}{\alpha_1} \frac{\partial}{\partial \xi^1} \left( -\frac{1}{\alpha_1} \frac{\partial w}{\partial \xi^1} + \frac{u_2}{R_{22}} + \frac{u_1}{R_{12}} \right)$$

$$\kappa_{12} = \frac{1}{2 \alpha_1} \frac{\partial}{\partial \xi^1} \left( -\frac{1}{\alpha_1} \frac{\partial w}{\partial \xi^2} + \alpha_2 \frac{\partial^2 u_2}{\partial \xi^2} \frac{\partial \alpha_1}{\partial \xi^1} + \frac{\partial w}{\partial \xi^2} + \frac{u_2}{R_{12}} + \frac{u_1}{R_{11}} \right) - \frac{\Omega}{R_{12}}$$

$$\kappa_{22} = \frac{1}{2 \alpha_2} \frac{\partial}{\partial \xi^2} \left( -\frac{1}{\alpha_1} \frac{\partial w}{\partial \xi^2} + \alpha_2 \frac{\partial^2 u_2}{\partial \xi^2} \frac{\partial \alpha_1}{\partial \xi^1} + \frac{\partial w}{\partial \xi^2} + \frac{u_2}{R_{12}} + \frac{u_1}{R_{11}} \right) - \frac{\Omega}{R_{22}}$$

(A.8)
It is to be observed that even when the Kirchhoff assumption is made, the couple strains $\lambda$ do not vanish identically. Thus,

$$\lambda_1 = \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi_1} - \frac{1}{R_{12}} \left( - \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi_1} + \frac{u_1}{R_{11}} + \frac{u_2}{R_{12}} \right) + \frac{1}{R_{11}} \left( - \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{R_{22}} + \frac{u_1}{R_{12}} \right)$$

$$\lambda_2 = \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi_2} + \frac{1}{R_{12}} \left( - \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{R_{22}} + \frac{u_1}{R_{12}} \right) - \frac{1}{R_{22}} \left( - \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi_1} + \frac{u_1}{R_{11}} + \frac{u_3}{R_{12}} \right)$$

(A.9)

But by appropriate modifications in the constitutive relations, the effect of couple strains can be ignored.

C. Stress-Stress Function Relations for a Reissner-Type Theory

From the static geometric analogy as given by Eqs. (26) and (27), we can write, in component form,

$$N_{11} = -\frac{1}{\alpha_1} \frac{\partial F_2}{\partial \xi_2} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial x_2}{\partial \xi_1} + \frac{J}{R_{12}}; N_{22} = -\frac{1}{\alpha_1} \frac{\partial F_1}{\partial \xi_1} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial x_1}{\partial \xi_2} - \frac{J}{R_{12}}$$

$$N_{12} = \frac{1}{\alpha_2} \frac{\partial F_1}{\partial \xi_2} - \frac{1}{\alpha_2} \frac{\partial F_2}{\partial \xi_1} + \frac{J}{R_{12}}; N_{21} = \frac{1}{\alpha_1} \frac{\partial F_2}{\partial \xi_1} - \frac{1}{\alpha_1 \alpha_2} \frac{\partial x_1}{\partial \xi_2} - \frac{J}{R_{12}}$$

$$Q_1 = \frac{\partial J}{\alpha_2 \partial \xi_2} + \frac{F_2}{R_{12}} - \frac{F_1}{R_{12}}; Q_2 = -\frac{1}{\alpha_1} \frac{\partial J}{\partial \xi_1} + \frac{F_1}{R_{12}} - \frac{F_2}{R_{12}}$$

$$M_{11} = \frac{1}{\alpha_2} \frac{\partial H_2}{\partial \xi_2} + \frac{1}{\alpha_2 \alpha_2} \frac{\partial x_2}{\partial \xi_1} + \frac{K}{R_{22}}; M_{22} = \frac{1}{\alpha_1} \frac{\partial H_1}{\partial \xi_1} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial x_1}{\partial \xi_2} + \frac{K}{R_{11}}$$

$$M_{12} = \frac{1}{\alpha_2} \frac{\partial H_2}{\partial \xi_2} + \frac{1}{\alpha_2 \alpha_2} \frac{\partial x_2}{\partial \xi_1} - \frac{K}{R_{22}} - J; M_{21} = \frac{1}{\alpha_1} \frac{\partial H_1}{\partial \xi_1} + \frac{1}{\alpha_1 \alpha_2} \frac{\partial x_1}{\partial \xi_2} + \frac{K}{R_{11}} + J$$

$$P_1 = -F_2 + \frac{\partial K}{\partial \xi_1} \frac{1}{\alpha_2}; P_2 = F_1 + \frac{\partial K}{\partial \xi_1} \frac{1}{\alpha_1}$$

(A.10)

If one makes the usual hypothesis that $M_{12} = M_{21} = \frac{1}{2}(M_{12} + M_{21})$ and
\[ N_{12} = N_{21} = \frac{1}{2}(N_{12} + N_{21}), \text{ one can write} \]
\[
M_{12} = M_{21} = -\frac{1}{2} \left[ \frac{1}{\alpha_1} \frac{\partial H_2}{\partial \xi^1} - \frac{H_2}{\alpha_1 \alpha_2} \frac{\partial^2}{\partial \xi^1 \partial \xi^2} + \frac{1}{\alpha_2} \frac{\partial H_1}{\partial \xi^2} - \frac{H_1}{\alpha_1 \alpha_2} \frac{\partial^2}{\partial \xi^1 \partial \xi^2} \right] + \frac{2K}{R_{12}} \]  
\[ (A.11) \]

from which
\[
J = \frac{1}{2} \left[ \frac{1}{\alpha_1} \frac{\partial H_2}{\partial \xi^1} + \frac{H_2}{\alpha_1 \alpha_2} \frac{\partial^2}{\partial \xi^1 \partial \xi^2} \right] - \frac{1}{2} \left[ \frac{1}{\alpha_2} \frac{\partial H_1}{\partial \xi^2} + \frac{H_1}{\alpha_1 \alpha_2} \frac{\partial^2}{\partial \xi^1 \partial \xi^2} \right] \]  
\[ (A.12) \]

and
\[
N_{12} = N_{21} = \frac{1}{2} \frac{\alpha_2}{\alpha_1} \frac{\partial}{\partial \xi^1} \left( \frac{F_2}{\alpha_2} \right) + \frac{1}{2} \frac{\alpha_1}{\alpha_2} \frac{\partial}{\partial \xi^2} \left( \frac{F_1}{\alpha_1} \right) - J \left( \frac{1}{R_{11}} - \frac{1}{R_{22}} \right) \]  
\[ (A.13) \]

Thus, in a modified Reissner-type theory when the stress tensor is made symmetric, the only independent stress functions are \(H_z, K\), and \(F_z\), with the function \(J\) being expressed in terms of \((H_z, K)\) as above.

**D. Stress-Stress Function Relations for the Kirchhoff-Type Theory**

Here, under the constraint that the couple-stress stress couples \(P_z\) vanish, it follows from Eq. (34) that
\[
F_1 = -\frac{1}{\alpha_1} \frac{\partial K}{\partial \xi^1} + \frac{H_1}{R_{11}} + \frac{H_2}{R_{12}}; \quad F_2 = -\frac{1}{\alpha_2} \frac{\partial K}{\partial \xi^2} + \frac{H_1}{R_{12}} + \frac{H_2}{R_{22}} \]  
\[ (A.14) \]

Again if \(M_{zf}\) is made symmetric, one sees, as above, that
\[
J = \frac{1}{2} \left[ \frac{1}{\alpha_1} \frac{\partial H_2}{\partial \xi^1} + \frac{H_2}{\alpha_1 \alpha_2} \frac{\partial^2}{\partial \xi^1 \partial \xi^2} \right] - \frac{1}{2} \left[ \frac{1}{\alpha_2} \frac{\partial H_1}{\partial \xi^2} + \frac{H_1}{\alpha_1 \alpha_2} \frac{\partial^2}{\partial \xi^1 \partial \xi^2} \right] \]  
\[ (A.15) \]

Thus, the only independent stress functions are \(H_z\) and \(K\). Thus,
\[
N_{11} = -\frac{1}{\alpha_2} \frac{\partial}{\partial \xi^2} \left( -\frac{1}{\alpha_1} \frac{\partial K}{\partial \xi^1} + \frac{H_1}{R_{12}} + \frac{H_2}{R_{22}} \right) \]
\[
- \frac{1}{\alpha_1 \alpha_2} \frac{\partial}{\partial \xi^1} \left( +\frac{1}{\alpha_2} \frac{\partial K}{\partial \xi^2} + \frac{H_1}{R_{12}} + \frac{H_2}{R_{22}} \right) + J \]  
\[ \frac{1}{R_{12}} \]
\[ N_{22} = - \frac{1}{a_1} \frac{\partial}{\partial \xi^1} \left( \frac{1}{a_1} \frac{\partial K}{\partial \xi^1} + \frac{H_1}{R_{11}} + \frac{H_2}{R_{12}} \right) \]
\[ + \frac{1}{a_1} \frac{\partial}{\partial \xi^2} \left( \frac{1}{a_2} \frac{\partial K}{\partial \xi^2} + \frac{H_1}{a_2 R_{12}} + \frac{H_2}{a_2 R_{22}} \right) - \frac{J}{R_{12}} \]
\[ N_{12} = \frac{1}{a_2} \frac{\partial}{\partial \xi^2} \left( \frac{1}{a_2} \frac{\partial K}{\partial \xi^2} + \frac{H_1}{a_2 R_{12}} + \frac{H_2}{a_2 R_{22}} \right) \]
\[ + \frac{1}{a_2} \frac{\partial}{\partial \xi^1} \left( \rac{1}{a_1} \frac{\partial K}{\partial \xi^1} + \frac{H_1}{a_1 R_{11}} + \frac{H_2}{a_1 R_{21}} \right) - \frac{J}{R_{11}} \]
\[ M_{11} = \frac{1}{a_2} \frac{\partial}{\partial \xi^2} \left( \frac{H_1}{a_2} + \frac{K}{R_{22}} \right) \]
\[ M_{22} = \frac{1}{a_1} \frac{\partial}{\partial \xi^1} \left( \frac{H_1}{a_1} + \frac{K}{R_{11}} \right) \]
\[ M_{12} = - \frac{1}{a_2} \frac{\partial}{\partial \xi^2} \left( \frac{H_2}{a_2} \right) - \frac{1}{a_1} \frac{\partial}{\partial \xi^1} \left( \frac{H_1}{a_1} \right) - \frac{K}{R_{12}} \]  

(A.16)

REFERENCES

Theoretical Formulation of Finite-Element Methods


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