ANALYSIS OF FLEXIBLE MULTIBODY SYSTEMS WITH SPATIAL BEAMS USING MIXED VARIATIONAL PRINCIPLES

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ABSTRACT

A general finite element formulation is presented for dynamic analysis of spatial elastic beams, for small strains, in a multi-body configuration. The tangent maps associated to the finite rotation vector are used to compute the tangent matrices used to integrate implicitly the equations of motion in descriptor form. A corotational method and a mixed variational method are used to compute the tangent stiffness matrix. The tangent constraint matrices are obtained using consistent linearization of the constraint equations. The tangent inertia matrices, including the gyroscopic and centrifugal terms, are also obtained by using the tangent maps of rotation. The numerical examples analyzed in this paper include: dynamic analysis of flexible beam structures and multi-flexible body systems with open and closed kinematic loops. A comparison with the previous results in the literature shows a very good performance in terms of time integration step and number of elements used. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: flexible multibody dynamics; finite rotation; beam finite elements; variational principles

1. INTRODUCTION

This paper deals with the modelling of the spatial dynamic behaviour of homogeneous, isotropic and linear elastic one-dimensional deformable bodies, such as beams or rods, undergoing arbitrarily large rotations and translations, and small strains. The beam may be connected to other beams by means of kinematic constraints to form an open or closed-loop multi-flexible body mechanical system. A previous paper,1 of which the present one is a sequel, has investigated two different methods to compute the tangent stiffness of the elastic beam. In particular, the elastic tangent stiffness matrix and residual vector for a space beam were derived with two different approaches: a primal, Total Lagrangean (TL), corotational approach, and a mixed, Updated Lagrangean (UL) approach. The idea of that derivation was to formally adopt a variational setting to derive the field equations of motion of constrained flexible bodies. Previously, a consistent variational approach for multi-flexible body dynamics with beams has only been presented by Cardona and Geradin.2 In their work, they used the Principle of Virtual Work and methods of non-linear structural

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dynamics to devise an incremental/iterative procedure to integrate the tangent equations of motion of a non-linear beam. However, the first functional for finitely deformed beams, obtained in a consistent fashion from a general four-field mixed variational principle for three-dimensional finite elasticity was proposed by Iura and Atluri.\(^1\)\(^3\)\(^4\) In the field of rigid body dynamics, Borri \textit{et al.}\(^5\)\(^6\) used time finite elements to solve the equations of motion of a rigid body and of open- and closed-loop chains of multi-rigid bodies. Therefore, this paper extends to open and closed kinematic chains of multi-flexible beam-like bodies the effort initiated by Iura and Atluri.\(^3\)\(^4\) by Cardona and Geradin,\(^2\) and by Borri, Mello and Atluri,\(^5\)\(^6\) and is organized as follows. First, we overview the basic kinematics of rotations and the kinetics of the problem. Second, we describe primal and mixed variational functionals valid for non-linear elastodynamics of three-dimensional continua, and then the basic kinematic of rotations and the associated tangent rotational maps, routinely used during the iterative procedure of consistent linearization, may be derived in a straightforward manner (see, e.g. References 1 and 7). We have the Euler–Rodrigues formula for the rotation tensor \(R\): 

\[
\mathbf{R}(\mathbf{u}) = 1 + a_0(\mathbf{u} \times \mathbf{1}) + a_1[\mathbf{u} \times (\mathbf{u} \times \mathbf{1})] \\
\mathbf{\Gamma}(\mathbf{u}) = 1 + a_1(\mathbf{u} \times \mathbf{1}) + a_2[\mathbf{u} \times (\mathbf{u} \times \mathbf{1})]
\]

where \(a_0 = s\theta/\theta, a_1 = (1 - c\theta)\theta^2, a_2 = (1/\theta^2)(1 - a_0).\) In the following, we adopt the convention that \(d(\cdot), \delta(\cdot),\) and \(\Delta(\cdot)\) represent a differential, a virtual variation, and a finite, but small, increment of a generalized coordinate, whereas \(\cdot\), \(\cdot\), and \(\cdot\) represent a differential, a variation, and a finite increment of a quasi-co-ordinate, respectively.

2. KINEMATICS AND KINETICS

Consider a deformable one-dimensional continuum which is capable of undergoing large displacements and arbitrarily large rotations. To a material element in the undeformed configuration \(C_u\), we assign the triad of basis vectors denoted by \(E_i\). To the same material element, but in the deformed configuration \(C_d\), we assign the triad of basis vectors \(e_i\), which denotes the basis \(E_i\) after a purely rigid body rotation (similarly, \(\mathcal{F}_u, \mathcal{F}_d\) denote the undeformed and the deformed cross-section). The convected co-ordinates \(Y_i\), with \(i = 1, 2, 3\), denote a curvilinear system associated to a point centred in the material element. The displacement vector \(u\) describes the rigid body rotation of the material element, and can be parameterized as a function of the finite rotation vector \(\alpha = \theta e\), where \(e\) is the unit vector fixed in space, and around which the finite rotation of magnitude \(\theta\) takes place. A virtual variation of rotation \(q_\alpha\), the angular velocity vector \(\omega\), and the curvature vector \(I_3\) can be expressed, respectively, as \(q_\alpha \times 1 = \delta R \cdot R^T\), \(\omega \times 1 = R \cdot R^T\) and \(I_3 \times 1 = R \cdot 3 \cdot R^T\), where \(1\) represents the identity matrix, \(\omega\) the angular velocity vector, \(I_3\) the material curvature vector, and \((\cdot)_3\) the covariant derivative with respect to \(Y^3\) in \(C_u\). In addition \(\omega = \Gamma(\alpha)\dot{\alpha}\), and the associated tangent rotational maps, routinely used during the procedure of consistent linearization, may be derived in a straightforward manner (see, e.g. References 1 and 7). We have the Euler–Rodrigues formula for the rotation tensor \(R\) and the associated tensor \(\Gamma\):
The configuration of the beam is completely known when the inertial position of one point $P$ of the cross-section $S$, and the orientation of the cross-section itself with respect to a reference triad, are known. The inertial properties of the motion may be described with a Lagrangean approach, in which the rotational motion is referred to a fixed triad in space. In this paper, a material element labeled with the generalized co-ordinate vector $q = (u, \alpha)^T$ is endowed with a Lagrangean density $L = \mathcal{T}(q, \dot{q}) - f_1(v, R, U, t)$, where $\mathcal{T}$ is the kinetic energy density and $f_1$ is the mixed functional density described in more detail below. In our derivation, we make the assumption that the material is linearly elastic, and that the system is scleronomic, i.e. the kinetic energy density is independent of time. Then, our derivation is based on the functional:

$$E_1 = \int_{[t_i, t_{i+1}]} \left\{ \int_{\mathcal{Q}_0} \mathcal{T}(q, \dot{q}) dV - F_1(v, R, U, t) \right\} dt \quad (2)$$

2.1. A four-field principle

For a general elastic material, of density $\rho_0$ in the undeformed configuration, a four-field mixed principle, involving $v$, $R$, $U$, and $t$ as independent variables may be stated as the stationarity condition of the functional:

$$F_1(v, R, U, t) = \int_{\mathcal{Q}_0} \left\{ W_0(U) + t^T : [(I + \text{grad } v) - R \cdot U] - \rho_0 b \cdot v \right\} dV$$

$$- \int_{\mathcal{Q}_0} \tilde{i} \cdot v dA - \int_{\mathcal{Q}_0} N \cdot t \cdot (v - \tilde{v}) dA \quad (3)$$

where $F_1$ and $U$ are, respectively, the deformation gradient tensor, and the right stretch tensor; $t = i^T A_k \otimes a_i$ is the first Piola–Kirchhoff stress tensor ($A_i, a_i$: covariant base vectors at an arbitrary point of $\mathcal{Q}_0$ and at an arbitrary point of $\mathcal{Q}_0$); $b$ are applied body forces per unit mass; $\tilde{i}$ are prescribed tractions on $\mathcal{Q}_0$ and $\tilde{v}$ the prescribed displacements on $\mathcal{Q}_0$. The operator: stands for the double contraction of two second-order tensors, i.e., $A : B = A_{ij} B^{ij}$. In $F_1$: $v$ must be $C^0$ continuous, $R$ orthogonal, $U$ symmetric, and $t$ unsymmetrical, in order to be admissible trial fields. When the condition $\delta F_1 = 0$ is enforced for arbitrary and independent variations: $C^0$ continuous $\delta u$, $\delta R$ under the constraint $\delta R \cdot R^T = (\delta R \cdot R^T)_{a}$, symmetric $\delta U$ and unsymmetrical $\delta t$, the following Euler–Lagrange Equations are obtained:

Constitutive law: $\partial W_0/\partial U = \frac{1}{2}(t \cdot R + R^T \cdot t^T) = (t \cdot R)_{s}$.

Compatibility condition: $(I + \text{grad } v) = R \cdot U$.

Angular momentum balance: $(t^T \cdot U \cdot R^T)_{a} = 0$.

Linear momentum balance: $\nabla \cdot \mathbf{t} + \rho_0 \mathbf{b} = 0$ together with the natural

Traction boundary conditions: $\mathbf{N} \cdot \mathbf{t} = \mathbf{t}$ on $\mathcal{Q}_0$ and

Displacement boundary conditions: $v = \tilde{v}$ on $\mathcal{Q}_0$. Here, $\nabla = g^i \partial / \partial \zeta^i$ denotes the gradient operator along the base vectors $g^i$ associated with the curvilinear co-ordinates $\zeta^i$, and $(\cdot)_a, (\cdot)_s$ represent the symmetrical and unsymmetrical part of a tensor.

2.2. A three-field principle

One may derive a complementary variational principle involving only v, R and t. To do this, U must be eliminated from F1 by applying the following contact transformation:

\[ W_s(U) + W_c(r) = \frac{1}{2} (t \cdot R + R^T \cdot t^T) : U \]

where \( W_s \) and \( W_c \) are the strain and complementary strain energy density per unit undeformed volume, respectively, and \( r = \frac{1}{2} (t \cdot R + R^T \cdot t^T) \) is the symmetrized Biot–Lur'e stress tensor or Jaumann stress tensor. Physically, the Jaumann stress is the stress tensor associated with the force vector acting on the stretched, but not yet rotated, differential element of area, and measured per unit of undeformed area. Making use of this contact transformation is equivalent to adopting the hypothesis that, \textit{a priori}, the constitutive law \( \partial W_c / \partial r = U \) is met. When this contact transformation is substituted into the expression of functional \( F_1 \), the following Hellinger–Reissner-type three-field functional is obtained:

\[ F_3(v, R, t) = \int_{\gamma_0} \left\{ -W_c[\frac{1}{2}(t \cdot R + R^T \cdot t^T)] \right\} \]

\[ + t^T : (I + \text{grad} v) - \rho_0 b \cdot v \} \, dV - \int_{\gamma_{s0}} I \cdot v \, dA - \int_{\gamma_{n0}} N \cdot t : (v - \nu) \, dA \]

To be admissible, the independent fields in functional \( F_3 \) have to satisfy the following requirements: \( v \) must be \( C^0 \) continuous, \( R \) orthogonal, and \( t \) unsymmetrical. When the condition \( \delta F_3 = 0 \) is imposed for arbitrary and independent \( \delta v, \delta R, \delta t \), subject only to the additional constraint \( (R \cdot \delta R^T)_s = 0 \), the following Euler–Lagrange equations are recovered: Compatibility condition \( R \cdot (\partial W_c / \partial r)_s = (I + \text{grad} v) \). Angular momentum balance \( [R \cdot (\partial W_c / \partial r)_s, t]_a = 0 \), Linear momentum balance \( \nabla_0 \cdot t + \rho_0 b = 0 \), Traction boundary condition \( N \cdot t = I \) on \( \gamma_{s0} \). Displacement boundary condition \( v = \nu \), on \( \gamma_{n0} \). A property of the functional \( F_3 \) (or of \( F_1 \) before the contact transformation) is that one of its Euler–Lagrange Equations is the angular momentum balance for the 1st Piola–Kirchhoff stress tensor. This angular momentum balance is therefore embedded in the complementary energy density defined in terms of the Jaumann stresses. The Jaumann stress is an objective stress measure, which is very useful in finite deformation elasticity.

2.3. The twist and wrench vectors

For a material element of the beam, the Internal Virtual Work (IVW) can be written as

\[ IVW = \int_Y [T \cdot \delta \mathbf{h} + \mathbf{M} \cdot \delta \mathbf{k}] \, dY^3 \]

where \( \delta \mathbf{h} = \delta \mathbf{u}_3 - \mathbf{q}_0 \times (\mathbf{X} + \mathbf{u})_3 \) is the co-rotational variation of the stretch vector \( \mathbf{h} \), \( \delta \mathbf{k} = \mathbf{q}_{0,3} = \delta \mathbf{I}_3 \) is the co-rotational variation of the curvature vector, and \( \mathbf{T} \) and \( \mathbf{M} \) denote the internal stress, and moment, resultants. The kinetic energy may be written as

\[ T = \frac{1}{2} \int_Y \rho \dot{\mathbf{v}} \cdot \mathbf{v} \, dV = \frac{1}{2} \int_Y (\mathbf{I} \cdot \mathbf{\dot{u}} + \mathbf{H} \cdot \mathbf{\omega}) \, dV^3 \]

where \( \mathbf{I} \) and \( \mathbf{H} \) are, respectively, the linear and angular momentum densities. In terms of the finite rotation vector, we may write that: \( \mathbf{I} = \mathbf{A}_p \mathbf{\dot{u}} + \mathbf{C}^T \mathbf{\omega} \), and that \( \mathbf{h} = \mathbf{C} \mathbf{u} + \mathbf{I}^T (\mathbf{z}) \mathbf{I} (\mathbf{z}) \mathbf{\omega} \), where \( \mathbf{A}_p, \mathbf{C}, \) and \( \mathbf{I}_p \) are the mass density, the static moment density, and the cross-sectional inertia tensor, respectively. For simplicity, in the computations we assume the body frame to be a principal axis frame, hence we take the matrix of first moments of inertia \( \mathbf{C} \) to be zero. Using the definition of adjoint of a vector space operator, we may justify the fact that if the intrinsic \textit{wrench} \( \langle \mathbf{I}, \mathbf{H} \rangle \) is conjugate to the intrinsic \textit{twist} \( \langle \mathbf{\dot{u}}, \mathbf{\omega} \rangle \), then \( \mathbf{h} = \mathbf{I}^T \cdot \mathbf{H} \) is conjugate to \( \mathbf{z} \). By taking the variation of the kinetic energy density, we also
obtain that
\[ \frac{\partial}{\partial t} /SOT = RL [ l /SO v + /p71 /p68 ] dY^3, \]
where \( /SOT = /\dot{u} - /\omega \times /u \) is the corotational variation of the (absolute) velocity, and \( /\omega = /\dot{\omega} - /\omega \times /\omega = /\omega \) is the corotational variation of the angular velocity. Finally, it can be shown that the External Virtual Work (EVW) may also be written as
\[ EVW = RL [ q /SO u + m /p68 ] dY^3 \]
where now \( q \) and \( m \) denote the vector of external forces and external moments distributed along the length of the beam, respectively. For simplicity, and in order to preserve the symmetry of the resulting matrices, distributed moments are not considered.

We may now collect the increment of the displacement vector and the increment of the finite rotation vector in the incremental twist vector (also describing quasi-coordinates) \( \Delta /SO q = (\Delta u, /\Delta /SO h)^T \), and the vector collecting the increments of generalized coordinates in vector \( \Delta q = (\Delta u, \Delta /SO h)^T \). We may conclude that \( \Delta /SO q = \mathbf{X} \cdot \Delta /SO q \), where \( \mathbf{X} \) is the non-linear operator denoted by:
\[
\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & \Gamma(z) \end{pmatrix}
\]

(6)

The connection is now clear with the field of traditional kinematics (whose terminology is commonly adopted in the multibody dynamics literature), and conclude that \( \mathbf{X} \) is nothing else that the Jacobian mapping a vector \( \Delta /SO q \) in joint space to a vector \( \Delta /SO h \) in the Cartesian workspace of a mechanical linkage. In a similar way, one may collect the increment of the linear momentum vector and the material increment of the angular momentum vector in the incremental momentum wrench vector (also describing quasi-momenta) \( \Delta /SO h = (\Delta l, /SO h)^T \), and the vector collecting the increments of generalized momenta in vector \( \Delta /p66 = (\Delta l, /\Delta /SO h)^T \). We may conclude that \( \Delta /SO h = \mathbf{X}^{-T} \cdot \Delta /SO h \). The increment of kinetic energy density of the material element now becomes
\[ 2 \cdot \Delta \mathcal{F} = \Delta l \cdot \Delta /SO h + /SO h \cdot /\Delta h = \Delta l \cdot \Delta /SO h + \Delta /SO h \cdot /\Delta h \]

(7)

which shows the energetically conjugate pairs in joint space (or twist-wrench notation) and in Cartesian space (or in generalized coordinate notation).

2.4. The eight field primal functional

Given the definitions of the previous section, a weak form of Hamilton’s principle would state that the only feasible trajectory of the system is that which satisfies:
\[
\int_{t_i}^{t_{i+1}} \left\{ \int_L [ \delta T - IVW + EVW ] \, dY^3 \right\} \, dt = [ /\mathbf{l}^{b} \cdot /\delta u + /\mathbf{H}^{b} \cdot /\phi_{b} ]_{t_i}^{t_{i+1}}
\]

(8)

if the Principle of Virtual Work is used. Alternatively, if the weak form \( F_1 \) is used:
\[
\int_{t_i}^{t_{i+1}} \left\{ \int_L [ \delta T - \delta F_1 ] \, dY^3 \right\} \, dt = [ /\mathbf{l}^{b} \cdot /\delta u + /\mathbf{H}^{b} \cdot /\phi_{b} ]_{t_i}^{t_{i+1}}
\]

(9)

which may be called a primal–primal eight field form of Hamilton’s weak principle. In the case of a space beam, the eight fields are: \( u, R, \dot{u}, /\omega, /h, l_3, T, M \). By making use of the above definitions
of twist and wrench vectors, equation (8) may be compactly written as

$$\int_{t_i}^{t_{i+1}} \int_L \left\{ (\mathbf{\sigma}_i \cdot \mathbf{\sigma}_x) \cdot J \cdot \delta^z \right\} \mathbf{d}Y^3 \mathbf{d}t = [\mathbf{\sigma}^b \cdot \delta \mathbf{\eta}]_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \int_L [\mathbf{\sigma}_e \cdot \delta \mathbf{\sigma}] \mathbf{d}Y^3 \mathbf{d}t \quad (10)$$

where $\mathbf{\sigma}_e$ is the internal force wrench, $\mathbf{\sigma}_s$ is the distributed external wrench, $\mathbf{\sigma}_t$ is the wrench of momenta density, and $\mathbf{\sigma}^b$ is the wrench of momenta density at the boundary; $\mathbf{v}_t$ is the twist of quasi-velocities, $\mathbf{v}_x$ is the twist of quasi-stretches, $\delta \mathbf{\eta}$ is the twist of variation of quasi-coordinates; and $J$ is the symplectic matrix:

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (11)$$

Adjoining $F_1$ in Hamilton’s principle would result in an eight-field functional, which after elimination of stretches and curvatures would lead to equation (10). Therefore, the eight-field principle is a dynamic generalization of the three- and four-field principles, and which includes the Hamiltonian nature of non-linear elasticity in an explicit form. The advantages of the three- and four-field functionals $F_3$ and $F_1$ in non-linear elasticity are, consequently, extended to the non-linear dynamic case.

2.5. Contact transformation on kinetic energy: the five-field functional

In a completely equivalent way to the static case presented in Quadrelli and Atluri, we may also invoke a contact transformation on the kinetic energy. This is allowed because the inertia matrix $M$ is always positive definite and invertible during the motion of the mechanical system. The contact transformation, in this case, may be written as

$$\mathbf{\mathcal{F}} + \mathbf{H} = \mathbf{1} \cdot \mathbf{\dot{u}} + \mathbf{h} \cdot \mathbf{\dot{\alpha}} \quad (12)$$

Using $\mathbf{1} = \mathbf{A}_p \mathbf{1} \cdot \mathbf{\dot{u}}$ and $\mathbf{h} = \Gamma^T(\mathbf{\xi}) \mathbf{1}_p \mathbf{\Gamma}(\mathbf{\xi}) \cdot \mathbf{\dot{\alpha}}$, and their inversions $\mathbf{\dot{u}} = \mathbf{A}_p^{-1} \mathbf{1} \cdot \mathbf{\dot{u}}$, and $\mathbf{\dot{\alpha}} = \Gamma^{-1}(\mathbf{\xi}) \mathbf{1}_p^{-1} \Gamma^{-T}(\mathbf{\xi}) \cdot \mathbf{\dot{h}}$, we may write that

$$\mathbf{H} = \mathbf{1} \cdot \mathbf{\dot{u}} + \mathbf{h} \cdot \mathbf{\dot{\alpha}} - \mathbf{\mathcal{F}} = \frac{1}{2} \mathbf{\Gamma}^T \mathbf{A}_p^{-1} \mathbf{1} \cdot \mathbf{\dot{u}} + \frac{1}{2} \mathbf{h}^T \mathbf{\Gamma}^{-1}(\mathbf{\xi}) \mathbf{1}_p^{-1} \mathbf{\Gamma}^{-T}(\mathbf{\xi}) \cdot \mathbf{\dot{h}} \quad (13)$$

We conclude that the contact transformation leads to the (configuration-dependent) Hamiltonian density:

$$\mathbf{H}(\mathbf{1}, \mathbf{h}, \mathbf{u}, \mathbf{\alpha}) = \frac{1}{2} \mathbf{\tau}^T \cdot \mathbf{\mathcal{H}}^{-1}(\mathbf{q}) \cdot \mathbf{\tau} \quad (14)$$

and to the dual elastodynamic functional:

$$A_1 = \int_{t_i}^{t_{i+1}} \int_V \left\{ \mathbf{1} \cdot \mathbf{\dot{u}} + \mathbf{h} \cdot \mathbf{\dot{\alpha}} - \mathbf{H}(\mathbf{1}, \mathbf{h}, \mathbf{u}, \mathbf{\alpha}) \right\} \mathbf{d}V \mathbf{d}t - \int_{t_i}^{t_{i+1}} \int_{\Gamma} F_3(\mathbf{v}, \mathbf{R}, \mathbf{t}) \mathbf{d}t + \mathbf{BT} \quad (15)$$

where BT represent the natural boundary conditions. This form may be called a mixed-mixed form of Hamilton’s weak principle. In this paper, we will not deal with $\mathbf{\mathcal{H}}$, but with $\mathbf{\mathcal{F}}$ only, in the inertia contribution to $A_1$. However, the stationarity condition represented by equation (10) for a non-linear beam element can also be subjected to a contact transformation so as to obtain
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a five-field stationarity condition equivalent to the five-field functional $A_1$. The five fields are: $l, h, v, R, t$. In conclusion, we may synthesize the above derivation as follows. Consider the contact transformation

$$ T + \mathcal{H} = l \cdot \dot{u} + h \cdot \dot{z} $$

and the contact transformation

$$ W_s(U) + W_c(r) = \tfrac{1}{2} (t \cdot R + R^T \cdot t^T) : U $$

Then consider the primal–primal weak principle:

$$ \int_{t_i}^{t_{i+1}} \left\{ \int_{L} \delta T \, dY^3 - \delta F_1 \right\} \, dt = [l^b \cdot \delta u + H^b \cdot \delta q]_{t_i}^{t_{i+1}} $$

Then, from equation (18), using equation (16), we obtain the primal–dual principle:

$$ \int_{t_i}^{t_{i+1}} \left\{ \int_{L} \delta T \, dY^3 - \delta F_3 \right\} \, dt = [l^b \cdot \delta u + H^b \cdot \delta q]_{t_i}^{t_{i+1}} $$

From equation (19), using equation (17), by integrating by parts with respect to time, we obtain the dual–dual principle:

$$ \int_{t_i}^{t_{i+1}} \left\{ \int_{L} [p^b \cdot \delta q - q \cdot \dot{p} - \delta \mathcal{H}] \, dY^3 - \delta F_3 \right\} \, dt = [p^b \cdot \delta q - q^b \cdot \delta p]_{t_i}^{t_{i+1}} $$

where $p$ is the vector of momenta, and $q$ the vector of generalized co-ordinates. Similarly, we obtain the dual-primal principle:

$$ \int_{t_i}^{t_{i+1}} \left\{ \int_{L} [p \cdot \delta q - q \cdot \dot{p} - \delta \mathcal{H}] \, dY^3 - \delta F_1 \right\} \, dt = [p^b \cdot \delta q - q^b \cdot \delta p] $$

We have therefore obtained a family of stationarity conditions which still lead to the same field equations as the original functionals, but which extend the advantages of the three- and four-field variational principles to the dynamic case. These conditions are expressed in weak form, thereby obtaining a set of mixed variational principles in which stresses and displacements, as well as momenta and velocities, appear as the independent variables. By repeated use of Green’s theorem, the continuity requirements on the field variables can be relaxed, and low-order interpolation functions may be chosen when discretizing the problem. The kinematics of finite rotation appears implicitly in the formulation by means of the operator described in equation (6). Therefore, consistent linearization of these weak forms requires the use of the tangent maps of rotation as described in the next section. Furthermore, as we will show below, kinematic constraints can also be dealt with by incorporating their effect into the original functional, by means of an Augmented Lagrangean procedure. This is equivalent to adding to the Hamiltonian density a term proportional to the constraint violation, but weighted by a delta function.

The motivation for introducing these functionals lies in the fact that mixed variational forms of Hamilton’s weak principle have been proven\textsuperscript{9} to lead to unconditionally stable numerical integration schemes for rigid body dynamics and linear structural dynamics. Despite the redundancy of independent variables, the advantages of using complementary energy-based finite elements in
space, and of using mixed finite elements in time, for geometrically (or materially) non-linear structural elements, might lead to very efficient incremental solution schemes for non-linear structural dynamics. This issue remains to be explored.

3. TANGENT MATRICES

3.1. Tangent inertia matrices

By taking partial derivatives of the increment of kinetic energy density in terms of the independent variables, described in \( \mathbf{q} \), we may construct the left-hand side of the Lagrange equations. After some extensive manipulation, and operating with the tangent maps of rotation \( \mathbf{L}_T \) and \( \mathbf{L}_R \) which are described in,\(^1\) we obtain the following symmetric tangent inertia matrix:

\[
\mathbf{M} = \int \begin{pmatrix} 0 & \Gamma^T \mathbf{I}_p \Gamma \\ \Gamma^T \mathbf{I}_p & \mathbf{I}_3 \end{pmatrix} \, dY^3
\]

the following tangent gyroscopic contribution:

\[
\mathbf{G} = \int \begin{pmatrix} 0 & 0 \\ 0 & 2 \mathbf{L}_T^T (\mathbf{L}_p \Gamma \mathbf{a}) + 2 \Gamma^T \mathbf{I}_p \Gamma \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 \mathbf{L}_T^T (\mathbf{L}_p \Gamma \mathbf{a}) + 2 \Gamma^T \mathbf{I}_p \Gamma \\ \end{pmatrix} \, dY^3
\] (22)

and the following tangent centrifugal contribution:

\[
\mathbf{X} = \int \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \end{pmatrix} \Gamma^T \mathbf{I}_p \mathbf{L}_T (\mathbf{L}_p \mathbf{a}) + \Gamma^T \mathbf{I}_p \mathbf{L}_T (\mathbf{L}_p \mathbf{a}) + \Gamma^T \mathbf{I}_p \mathbf{L}_T (\mathbf{L}_p \mathbf{a}) \end{pmatrix} \, dY^3
\] (23)

The residual inertia vector becomes \( \mathbf{g} = \int \{ \mathbf{M} (\mathbf{a}) \cdot \dot{\mathbf{q}} + \mathbf{G} (\mathbf{a}, \mathbf{a}) \cdot \dot{\mathbf{q}} + \mathbf{X} (\mathbf{a}, \mathbf{a}) \cdot \dot{\mathbf{q}} \} \, dY^3 \). Since we assume that deformations within an incremental step are of small magnitude, all integrations are done over the initial length of the element. We can now introduce a finite element interpolation scheme such as \( \mathbf{q} = \mathbf{N}^T(Y^3) \mathbf{q}_N \), where \( C^0 \)-linear (two-node) interpolation functions \( \mathbf{N}^T(Y^3) \) are adopted both for the displacement vector and for the finite rotation vector. After interpolation of the nodal variables, the quantities defined above can be assembled in the usual way to form global quantities.

3.2. Tangent matrices of constraint

A multibody problem is qualitatively different that a structural dynamics problem: instead of having to deal with a set of (semidiscretized) Ordinary Differential Equations (ODE), the adjoined algebraic constraint equations lead to a set of Differential Algebraic Equations (DAE).\(^{10}\) A linearization procedure may be used to obtain the tangent matrices for the kinematic constraints. In this derivation, we go as further as deriving the constraint Hessian. This is possible since we have compact expressions for the tangent maps which involve the incremental rotation fields. Different types of algebraic equations describing kinematic joints may be introduced, but the primary ones, i.e. the ones referring to the lower kinematic pairs of major use in multibody dynamics, can be shown to reduce to combinations of basic algebraic equations. These are (considering for simplicity only rotational-type of joints) the equations pertaining to the spherical joint (or loop closure equation), and the dot-\(^{11}\) equations, described below. A Spherical joint requires the condition that the two connected nodes share the same position vector. A Universal joint can be thought of...
as a spherical joint which also must satisfy a dot-1 condition. A Revolute joint can be thought of as a universal joint which must also satisfy two dot-1 conditions. All these type of holonomic constraints, once a virtual variation is taken, reduce to the (Pfaffian) form $\mathcal{Z}_{ij}(q) \cdot \delta q = 0$, where by $\mathcal{Z}_{ij}(q)$ we have denoted the configuration dependent constraint Jacobian. There is no conceptual difficulty in extending these derivations to the case of non-holonomic constraints as well, since they may be written in varied form as well. We denote by $\oplus$, $\mathcal{J}$, and $\mathcal{H}$ the violation, the Jacobian, and the Hessian of the algebraic constraint. For a spherical joint constraint connecting nodes $p$ and $q$, the constraint violation $\oplus$ is a $(3 \times 1)$ vector which can be written as the following loop-closure equation $\oplus_{\text{sphe}} = X^p + u^p + R^p \cdot s^p - (X^q + u^q + R^q \cdot s^q) = 0$ where $s^p$ and $s^q$ represent the distance from the nodes $p$ and $q$ to the centre of the hinge, and $X^p$ and $X^q$ the initial nodal co-ordinates. A dot-1 condition represents the orthogonality of two directors, one at node $p$ and the other at node $q$. In symbols, if $e^p_1$ and $e^q_1$ represents unit vectors at point P in the 1-direction and at point Q in the two-direction, we have the scalar equation $\oplus_{\text{dot-1}} = e^p_1 \cdot e^q_1 = 0$, and, in component form, in terms of the rotation tensors at nodes $p$ and $q$ $\oplus_{\text{dot-1}} = R^p_{11} \cdot R^q_{11} = 0$. Using the procedure outlined above when the tangent maps were defined for the rotation vector, the constraint Jacobian $\mathcal{J}_{\text{sphe}}$ for the Spherical joint can be obtained as

$$\mathcal{J}_{\text{sphe}} = \begin{pmatrix} 1 & L^p_{R}(\alpha^p, s^p) & -1 & -L^q_{R}(\alpha^q, s^q) \\ 0 & E^T \cdot R^q T \cdot L^p_{R}(\alpha^p, E_1) & 0 & E^T \cdot R^p T \cdot L^p_{R}(\alpha^p, E_2) \end{pmatrix}$$

(24)

for the Universal joint as

$$\mathcal{J}_{\text{univ}} = \begin{pmatrix} 1 & L^p_{R}(\alpha^p, s^p) & -1 & -L^q_{R}(\alpha^q, s^q) \\ 0 & E^T \cdot R^q T \cdot L^p_{R}(\alpha^p, E_1) & 0 & E^T \cdot R^p T \cdot L^p_{R}(\alpha^p, E_2) \end{pmatrix}$$

(25)

and for the Revolute joint as

$$\mathcal{J}_{\text{rev}} = \begin{pmatrix} 1 & L^p_{R}(\alpha^p, s^p) & -1 & -L^q_{R}(\alpha^q, s^q) \\ 0 & E^T \cdot R^q T \cdot L^p_{R}(\alpha^p, E_1) & 0 & E^T \cdot R^p T \cdot L^p_{R}(\alpha^p, E_2) \end{pmatrix}$$

(26)

where $1$ represents the identity matrix and $L_{R}(\alpha, b)$ describes the tangent operator of $R$ applied to a general vector $b$. The constraint Hessian $\mathcal{H}_{\text{sphe}}$ for the Spherical joint becomes

$$\mathcal{H}_{\text{sphe}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\partial L^p_{R}}{\partial \alpha^p} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial L^q_{R}}{\partial \alpha^q} \end{pmatrix}$$

(27)

and, similarly, it can be computed for the other types of joint. An additional algebraic condition may be added to the constraint of the revolute joint by imposing a time profile for the relative angle of the joint. This can be accomplished by introducing the equality of relative angular velocity of the two nodes, in the direction of the allowable motion, to the imposed angular velocity. This results in a constraint equation of the type $\mathcal{Z}_{ij}(q) \cdot \delta q + b(i) = 0$ where $b(i)$ represents the effect of
the driving time-dependent term. Mechanisms may have a kinematic topology leading to open- and closed-loop kinematic chains. In both cases, the equations of constraint have the same differential form. For closed kinematic chains, one first has to select which joints to cut (perhaps in such a way as to ensure some bandedness in the form of the constraint Jacobian) in order to reduce the closed mechanism to an open-loop one. The analysis then proceeds regularly as described above.

3.3. Tangent stiffness matrix

We use two different approaches to compute the tangent stiffness of the beam element. In the first case, we use a TL corotational approach with a primal, pure displacement based finite element directly obtained by specializing $F_1$ to the case of a one-dimensional continuum and enforcing the compatibility equation in strong form, and in which the incremental variables are referred to the initial configuration. In the second case, we develop a new element which, instead, makes use of an UL point of view, is based on the mixed functional $F_3$ described above, and in which the incremental variables are referred to the previously converged configuration. The details of these derivations may be found in Quadrelli and Atluri, and will be omitted here.

3.4. Tangent iteration matrix

We have selected the implicit Hilber–Hughes–Taylor (HHT) algorithm to integrate in time the tangent equations of motion. The HHT method allows for some tunable degree of artificial viscosity, which is beneficial when one treats systems of DAE, as when kinematic constraints are present. In addition, the unconditional stability characteristics of the Newmark’s integrator for linear systems are preserved. The tangent iteration matrices are solved for the incremental variables with Newton–Raphson method until a specified convergence criterion is satisfied. The algorithm has been extended to include, for the static case, a continuation method (arc-length method) able to trace the response in the post-buckling regime. The structure of the system tangent iteration matrix is unsymmetric due to the fact that the skewsymmetric tangent gyroscopic matrix is also present. The HHT algorithm operates by redefining the residual vector in such a way that a weighted time average of the internal forces appears in the equation. Therefore, with the contribution of the elastic forces of the $i$th element, the new residual vector at time $t + \Delta t$ is

$$\mathbf{R}^{t+\Delta t} = -\mathbf{M}_{i\text{el}}(\mathbf{q}) \cdot \mathbf{q}_{i\text{el}}^{t+\Delta t} + \mathbf{K}_{i\text{el}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}}_{i\text{el}}^{t+\Delta t}$$

$$+ (1 + \hat{\gamma})(\mathbf{X}_{i\text{el}}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_{i\text{el}}(\mathbf{q}) \cdot \mathbf{q}_{i\text{el}}^{t+\Delta t} + \mathbf{F}_{i\text{el}}^{t+\Delta t})$$

$$- \hat{\gamma} \mathbf{Y} \cdot [(\mathbf{X}_{i\text{el}}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{K}_{i\text{el}}(\mathbf{q}) \cdot \mathbf{q}_{i\text{el}}^{t} + \mathbf{F}_{i\text{el}}^{t})]$$

(28)

where $\hat{\gamma}$ is the HHT damping parameter, and the matrix $\mathbf{Y}$ represents the projection of the rotational forces at the previous time step onto the tangent space at the current time step. This projection can be accomplished by the matrix $\mathbf{I}^{-T}$, and therefore, the nodal rotation parameters of the previous step need to be used to compute the vector of residual forces. We adopt an Augmented Lagrangean approach to adjoint the constraints to the Lagrangean of the system. This method has been proven to give excellent convergent properties for constrained dynamic systems. A scaling parameter $s$ is necessary to remove the ill-conditioning of the tangent iteration matrix. The penalty factor $p$ ensures the positive definiteness of the displacement-related sub-partition of the tangent iteration.
matrix, thereby improving the convergence to the solution. The variation of the Lagrangean thus becomes

\[ \delta \mathcal{L} = \int_L \left[ 1 \cdot \delta^\circ v + \mathbf{H} \cdot \delta^\circ \mathbf{w} - \mathbf{T} \cdot \delta^\circ \mathbf{h} - \mathbf{M} \cdot \delta^\circ \mathbf{k} - \mathbf{q} \cdot \delta \mathbf{u} - \mathbf{m} \cdot \delta \mathbf{q}_0 \right] \mathrm{d}Y^3 + \delta \left[ \frac{1}{2} \mathbf{p} \odot \odot - s \odot \cdot \lambda \right] \]

(29)

Although the choice of these scaling and penalty parameters is quite problem-dependent, they were both set equal to the maximum of the diagonal terms in the tangent stiffness and inertia matrices. Consequently, with the contribution of the tangent inertia matrices, the tangent iteration matrix of the HHT scheme is as follows:

\[ \mathbf{Z}_i = \left( \begin{array}{cc} \frac{\mathbf{d}_g}{\beta h^2} + \gamma \mathcal{K}_g + \mathcal{X}_g + s \mathcal{M}(\tilde{\lambda}) + \mathbf{p} \mathcal{J} \cdot \mathcal{J}^T & -s \mathcal{J} \\ -s \mathcal{J}^T & 0 \end{array} \right) \]

(30)

and the contribution to the residual vector is as follows:

\[ \mathbf{\tilde{r}}_i = \left( \begin{array}{c} -\mathcal{R}_g + \mathcal{J} \cdot (p \odot - s \tilde{\lambda}^g) - \mathbf{a}_g \\ -\odot \end{array} \right) \]

(31)

where \( \lambda \) is a vector of Lagrange multipliers, and all arrays are computed at the previous step. The arrays \( \mathcal{R} \) and \( \mathcal{F} \) are, respectively, the tangent stiffness matrix and the residual vector, discussed in the previous section.

4. THE TL INCREMENTAL ITERATIVE SCHEME

The following corotational incremental/iterative strategy is adopted within a Total Lagrangean approach. This computational scheme requires minor changes when material non-linearities need to be analyzed, so that inelastic or rate-dependent constitutive equations can be treated incrementally. A similar procedure may be adopted when the tangent stiffness is obtained in the Updated Lagrangean form, and this is discussed in Quadrelli and Atluri.\(^1\) The variables \((\mathbf{u}_g, \mathbf{z}_g)\) represent the nodal displacement vector and the finite rotation vector of the \(i\)th element, measured in global co-ordinates. For the \(i\)th element, given \((\mathbf{u}_g, \mathbf{z}_g), (\mathbf{u}_g, \mathbf{z}_g), (\mathbf{u}_g, \mathbf{z}_g)\), and \((\mathbf{u}_g, \mathbf{z}_g)\) at the \(N\)th time (or load step),

(1) predict displacements, velocities, accelerations, and multipliers at the next time (or load step) with the explicit predictor of the HHT-scheme:

\[ \mathbf{q}_{N+1}^0 = \mathbf{q}_N \]

\[ \mathbf{\dot{q}}_{N+1}^0 = -\frac{1}{\beta h} \mathbf{q}_N - \frac{1 - 2\beta}{2\beta} \mathbf{\dot{q}}_N \]

\[ \mathbf{\ddot{q}}_{N+1}^0 = \mathbf{q}_N + h \cdot [(1 - \gamma) \mathbf{q} + \gamma \mathbf{\ddot{q}}_{N+1}^0] \]

(32)

\[ \lambda_{N+1}^0 = \lambda_N \]

where \( h \) is the time step, and \( \beta \) and \( \gamma \) are the parameters of the integrator. Note that both the displacement vector and the finite rotation vector are treated in the same way;
(2) compute the tangent inertia and constraint matrices, defined with respect to the fixed frame;
(3) compute the finite rotation tensor \( \mathbf{R}^{i}([\mathbf{u}_{a}]) \), the current rigid body (average) rotation matrix \( \mathbf{E}([\mathbf{u}_{a}]) \), and the initial orientation matrix \( \mathbf{E}_{0}(\mathbf{X}) \);
(4) extract the deformation displacements \( \mathbf{u}^{e} \) and rotations \( \mathbf{0}^{e} \) associated to each node, i.e. \( \mathbf{u}^{e} = \mathbf{E}^{T} \cdot \{([\mathbf{u}_{a} + \mathbf{X}_{a}] - ([\mathbf{u}_{a} + \mathbf{X}_{e}]) - \mathbf{X}^{e}, \mathbf{T}^{e}([\mathbf{0}^{e}]) = \mathbf{E}^{T} \cdot \mathbf{R}^{e} \cdot \mathbf{E}_{0}, \) and \( \mathbf{0}^{e} = R2frv(T^{e}) \), where the pseudo-code operator \( R2frv(\cdot) \) denotes the operation of extraction of a finite rotation vector from a rotation tensor. Also, \( \mathbf{u}_{a}^{e} \) and \( \mathbf{X}^{e} \) represent the displacement vector and the initial co-ordinate vector of a reference point, typically coincident with the left node of our two-node beam formulation, respectively, and \( \mathbf{X} \) is the initial co-ordinate vector of the current node rigidly rotated to the current configuration;
(5) compute the tangent stiffness matrix \( \mathbf{K}(\mathbf{u}^{e}, \mathbf{0}^{e}) \) and the residual internal force vector \( \mathbf{f}(\mathbf{u}^{e}, \mathbf{0}^{e}) \) measured with respect to the corotated frame. We implicitly assume that the co-rotational tangent approach\(^1\) has been used, therefore the deformation displacement and rotation vectors at the reference node are zero. Consistently, only a partition of the internal force vector must be used. Since the internal strain energy can be written as \( W_{s} = W_{s}(\mathbf{d}) \), where in symbolic form \( \mathbf{d} = \mathbf{d}(\mathbf{D}) \) represents the mapping between the local displacements \( \mathbf{d} = ([\mathbf{u}_{a}^{e}, \mathbf{0}_{e}^{e}] \) of one end of the beam with respect to the other, measured by an observer with respect to a reference point located in the corotated frame, and the global displacements \( \mathbf{D} = ([\mathbf{u}_{e}^{e}, \mathbf{0}_{e}^{e}, \mathbf{u}_{a}^{e}, \mathbf{0}_{a}^{e}] \) (in this case, for a two-noded element) measured by the inertial observer (note that this mapping is highly non-linear because of the rotations), then, locally, the internal force vector is \( \mathbf{f}^{e}_{a}(\mathbf{u}_{a}^{e}, \mathbf{0}_{a}^{e}) = \partial W_{s}/\partial \mathbf{d}_{a} \), and the elasticity matrix is \( \mathbf{K}^{e}_{ab} = \partial^{2} W_{s} / (\partial \mathbf{d}_{a} \partial \mathbf{d}_{b}) \), and globally

\[
\begin{align*}
\mathbf{K}^{e}_{a} &= \left( \frac{\partial \mathbf{d}}{\partial \mathbf{D}_{a}} \right)^{T} \cdot \partial W_{s} / \partial \mathbf{d} \cdot \left( \frac{\partial \mathbf{d}}{\partial \mathbf{D}_{a}} \right) \quad (33) \\
\mathbf{K}^{e}_{ab} &= \left( \frac{\partial \mathbf{d}}{\partial \mathbf{D}_{a}} \right)^{T} \cdot \frac{\partial^{2} W_{s}}{\partial \mathbf{d} \partial \mathbf{d}} \cdot \left( \frac{\partial \mathbf{d}}{\partial \mathbf{D}_{b}} \right) + \left[ \frac{\partial (\partial \mathbf{d} / \partial \mathbf{D}_{a})}{\partial \mathbf{D}_{b}} \right] : \mathbf{I}^{e}_{a} \Rightarrow \\
\mathbf{K}^{e} &= \mathbf{W}^{T} \cdot \frac{\partial^{2} W_{s}}{\partial \mathbf{d} \partial \mathbf{d}} \cdot \mathbf{W} + \sum_{i=1}^{nif} D\mathbf{W}_{i} \cdot \left( \frac{\partial W_{s}}{\partial \mathbf{d}} \right)_{i} \quad (34)
\end{align*}
\]

where \( nif \) denotes the number of internal forces, and the third-order tensor \( D\mathbf{W} \) is contracted (symbolized by \( : \) ) with the vector of internal forces to yield the initial stress matrix in the global frame. Note that at convergence, some suitable norm of the internal force vector is a very small number, hence the second term drops out. Then, this formulation has been adopted, with the provision that whenever some norm (rigorously, it should be a weighted norm of the internal forces, because displacements and rotations have dissimilar units) of the vector of the internal forces is below a certain tolerance \( \varepsilon (\varepsilon < 10^{-3}, \) in the numerical examples), then the contribution of the second term is not included. We have called this procedure the co-rotational tangent procedure\(^1\) because we make use of the rigidly rotated spatial basis rigidly attached to the cross-section of the reference node. A note of caution is needed at this point: the relative rotation of any cross-section with respect to the reference one might not be small (i.e. rotations cease to be additive), which implies that in certain instances to obtain convergence we have to refine the mesh. Since \( \mathbf{0}_{e}^{e} \) represent local rotations (i.e. local infinitesimal rotation vector) at node \( a \), and the independent variables representing the rotation field are the components of the global finite rotation vector, we must use make of the tangent maps and their inverse (and go through the material increments of rotation) to
operate the mapping back to global generalized co-ordinates. Since the independent field is the finite rotation vector, a symmetric tangent stiffness matrix is obtained.

(6) assemble the arrays and, for an \( n \)th incremental external load vector \( \mathbf{F}_{\text{ext}} \), solve for the increments of global variables with \( i \) Newton–Raphson iterations until a specified convergence criterion such as max

\[
\left\{ \left\| \mathbf{F}_{\text{ext}} \right\|, \left\| \mathbf{\Delta q}_g \right\| \right\} \leq \min \left\{ \varepsilon_F, \varepsilon_{\text{violation}} \right\}
\]

is satisfied. The system of equations may be rewritten as

\[
\begin{pmatrix}
\mathbf{T}_{qq} & \mathbf{T}_{q\lambda} \\
\mathbf{T}_{\lambda q} & \mathbf{T}_{\lambda\lambda}
\end{pmatrix}
\begin{pmatrix}
\mathbf{\Delta q} \\
\mathbf{\Delta \lambda}
\end{pmatrix}
= 
\begin{pmatrix}
\mathbf{\lambda}_g \\
\mathbf{\lambda}_\lambda
\end{pmatrix}
\]

and can be condensed by first solving for the increments of Lagrange multipliers with \( \mathbf{\Delta \lambda} = (\mathbf{T}_{qq}^{-1} \cdot \mathbf{\lambda}_g) \), and then by computing the increments of generalized co-ordinates as \( \mathbf{\Delta q} = -\mathbf{T}_{qq}^{-1} \cdot \{ \mathbf{\lambda}_g - \mathbf{\lambda}_\lambda \} \). Note that \( \mathbf{T}_{qq} \) is block-diagonal and positive definite, hence always invertible. In the original formulation of the problem the block \( \mathbf{T}_{\lambda\lambda} \) is a null array, however a Perturbed Lagrangean approach would render it diagonal and scaled by a regularization constant which would make the solution by classical Newton–Raphson methods a possible task. An additional comment may be necessary with regards to the invertibility of \( \mathbf{T}_{\lambda\lambda} \). In particular, one may notice that this matrix may lose rank when the constraint Jacobian \( \mathbf{J}_\lambda \) loses rank. This may occur when, in a multibody system, a singular configuration is encountered, i.e., a configuration for which at least two rows (or columns) of the Jacobian become linearly dependent. A more refined analysis introduces the use of the Augmented Lagrangean approach, in which the invertibility of \( \mathbf{T}_{\lambda\lambda} \) is assured by augmenting, in the sense of penalty methods, the Lagrangean of the system with terms depending on the constraint Jacobian which make the metric of the configuration space always positive definite, until convergence is attained.

(7) update global nodal displacements and rotations, their time derivatives, and the multipliers, as follows:

\[
\mathbf{u}_{g}^{i+1} = \mathbf{u}_{g}^i + \mathbf{\Delta u}_g \\
\mathbf{R}_i^{i+1} = \mathbf{R}(\mathbf{\Delta x}_g) \cdot \mathbf{R}_i^d \\
\mathbf{q}_{N+1}^{i+1} = \mathbf{q}_N^i + \frac{\gamma}{\beta h} \mathbf{\Delta q} \\
\mathbf{q}_{N+1}^{i+1} = \mathbf{q}_N^i + \frac{1}{\beta h^2} \mathbf{\Delta q} \\
\mathbf{\lambda}_N^{i+1} = \mathbf{\lambda}_N^i + \mathbf{\Delta \lambda}
\]

(8) extract \( (\mathbf{u}_g, \mathbf{x}_g) \), with \( \mathbf{x}_g = R2frv(\mathbf{R}_g) \), and continue with the next load step.

---

5. NUMERICAL RESULTS

Next, we discuss some numerical results obtained using the displacement-based corotational element in Total Lagrangean form (case a), and the mixed element in Updated Lagrangean form (case b). Two Gauss points were used for the numerical integration of the integrals involving the tangent inertia matrices. The convergence tolerance used was $10^{-8}$. In Quadrelli and Atluri, several numerical examples obtained using both cases a and b were reported, with excellent results, for static and dynamic problems involving large deformations. Therefore, in this paper we focus only on the large dynamic deformation and on the multi-flexible body response of linear elastic materials. The material properties are shown in Table I.

5.1. Dynamic problems

Figure 1 shows the initial dynamic response of a highly flexible beam, modeled with five elements, driven by a time-dependent torque. The torque applied is a constant loading of 80 Nm of

<table>
<thead>
<tr>
<th>Table I. Beam properties</th>
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<tbody>
<tr>
<td><strong>Length</strong> $L = 10$ m</td>
</tr>
<tr>
<td><strong>Mass density</strong> $A_y = 1$ kg/m$^2$</td>
</tr>
<tr>
<td><strong>Young’s modulus</strong> $E = 10^5$ N/m$^2$</td>
</tr>
<tr>
<td><strong>Shear modulus</strong> $G = 38462$ N/m$^2$</td>
</tr>
<tr>
<td><strong>Poisson’s ratio</strong> $= 0.30$</td>
</tr>
<tr>
<td><strong>Polar moment of inertia</strong> $J_x = 2$ m$^4$</td>
</tr>
<tr>
<td><strong>Transverse moments of inertia</strong> $J_y = J_z = 1$ m$^4$</td>
</tr>
<tr>
<td><strong>Cross-sectional area</strong> $A = 0.1$ m$^2$</td>
</tr>
<tr>
<td><strong>Shear correction factor</strong> $= 5/6$</td>
</tr>
<tr>
<td><strong>Material density</strong> $= 10$ kg/m$^3$</td>
</tr>
</tbody>
</table>

Figure 1. Torque driven rotating beam, with revolute joint. Initial phase of motion
magnitudes for 2.5 s. We can appreciate the very large deformation superimposed onto the large rigid body rotation. We can say that the deformation is a significant percentage of the beam dimension. The time step used in the simulation was 0.05 units of time. This example is representative of flexible robotic manipulators. The hinge is modelled with a revolute joint. Figure 2 shows the same problem, modelled with only two beam elements. The comparison between the two solutions (case a and case b) is very good, and even the problem with two elements captures very well the initial phase of the motion, in which the effect of the inertia of the beam is preponderant. Figure 3 shows the time history of the Euclidean norm of the constraint violation, which, remarkably, always

remains of the order of $1 \cdot E - 30$. Figure 4 depicts the behavior of a free-flying cross with one element per leg. This example is taken from,\textsuperscript{15} including the loading conditions. The difference between this example and those in the paper is that in the paper the structure is modelled with brick elements. The reason to show these examples was to confirm that the beam element is capable of undergoing arbitrarily large rotations and small strains in full three-dimensional problems. Figure 5 shows the dynamical response of a tumbling tetrahedron modelled with one element per member, which undergoes vibration and tumbling motion of significant magnitude in the ambient space. Although large rotations are involved, the average number of iterations per time step in most of these problems was 3. Convergence was always attained when the full consistent tangent inertia contributions (i.e. inertia, centrifugal, gyroscopic) described in the previous chapters were used.
We can then conclude that, for large rotations of three-dimensional problems, the full inertia contribution is required in numerical simulation. Furthermore, the very flexible cases required the initial stress terms to converge.

5.2. Multibody dynamics problems

Figure 6 shows the initial configuration of a fifteen link chain hinged at the outer nodes, and allowed to fall in the plane under the action of gravity. This problem was solved in Reference 16 using space–time finite elements and rigid links. The problem is now solved with one flexible element per link. The problem has 30 nodes (for a total of 180 degrees of freedom), 16 revolute joints (so that a total of 80 nodal co-ordinates are constrained). Figure 7 shows the time history of the vertical displacement of the right node of the centre link during the motion. When the system is modelled with rigid links, the rebound occurring when the chain reaches the maximum extension causes a sudden change in the velocity of the centre link, introducing a discontinuity. When the system is modelled with flexible links, the rebound is much less discontinuous, because the elasticity of the links begins to act before the rebound occurs, and the links absorb elastic energy more gradually. Therefore, the simulation with flexible links is more realistic. In Figure 8, the initial configuration of a 12 bar mechanism, connected to a wall at the top joints, is shown. This problem
was also solved in Reference 16 using space–time finite elements and rigid links. The problem is now solved with one flexible element per link. The problem has 24 nodes (for a total of 144 degrees of freedom), 16 revolute joints (so that a total of 80 nodal co-ordinates are constrained). The loading consists of a constant vertical force at joint 6 of magnitude $50\,\text{N}$, and, at the same joint, an horizontal force acting up to $1.5\,\text{s}$, with a magnitude equal to $f_1 = 20\sin(2\pi t/1.5)\,\text{N}$. The time step used was 0.01, and the problem was solved with an average of 4 iterations per time step. Figure 9 shows the vertical component of the displacement of node 8. The initial trend of the motion compares excellently with the initial trend of the motion found in Reference 16. From an animation of the entire sequence of motion, one can appreciate the extension and contraction of all the members. Therefore, a significant elastic oscillation is superimposed to the oscillatory

Figure 8. Initial configuration of 12 bar mechanism
motion in the horizontal and vertical directions, with alternate reinforcement and weakening of the amplitude, denoting the presence of a parametric resonance phenomenon. Again, the simulation with flexible members is much more realistic than with rigid members.

6. CONCLUSIONS

A formulation to simulate the dynamic behaviour of elastic, geometrically non-linear beams for multi-flexible body applications has been presented. First, the use of the tangent maps of the finite rotation vector to correctly linearize the inertia forces and the constraint forces ensures convergence of the incremental/iterative procedure. Second, we introduce an Augmented Lagrangean procedure, and two new finite elements: an element based on the Total Lagrangean tangent co-rotational method, and an element based on the Updated Lagrangean form of a mixed variational principle for finite elasticity, to compute the tangent stiffness matrix of the space beam, obtaining excellent numerical results. These elements are very promising for applications in the field of large-scale flexible multibody dynamics, in which accuracy and coarser meshes are simultaneously needed. Third, we have introduced new functionals for non-linear elastodynamics problems, which can be obtained from the action integral by introducing a contact transformation on the kinetic energy density, and a contact transformation on the strain energy density. The issue of the finite element implementation of these functionals remains open. Despite the redundancy of independent variables, there could exist a combined advantage of using complementary energy-based finite elements in space, and mixed finite elements in time, for geometrically (or materially) non-linear structural elements, leading to time-integration schemes for non-linear structural dynamics with excellent invariant properties.

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