Development of a new frame finite element for crash analysis, using a mixed variational principle and rotations as independent variables

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Abstract

This paper deals with the development and application of a new space curved frame finite element to be used for crash analysis (non-linear). The frame finite element has been developed using a mixed variational principle (complementary form) and using rotations as independent variables. The formulation has been validated for problems of large deflection and rotation, and for problems involving initially curved members. Based on the validation performed, it is expected that crash problems may be modelled using a single element per member thus retaining computational efficiency while performing an accurate analysis. An illustrative example (modelling of an S-leg seat) is presented here to illustrate the benefits of the proposed approach to a designer.

1. Introduction

Most existing crashworthiness codes fall into two broad categories. The first, discrete element models (Ex. KRASH), are non-linear spring-mass models that are well suited to perform a computationally efficient (high speed, low computational resource requirement) analysis, but are however limited in the level of detail and accuracy. The second, detailed finite element models (Ex. Dyna3D, DYCAST), are well suited to perform a detail and accurate analysis but require high computational resources and involved meshing. The present effort aims at combining the advantages of the two, that is to realize the twin objectives of accuracy and computational effectiveness. Based on the validation performed, it is expected that the new frame finite element will enable a computationally efficient crash analysis without sacrificing accuracy. This may be implementable in a existing code (Ex. KRASH) to achieve the same objectives.

The following salient features help to realize the twin objectives of accuracy and reduction of computational effort.

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1. Overview of technical approach

(1) The kinematics model Iura and Atluri [1] accounts for large deflection and rotation (rigid and relative) as well as shear deformation.

(2) A Mixed Variational Principle (In Complementary Form) is used which enables modelling non-linear problems with just a single element per member where conventional displacement based finite element approaches would require several elements (5–20) for achieving the same degree of accuracy.

(3) Structural collapse is accounted for by using plastic-hinge approach.

2. Theory (kinematics and variational principle)

2.1. Geometry of the undeformed and deformed frame

For the sake of clarity, the kinematical relations of the curved member are summarized below. It is discussed in greater detail in Ref. [1].
Let \( Y^m(m = 1, 2, 3) \) be a convected orthogonal curvilinear coordinate system. The coordinates \( Y^x(\alpha = 1, 2) \) are taken in the cross-section, while the coordinate \( Y^3 \) is taken along the beam axis, as shown in Fig. 1. The undeformed base vectors at an arbitrary point in the beam axis, expressed in terms of the undeformed unit based vectors \( E_m \) at the beam axis, as shown below:

\[
\begin{align*}
A_x &= E_x, \\
A_3 &= -Y^2K_1E_1 + Y^2K_2E_2 + g_0E_3,
\end{align*}
\]

where

\[
g_0 = 1 - Y^1K_2 + Y^2K_1.
\]

The quantities \( e_m \) are the maps of the base vectors after a purely rigid rotation denoted by the finite rotation tensor \( R \), and are given by \( e_m = R \cdot E_m \) \([4, 5]\). It should be noted here that \( e_3 \) in general does not coincide with a unit tangent to the deformed beam axis \( (e_3^0) \), due to the presence of shear deformation. The angles of shear deformation, \( \beta_1, \beta_2 \) and \( \beta_3 \), are given by \( \sin \beta_\alpha = e_\alpha \cdot e_3^0 \) \((\alpha = 1, 2)\), and \( \beta_3 = e_3^0 \cdot e_3 \).

The displacement vector of an arbitrary point in the cross-section, \( U \) may be written as shown below:

\[
U = u + Y^x(e_x - E_x)
\]

where, \( u \), the displacement vector at the beam axis, is written as \( u^mE_m \).

Fig. 1. Kinematic scheme for large deformation and large rotation analysis.
Using the Frenet-Serret formulae, the derivatives of the base vectors with respect to the length parameter may be written in terms of the curvatures as shown below:

\[ E_{m,3} = K \times E_m, \quad e_{m,3} = k \times e_m. \]  

(3)

\( K_x \) and \( k_x \) \((x = 1, 2)\) are components of undeformed and deformed curvatures; and \( K_3, k_3 \) are the twists.

\[(\cdot)_3 = \frac{d(\cdot)}{dL}\] where \( L \) is the parameter of the length of an arc along the line of origin of the coordinate system \( Y^m \), in the reference configuration.

### 2.2. Constitutive relation

The relationships between the stress (moment) resultant and the changes in stretch (curvature) at a point in the frame (along its length), can be written, based on theory of semi-linear elastic solids. As will be mentioned later, the influence of localized non-linear deformation can be dealt with by a method of plastic hinge. In the event of a structural collapse, the localized zone of deformation (plastic hinge), has a greater effect on the deformation behavior than the distributed plasticity in the larger portions of the frame. Hence we shall use elastic relationships between stress/moment resultants and changes in stretch/curvature in this paper. Furthermore, collapse behavior is modelled using plastic hinge method.

It is also mentioned here that for the case of distributed plastic deformation in a frame, only the stress/moment resultant – stretch/curvature relationships will alter in the context of present frame element development.

The constitutive relation for the beam, relating the components of the change in curvature and the stretch in \( E_m \) coordinates to those of stress and moment resultants is given by the following relationship \([1]\).

\[
\begin{bmatrix}
T^1 \\
T^2 \\
T^3 \\
M^1 \\
M^2 \\
M^3
\end{bmatrix}
= 
\begin{bmatrix}
GA_0 & 0 & 0 & 0 & -GI_1 \\
GA_0 & 0 & 0 & 0 & GI_2 \\
EA & EI_1 & -EI_2 & 0 \\
EI_{11} & -EI_{12} & 0 & 0 \\
EI_{22} & 0 & 0 & 0 \\
0 & GJ & 0 & 0
\end{bmatrix}
\begin{bmatrix}
h_1 \\
h_2 \\
h_3 \\
\bar{k}_1 \\
\bar{k}_2 \\
\bar{k}_3
\end{bmatrix}
\]  

(4)

where

\[ h_1 = g \sin \beta; \quad h = g \sin \beta_2; \quad h_3 = g\beta_3 - 1. \]  

(5)

\[ A = \int Y^1 Y^2 g_0 \, dA, \quad A_0 = kA, \quad I_x = \int Y^x g_0 \, dA, \]  

(6)

\[ I_{12} = \int Y^1 Y^2 g_0 \, dA, \quad I_{22} = \int (Y^2)^2 g_0 \, dA, \quad J = \int \rho g_0 \, dA. \]
Note, the stress and moment resultants are defined as [4].

\[
T = T^m e_m = \int g_0 A^3 (S_1 \cdot F^T) \, dA,
\]

\[
M = M^m e^m = \int Y^a e_a g_0 A^3 (S_1 \cdot F^T) \, dA
\]

where \( A^m \) is the reciprocal basis of \( A_m \), \( S_1 \) is the second Piola-Kirchoff stress tensor; \( F \) is the deformation gradient tensor; \( (\cdot)^T \) is a transpose and \( dA = dY^1 \, dY^2 \).

The factor \( k \) is a shear-correction factor [6]. It is important to note here that some of the inertia terms (such as \( I_{12} \)) vanish on choosing a suitable coordinate system.

2.3. General mixed variational principle

As first shown by Fraeijis de Veubeke [7] and later generalized by Atluri and Murakawa [8] a general mixed principle, for a 3-dimensional elastic material, and involving the first Piola-Kirchoff stress tensor \( t_1 \), the right stretch tensor \( U \), the finite rotation tensor \( R \) and the displacement vector \( v \) as variables, can be stated as the stationary condition of the functional \( F_1 \):

\[
F_1(t_1, U, R, v) = \int [W_0(U) + t_1^1 : ((I + \nabla_0 v)^T - R \cdot U) - \rho_0 \bar{b} \cdot v] \, dV_0
\]

\[
- \int \bar{t} \cdot v \, ds - \int \bar{t} \cdot (v - \bar{v}) \, ds
\]

where \( W_0 \) is the strain energy function, \( \rho_0 \) the mass density in the undeformed state, \( \bar{b} \) the body force vector per unit mass, \( \bar{t} \) the traction on the boundary per unit undeformed area and \( \nabla_0 \) the gradient operator in the undeformed state. It should be noted that the rotation tensor \( R \) may be written in terms of a rotation vector as \( e^{\alpha x} \) [5, 9].

Using the kinematic and constitutive relations mentioned earlier, Iura and Atluri [1] derived the functional for the beam, and on imposing the stationary condition obtained the following equation:

\[
\delta F_1 = \int \left[ \left( \frac{\partial W_s}{\partial h_m} - T^m \right) \delta h_m + \left( \frac{\partial W_s}{\partial \bar{h}_m} - M^m \right) \delta T^m + \delta T \cdot ((x + u)_3 - R \cdot (h + E_3)) \right]
\]

\[
\int \nabla_0 \delta \cdot \bar{t} - (T, q \cdot \delta u - (M_3 + (x + u)_3 \times T) \cdot d\phi) \, dL
\]

\[
- [s, \delta T \cdot (u - \bar{u}) + \delta M \cdot (\phi - \bar{\phi})]_{L=0} \left[ \int_{L=0} \delta \phi \cdot M \right]_{L=0} = 0
\]

where \( \delta \phi \) and \( I_3 \) are defined so as to satisfy \( \delta \phi \times I = \delta R \cdot R^T \) and \( I_3 \times I = R_{3,} \cdot R^T \). (following Atluri [4] and Pietraskiewicz and Badur ([5], respectively).

3. Element development

The advantage in using the variational principle in Eq. (8) is that on imposing the stationary condition the condition of balance of angular momentum is independent of the constitutive relation.
One may obtain the changes in curvatures and stretch by differentiating the displacement or rotation field. It is however noted that such operations (leading to a displacement finite element) would not give a singular nature for curvatures. Furthermore, the variations would be one order lower than that of rotations and two orders lower than that of displacements. Thus many elements are needed to represent required deformation.

On the other hand, the variations of stress and moment resultant are determined by the current configuration of the frame. Especially, the stress resultants do not vary in the absence of body, inertia or any distributed forces. Therefore, it is far easier to assume the stress and moment field than deriving those from the kinetic variables. Also, it is quite simple to derive the changes in curvatures and stretches from the stress and moment resultants as done below.

Thus we use the strong form of the constitutive relation to eliminate the stretch and the curvature (in terms of the moment and stress resultants by inverting the constitutive equation) while imposing compatibility, linear and angular momentum balance conditions in their weak forms. Consequently the stationary condition was simplified as shown below:

\[
\delta F_1 = \int [\delta T \cdot (x + u)_3 - R \cdot (h + E_3)] + \delta M \cdot (l_3 - R \cdot \tilde{k})
- (T, + q) \cdot \delta u - (M, + (x + u)_3 \times T) \cdot d\phi \, dL
- [s, \delta T \cdot (u - \bar{u}) + \delta M \cdot (\phi - \bar{\phi})]^{L=L_0}_{L=0}
\]

On inverting the constitutive relation, the stretch and curvature may be written as

\[
h = H_T \cdot T + H_M \cdot M,
\]
\[
\tilde{k} = K_T \cdot T + K_M \cdot M,
\]

thus reducing the number of independent variables to four. Note \(H_T, H_M, K_T\) and \(K_M\) are \((3 \times 3)\) obtained by inverting the \((6 \times 6)\) matrix in Eq. (4).

3.1. Discretization procedure and derivation of tangent stiffness matrix

3.1.1. Selection of trial functions

The discretization procedure is summarized here. It is discussed in detail in an earlier paper by the authors [2].

The development below is quite general and assumes linear trial functions for the three components of all the four variables (stress resultant, moment resultant, displacement and rotation) as indicated below. In the absence of distributed loading the procedure is simplified greatly as the stress resultant is a constant.

\[
T^i = T^i_1 \frac{L}{l} + T^i_2 \left(1 - \frac{L}{l}\right), \quad M^i = M^i_1 \frac{L}{l} + M^i_2 \left(1 - \frac{L}{l}\right)
\]
\[
u^i = u^i_1 \frac{L}{l} + u^i_2 \left(1 - \frac{L}{l}\right), \quad \omega^i = \omega^i_1 \frac{L}{l} + \omega^i_2 \left(1 - \frac{L}{l}\right) \text{ for } i = 1, 2, 3.
\]
It is important to note here that \( T' \) and \( M' \) refer to the components in \( E_i \) coordinates while \( u' \) and \( \omega' \) refer to the components in \( E_i \) coordinates. The subscripts 1 and 2 indicate that the variable is defined at the end nodes 1 and 2 of the frame element, respectively.

The behavior of \( T, M, u \) and \( \omega \) and the suitability of the trial functions are described below.

(a) \( T \) (Stress resultant): Constant trial functions give the exact solutions for a problem where there is no distributed loading.

(b) \( M \) (Moment resultant): From the strong form of the angular momentum balance condition, it may be seen that the order of \( M \) is one more than the order of \( T \) when the displacements are small. Hence linear trial functions are suitable for the moment resultants.

(c) \( u \) (Deflection of the beam axis): It must be emphasized here, that due to the mixed variational approach (complementary form) used here the differentiability requirement on the displacements are substantially reduced. Displacement based approaches, which are the most used and seen in literature, use higher-order shape functions for displacements (cubic) and require several elements for modelling a single member.

(d) \( \omega \) (Finite rotation vector): From the definition of \( u (R = e^{\omega x'}) \), Section 2.3) it can be seen that with a linear variation of \( \omega \), we obtain a higher-order variation of \( R \). It should also be emphasized here that the rotations are interpolated independent of the displacement.

Consider the example of a cantilever beam loaded at one end by a constant force normal to the axis and undergoing small deformations. It is well known that the variation of the deflection and the rotation are cubic and quadratic respectively. Using the formulation we have derived, treating the entire cantilever beam as one element, we obtained the exact end deflection using just linear functions for \( u, \omega, M \) and constant for \( T \). It should be noted that in the formulation derived, the rotation and the displacement are independent variables and so are the stress and moment resultants, thus we have a mixed formulation where lower-order elements are quite effective and locking is less likely.

### 3.1.2. Derivation of tangent stiffness matrix

(a) Expression (12) is now substituted into Eq. (10). The integrated terms and boundary terms are evaluated.

(b) The condition that the variation in each of the components of the four variables at both the nodes are arbitrary and independent (\( \delta T_1, \delta T_2, \delta M_1, \delta M_2, \delta u_1, \delta u_2, \delta \omega_1, \delta \omega_2 \)) is invoked, thus giving the system of Eqs. (13).

Consequently 24 equations are obtained in 24 unknowns (4 variables, 3 components, 2 unknowns) as shown in the subsequent discussions. Quite often, however, while dealing with 1D or 2D problems, problems with displacement specified boundary conditions, or problems without distributed loading, the number of components and equations involved reduces substantially. For instance, in the cantilever beam example [2], due to the absence of distributed loading, the stress resultant was assumed to be a constant, furthermore it was a planar problem, thus greatly simplifying the problem.

As mentioned earlier, the 24 equations are obtained by the collecting the terms multiplying the variations below and setting them to zero. It should be noted that the variations below are arbitrary and independent.
The 24 scalar equations may be arranged as 8 matrix equations (1 x 3 matrices). The system of equations is listed below

\[
\begin{bmatrix}
\delta T_1 \\
\delta T_2 \\
\delta T_3 \\
\delta M_1 \\
\delta M_2 \\
\delta M_3 \\
\delta u_1 \\
\delta u_2 \\
\delta u_3 \\
\delta \omega_1 \\
\delta \omega_2 \\
\delta \omega_3
\end{bmatrix} =
\begin{bmatrix}
\mu x_3 \\
\mu u_3 \\
\mu \omega_3 \\
Pl \\
Pr \\
\rho l \\
\omega_3 \\
\omega_2 \\
\omega_1
\end{bmatrix}.
\]

[13]

The details of the above equations and specific values for an example are described in an earlier paper by the authors [2]. It is worth emphasizing that the above row matrices are in general functions of the components of the nodal displacements, rotations, stress and moment resultants. It should also be noted that the $G_i$'s are weighted functions of the applied load.

It can be seen that the system of equations (13) are highly coupled. Hence, we use Taylor’s expansion (used as a linearization procedure) to relate the incremental displacements explicitly to the applied loading (through the tangent stiffness matrix) as explained in the following discussion.

By using Taylor’s expansion, the value of a function at a point $(x_1, x_2, \ldots, x_n)$ may be written in terms of the value at a point in its neighborhood $(x_1^0, x_2^0, \ldots, x_n^0)$ and the increments $(x_1 - x_1^0, x_2 - x_2^0, \ldots, x_n - x_n^0)$ as shown below:

\[
f(x_1, x_2, \ldots, x_n) = f(x_1^0, x_2^0, \ldots, x_n^0) + \left( \frac{\partial f}{\partial x_1} \right) (x_1 - x_1^0) + \left( \frac{\partial f}{\partial x_2} \right) (x_2 - x_2^0) + \cdots + \left( \frac{\partial f}{\partial x_n} \right) (x_n - x_n^0).
\]

Using Taylor’s expansion, we now express the values of the three scalar components of each of the row matrices ($A_i^*, B_i^*, D_i^*$ and $G_i^*$, $i = 1, 2$) in terms of their values at an initial state and the corresponding increments in the components of the nodal displacements, rotations, stress and moment resultants. Consequently, the system of equations (14) is rewritten to relate the incremental values (of the nodal components of the displacements, rotations, stress and moment resultants) to
the applied loading.

\[
\begin{bmatrix}
A(12 \times 12 \text{ Matrix}) & B(12 \times 12 \text{ Matrix}) \\
C(12 \times 12 \text{ Matrix}) & D(12 \times 12 \text{ Matrix})
\end{bmatrix}
\begin{bmatrix}
\Delta P \\
\Delta V
\end{bmatrix}
= 
\begin{bmatrix}
X \\
Y
\end{bmatrix}
\] (14)

The block matrices \( A, B, C, D \) and \( \Delta P, \Delta V, X \) and \( Y \) are listed as follows.

\[
A = \begin{bmatrix}
\partial B^*_{11} & \partial B^*_{12} & \partial B^*_{13} & \partial B^*_{14} & \partial B^*_{15} & \partial B^*_{16} & \partial B^*_{17} & \partial B^*_{18} & \partial B^*_{19} & \partial B^*_{110} & \partial B^*_{111} \\
\partial T^*_{11} & \partial T^*_{12} & \partial T^*_{13} & \partial T^*_{14} & \partial T^*_{15} & \partial T^*_{16} & \partial T^*_{17} & \partial T^*_{18} & \partial T^*_{19} & \partial T^*_{110} & \partial T^*_{111} \\
\partial B^*_{21} & \partial B^*_{22} & \partial B^*_{23} & \partial B^*_{24} & \partial B^*_{25} & \partial B^*_{26} & \partial B^*_{27} & \partial B^*_{28} & \partial B^*_{29} & \partial B^*_{210} & \partial B^*_{211} \\
\partial T^*_{21} & \partial T^*_{22} & \partial T^*_{23} & \partial T^*_{24} & \partial T^*_{25} & \partial T^*_{26} & \partial T^*_{27} & \partial T^*_{28} & \partial T^*_{29} & \partial T^*_{210} & \partial T^*_{211} \\
\partial B^*_{31} & \partial B^*_{32} & \partial B^*_{33} & \partial B^*_{34} & \partial B^*_{35} & \partial B^*_{36} & \partial B^*_{37} & \partial B^*_{38} & \partial B^*_{39} & \partial B^*_{310} & \partial B^*_{311} \\
\partial T^*_{31} & \partial T^*_{32} & \partial T^*_{33} & \partial T^*_{34} & \partial T^*_{35} & \partial T^*_{36} & \partial T^*_{37} & \partial T^*_{38} & \partial T^*_{39} & \partial T^*_{310} & \partial T^*_{311} \\
\partial B^*_{41} & \partial B^*_{42} & \partial B^*_{43} & \partial B^*_{44} & \partial B^*_{45} & \partial B^*_{46} & \partial B^*_{47} & \partial B^*_{48} & \partial B^*_{49} & \partial B^*_{410} & \partial B^*_{411} \\
\partial T^*_{41} & \partial T^*_{42} & \partial T^*_{43} & \partial T^*_{44} & \partial T^*_{45} & \partial T^*_{46} & \partial T^*_{47} & \partial T^*_{48} & \partial T^*_{49} & \partial T^*_{410} & \partial T^*_{411} \\
\partial D^*_{11} & \partial D^*_{12} & \partial D^*_{13} & \partial D^*_{14} & \partial D^*_{15} & \partial D^*_{16} & \partial D^*_{17} & \partial D^*_{18} & \partial D^*_{19} & \partial D^*_{110} & \partial D^*_{111} \\
\partial T^*_{11} & \partial T^*_{12} & \partial T^*_{13} & \partial T^*_{14} & \partial T^*_{15} & \partial T^*_{16} & \partial T^*_{17} & \partial T^*_{18} & \partial T^*_{19} & \partial T^*_{110} & \partial T^*_{111} \\
\partial D^*_{21} & \partial D^*_{22} & \partial D^*_{23} & \partial D^*_{24} & \partial D^*_{25} & \partial D^*_{26} & \partial D^*_{27} & \partial D^*_{28} & \partial D^*_{29} & \partial D^*_{210} & \partial D^*_{211} \\
\partial T^*_{21} & \partial T^*_{22} & \partial T^*_{23} & \partial T^*_{24} & \partial T^*_{25} & \partial T^*_{26} & \partial T^*_{27} & \partial T^*_{28} & \partial T^*_{29} & \partial T^*_{210} & \partial T^*_{211} \\
\partial D^*_{31} & \partial D^*_{32} & \partial D^*_{33} & \partial D^*_{34} & \partial D^*_{35} & \partial D^*_{36} & \partial D^*_{37} & \partial D^*_{38} & \partial D^*_{39} & \partial D^*_{310} & \partial D^*_{311} \\
\partial T^*_{31} & \partial T^*_{32} & \partial T^*_{33} & \partial T^*_{34} & \partial T^*_{35} & \partial T^*_{36} & \partial T^*_{37} & \partial T^*_{38} & \partial T^*_{39} & \partial T^*_{310} & \partial T^*_{311} \\
\partial D^*_{41} & \partial D^*_{42} & \partial D^*_{43} & \partial D^*_{44} & \partial D^*_{45} & \partial D^*_{46} & \partial D^*_{47} & \partial D^*_{48} & \partial D^*_{49} & \partial D^*_{410} & \partial D^*_{411} \\
\partial T^*_{41} & \partial T^*_{42} & \partial T^*_{43} & \partial T^*_{44} & \partial T^*_{45} & \partial T^*_{46} & \partial T^*_{47} & \partial T^*_{48} & \partial T^*_{49} & \partial T^*_{410} & \partial T^*_{411}
\end{bmatrix}
\]

similarly \( B, C, D \) \((12 \times 12)\) are given by

\[
B = \begin{bmatrix}
\frac{\partial B^*_{11}}{\partial u_1^1} & \ldots & \frac{\partial B^*_{11}}{\partial u_2^1} \\
\ldots & \ldots & \ldots \\
\frac{\partial B^*_{23}}{\partial u_1^1} & \ldots & \frac{\partial B^*_{23}}{\partial u_2^1}
\end{bmatrix}, \quad
C = \begin{bmatrix}
\frac{\partial A^*_{11}}{\partial T_1} & \ldots & \frac{\partial A^*_{11}}{\partial M_2^2} \\
\ldots & \ldots & \ldots \\
\frac{\partial G^*_{23}}{\partial T_1} & \ldots & \frac{\partial G^*_{23}}{\partial M_2^2}
\end{bmatrix}, \quad
D = \begin{bmatrix}
\frac{\partial A^*_{11}}{\partial u_1^1} & \ldots & \frac{\partial A^*_{11}}{\partial u_2^1} \\
\frac{\partial G^*_{23}}{\partial u_1^1} & \ldots & \frac{\partial G^*_{23}}{\partial u_2^1}
\end{bmatrix}
\] (15)
It should be noted that $B_{1}^1$, $B_{1}^2$, $B_{1}^3$, etc. are the scalar components of the row matrix $B_{1}^i$ (1 x 3). The same holds for the ($A_i^1$s, $D_i^1$s and $G_i^1$s, $i = 1, 2, j = 1, 2, 3$).

$$\Delta P = \begin{bmatrix} \Delta T_1^1 \\ \Delta T_1^2 \\ \Delta T_1^3 \\ \Delta M_1^1 \\ \Delta M_1^2 \\ \Delta M_1^3 \end{bmatrix}, \quad \Delta V = \begin{bmatrix} \Delta \omega_1^1 \\ \Delta \omega_1^2 \\ \Delta \omega_1^3 \\ \Delta u_1^1 \\ \Delta u_1^2 \\ \Delta u_1^3 \end{bmatrix}, \quad X = \begin{bmatrix} B_{11}^1 \\ B_{12}^1 \\ B_{13}^1 \\ B_{21}^2 \\ B_{22}^2 \\ B_{23}^2 \\ A_{11}^1 \\ A_{12}^1 \\ A_{13}^1 \\ A_{21}^2 \\ A_{22}^2 \\ A_{23}^2 \end{bmatrix}, \quad Y(q) = \begin{bmatrix} A_{11}^1 \\ A_{12}^1 \\ A_{13}^1 \\ A_{21}^2 \\ A_{22}^2 \\ A_{23}^2 \end{bmatrix}.$$ 

This system of equations is condensed to explicitly relate the incremental displacements to the applied loading through a tangent stiffness matrix [2] (note: block matrix $Y$ is also represented as $Y(q)$ to indicate that it is a weighted function of the end load and the distributed loading):

$$A \Delta P + B \Delta V = X, \quad C \Delta P + D \Delta V = Y.$$ 

In general an incremental procedure as described above is to be followed. However, for certain specific cases, for which the system of equations obtained are uncoupled, the incremental procedure need not be used.

4. Programming details

The equations are highly coupled and non-linear for problems involving large deformation. Hence, the number of gauss points required for numerical integration is increased significantly. Due to the above reasons and the possibility of errors associated with numerical integration and differentiation, the tangent stiffness matrix is evaluated symbolically. Furthermore, a symbolic tangent stiffness matrix is easier to implement in an existing code (e.g. KRASH). In the present effort, the Tangent Stiffness Matrix was evaluated using the symbolic mathematics system, Mathematica.

5. Validation

To test the accuracy and the computational effectiveness of the present formulation, in modelling non-linear problems and problems involving initially curved members, the following problems were solved using the proposed formulation. The problems below were modelled using a single element and the results thus obtained have been compared with those in the literature.
(1) Curved beam (circular arc through 90° shown in Fig. 2). The objective of this example was to verify the effectiveness of the proposed formulation in modelling curved members (large initial curvature) while maintaining computational effectiveness. This problem was modelled using a single curved element using the proposed formulation. As indicated in Fig. 2 the end point displacement thus obtained was within three percent of the analytical solution obtained by Timoshenko [10].

(2) Non-linear analysis of a cantilever beam (shown in Figs. 3(a) and (b)). (a) Subjected to an end load, (b) subjected to an end moment.

In the cantilever analysis subjected to the concentrated tip load, the effect of aspect ratio on the element performance in the non-linear regime was studied. As can be seen in Fig. 3(a) the end point deflections using the proposed formulation (with a single element) compared well with the results obtained by Bathe and Bolourchi [11] using three beam elements and was superior to those obtained using three isoparametric elements. Furthermore, the accuracy of the proposed element was unaffected by the aspect ratio over a wide range (aspect ratios 4–100).

Fig. 3(b) shows the results (end point deflections) obtained in the analysis of a cantilever beam subjected to an end-moment. The results obtained using a single element compared well with those obtained by Bathe and Bolourchi [11] (using 20 beam elements), indicating high computational effectiveness without sacrificing accuracy.

Various authors [12–14] have modelled the problem of the cantilever beam undergoing large deformations quite accurately. However several elements (5 or above) were used. Unlike the
displacement-based approaches, mostly followed in the literature, the proposed mixed variational approach (in complementary form) gives high accuracy with lower differentiability requirements on displacement and fewer elements.

Other examples like the non-linear analysis of a shallow arch subjected to an apex load were also carried out by the authors [2].

6. Practical applications/examples

An illustrative example of the benefits of the proposed approach is that of an S-leg seat. It is a geometry common in aircraft seats due to its enhanced energy absorption capacity.

![Diagram of ISOPARAMETRIC ELEMENT MODEL and BEAM ELEMENT MODEL](image)

Three Beam Element Model by Bathe and Bolourchi (1979)

**Large Defln. of Cantilever Beam**

*Validation of Proposed Methodology*

*End Load Applied*

![Graph showing Tip Deflection Ratio vs. Load Parameter](image)

Note: Aspect Ratio $a/h = 100$

Tip Deflection Ratio: $\frac{w}{L}$

Load Parameter: $\frac{P L^2}{EI}$

Fig. 3.
As has been shown in the validation, a 90° curved beam is very accurately modelled (within 3% error) using the present approach. Four such elements are used to model the S-leg. Any conventional displacement-based approach would require several elements (20 or more) to model this structure. Furthermore, since a symbolic tangent stiffness matrix has been derived, there is no need for extensive remeshing to perform a parametric analysis.

In this effort, the effect of varying the radius of the S-leg on its resilience (energy absorption in linear and non-linear elastic region) is studied. It is illustrative of the kind of parametric analysis that a designer may perform with little or no remeshing effort and substantially reduced computational effort with the present approach.

The load displacement behavior for different radii (3.25, 3.5, 3.75 and 4 in) are studied (Fig. 4(a)). Using this, the resilience characteristics are plotted. It is important to notice the trade-off the designer faces. A high radius, though resulting in a loss in stiffness (very pronounced in the linear region) permits a much larger displacement without the risk of stroking. It is for this reason that both the resilience–displacement and the resilience–displacement ratio \( \frac{u}{r} \) characteristics are studied for different radii. This parametric study is illustrative of the benefits that can be achieved.
from the proposed approach as a consequence of its suitability for modelling initially curved members, large deflections and symbolic processing (no remeshing effort).

Fig. 4(a) seems to indicate that lower radii is considerably superior. Consequently a more detailed analysis was performed with radii 3.25 and 4.0 in (highest and lowest in the given set). The resilience–displacement characteristics (Fig. 4(b)) were definitely superior for the lower radius (3.25 in). However, the resilience–displacement ratio characteristics (Fig. 4(c)), which accounts for the lower allowable displacements for smaller radii, indicates almost equal performance.

In conclusion, designs aimed at improved crashworthiness and optimal performance, require extensive parametric analysis. Consequently, it would substantially benefit the designer, to use computational methodologies that require fewer elements and no involved meshing, like the proposed approach.

\[
\begin{align*}
\text{radius of S-Leg 'r'}
\end{align*}
\]

Practical Application of Proposed Model

Load Deflection Characteristics of S-Leg Seat

S-Leg Modelled With Only Four Elements

Vertical Disp. at apex (in)

\[
\begin{align*}
\text{Square Section} \\
\text{Length } 7/16 \text{ in.} \\
\text{Thickness } 0.083 \text{ in.} \\
E = 10000000 \text{ lb/sq.in}
\end{align*}
\]

(a)

Fig. 4.
Structural Collapse is accounted for through plastic-hinge approach [3]. Distributed plasticity may be accounted for by modifying the relation between stretch-curvature and stress-moment resultants (constitutive relation in Eq. (4)).

7. Conclusion

The developments accomplished in the present course of study are listed below: A new frame finite element method has been derived using a mixed variational principle (complementary form). The formulation is capable of:
(a) handling large rotations and displacements; and the effect of transverse shear,
(b) modelling structural collapse through hinge formation,
(c) modelling members with initial curvature,
(d) running on a high end P.C or equivalent computational resource has been developed. Based on the validation for geometrically non-linear problems and initially curved members, it is expected that only one element will be required per member for modelling crash scenarios. It should be emphasized here that the mixed variational approach (complementary form) has a much lower differentiability requirement on displacements than the displacement based approaches found in literature and the computational effectiveness of the proposed approach is a consequence of this.

The proposed approach can be incorporated in an existing code (example KRASH), by utilizing the tangent stiffness matrices (symbolic form), to achieve a substantial gain in accuracy without sacrificing computational effectiveness.
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References