NOTES AND COMMENTS ON COMPUTATIONAL ELASTO-PLASTICITY: SOME NEW MODELS AND THEIR NUMERICAL IMPLEMENTATION

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ABSTRACT

The following topics are discussed in this paper: (i) the basic interactive nature of classical elasto-plasticity and a redefinition of elastic and plastic processes that facilitates numerical calculations, (ii) generalized mid-point or end-point algorithms to determine the stress increment in an elastic-plastic solid from a given strain increment, (iii) an endochronic (internal time) rate theory of time-independent plasticity which encompasses various multiple-yield-surface theories and nonlinear kinematic hardening theories as its specializations, and (iv) comments on finite element and boundary element methods for solving boundary value problems in elasto-plasticity.

INTRODUCTION

Classical plasticity theory is now about 35-40 years old and is one of the greatest achievements of mechanics of this century. Meaningful computational implementation of elasto-plasticity is about 15-20 years old (see Marciniak and King, 1967; Yamada and co-workers, 1968; Zienkiewicz and co-workers, 1969). It is easy to believe, on first glance, that computational elasto-plasticity is, at present, quite a routine task, and that a multitude of "general purpose" programs which can efficiently solve elasto-plastic problems exist; but experience has shown that both these beliefs may be fallacious (see a comparison of solutions to a round-robin problem of elasto-plastic deformation of a three-point bend bar, Wilson and Caias, 1978).

This paper, which is expository in the main, attempts to point out that much remains to be done in the areas of constitutive modeling and attendant computational (finite element/boundary element) methodologies for capturing observed behavior of metals/metallic structures under a general (non-proportional, cyclic) spectrum of loading. The main themes of this paper are: (i) the need for a basic understanding and reevaluation of loading/unloading criteria that define elastic and plastic processes at a material point in an elastic-plastic body, (ii) rational algorithms to determine the stress rate (and thus the stress history) from a specified
strain-rate (and thus the strain history) at a material point, (iii) non-classical plasticity theories which employ fairly general evolution equations for yield surface expansion and yield surface translation, and (iv) comments on finite element/boundary element methods in elasto-plasticity.

Attention in this paper is focused mainly on infinitesimal deformation problems (see Atluri, 1980 and 1984; Reed and Atluri, 1984 for discussions of finite deformation elasto-plasticity). In Section 2 we briefly review the classical plasticity theory and point out the need for a redefinition of loading/unloading processes suitable for determining the differential stress $\sigma_{ij}$ due to a given differential change of strain $\varepsilon_{ij}$. Computational algorithms, based on these redefined loading/unloading criteria, suitable for use in conjunction with small but finite strain increments, $\varepsilon_{ij}$, are then given. In Section 3, a summary of a recently developed endochronic (internal time) rate theory of plasticity is given. When the kernels in this theory are appropriately chosen, it encompasses all the earlier theories in literature involving multiple-yield-surfaces, nonlinear kinematic hardening, etc., as its specializations. In Section 4, comments are made on finite element/boundary element methods in elasto-plasticity. The iterative processes in finite element algorithms are succintly summarized. Problems associated with the incompressibility nature of deformation at large plastic flow, and the difficulties they pose in the usual displacement-based finite element method, are discussed. A case is made for the possible use of stress-based finite element methods to alleviate the above problems due to plastic incompressibility as well as to obtain better stress estimates in general.

CLASSICAL ELASTO-PLASTICITY

Here we consider, for simplicity, classical deviatoric (volume preserving) metal plasticity. Further, in this section, we restrict ourselves to the case of infinitesimal deformations. Let $\sigma$ be the stress, $\varepsilon$ the strain, $\sigma'$ the stress deviator, $\sigma_{0}$ the mean stress, $\sigma_{\varepsilon}$ the differential elastic strain, $d\sigma$ the differential plastic strain ($d\sigma = d\varepsilon + d\sigma_p$); $d\sigma = d\varepsilon$ (for volume preserving plastic flow). If $A$ and $B$ are second-order tensors, the notation $A:B = A_{ij}B_{ij}$ will be used. The yield function for a general isotropic and/or kinematic hardening plasticity, may be defined as:

$$f(\varepsilon_{ij}, \varepsilon_{ij}, \varepsilon) = 0$$

(2.1)

where $\varepsilon_{ij}$ is the current plastic strain, and $\varepsilon$ is a parameter that may characterize hardening/softening. For associative plasticity, the Drucker postulate leads to:

$$d\sigma_{p} = d\varepsilon \frac{\partial f}{\partial \varepsilon}$$

(2.2)

Henceforth, we define a "unit" normal to the yield surface as the tensor $N$, i.e.,

$$N = \frac{\partial f/\partial \varepsilon}{\partial f/\partial \varepsilon}$$

(2.3)

In the development of classical plasticity (Atluri, 1976), the following definitions of Elastic Process (no change in plastic strain) and Plastic Process (change in plastic strain) arise naturally:

(a) Elastic-Perfectly Plastic Material:

Elastic Process ($E$):

$$f < 0$$

(2.4a)

or (ii) $f = 0$ and $d\sigma = 0$ (2.4b)

Plastic Process ($P$):

$$f = 0$$

(2.5a)

in which case,

$$d\varepsilon = 0; d\sigma = 0$$

(2.5b)

(b) Elastic-Strain Hardening Material:

E: (i) $f < 0$

(2.6a)

or (ii) $f = 0$ and $d\sigma = 0$

(2.6b)

P: $f = 0$; $d\sigma = 0$ (2.6c)

(2.6d)

A comparison of (2.5a,b) with (2.6c,d) reveals that in the classical development of plasticity, the definition of a plastic process in the case of an elastic-perfect-plastic material is not a limiting case of that for a strain-hardening material. Also, from the point of view of solving boundary value problems, say using the finite element displacement method, the generic problem in elasto-plastic stress analysis is to determine the stress rate (and thus the stress history) from the given strain rate (and thus the stress history). Viewed in this light, the essential iterative nature of plasticity solutions is apparent from the definitions of plastic processes in (2.5a,b) and (2.6c,d): these definitions depend on a knowledge of $d\varepsilon$ (and its orientation with respect to $N$), which in fact is the quantity to be solved for! On the other hand, while using a complementary energy method with stresses as the basic unknowns, $d\sigma$, (2.5, 2.6) may be natural in determining $d\varepsilon$ for a specified $d\sigma$.

If the elastic and plastic processes can be redefined such that (i) the new definitions involve $d\varepsilon$ directly, and (ii) the case of perfect plasticity can be unambiguously treated as the limiting case of strain-hardening plasticity, it would be advantageous from a computational implementation point of view while using the finite element displacement method. It is shown (Atluri, 1976), and briefly discussed below, that this can be done easily in the case of elastic-plastic materials that behave isotropically in the elastic range, a situation which is not uncommon in metal plasticity.
(a) Elastic-Perfectly Plastic Material:

Elastic Process (E):

\[ (1) \quad f < 0 \]

\[ (11) \quad f = 0 ; \quad \varepsilon : d\varepsilon < 0 \quad \text{or} \quad \frac{\partial f}{\partial \varepsilon} : d\varepsilon < 0 \quad (2.4b) \]

In this case, \( d\varepsilon = \varepsilon^B : d\varepsilon \) \( (2.7) \)

where \( \varepsilon^B \) is the elasticity tensor. For an elastically isotropic material, Eq. (2.7) may be written:

\[ d\varepsilon = 2\mu d\varepsilon + \nu(\varepsilon: d\varepsilon) I \]

\[ (2.8) \]

where \( I \) is an identity tensor and \( \mu \) and \( \nu \) are Lamé parameters. Since \( (\partial f/\partial \varepsilon) \) is a deviatoric tensor for deviatoric plasticity, we have from (2.4b) and (2.8),

\[ \frac{\partial f}{\partial \varepsilon} : d\varepsilon = (\frac{\partial f}{\partial \varepsilon} : d\varepsilon) I \]

\[ (2.9) \]

Hence, the definition of the elastic process, (2.4b), may be restated as:

\[ (E): (11) \quad f = 0 ; \quad \varepsilon : d\varepsilon < 0 \quad \text{or} \quad \frac{\partial f}{\partial \varepsilon} : d\varepsilon < 0 \quad (2.4c) \]

Plastic Process (P):

\[ (2.5a) \]

Here, \( \varepsilon^P = d\lambda \frac{\partial f}{\partial \varepsilon} \), \( d\lambda > 0 \). Hence,

\[ d\varepsilon = \varepsilon^B : [d\varepsilon - d\lambda \frac{\partial f}{\partial \varepsilon}] \]

\[ (2.10) \]

Hence, (2.5a) implies:

\[ \frac{\partial f}{\partial \varepsilon} : \varepsilon^B : d\varepsilon - d\lambda \frac{\partial f}{\partial \varepsilon} = 0 \]

\[ (2.11) \]

Since \( \frac{\partial f}{\partial \varepsilon} : \varepsilon^B : d\varepsilon - d\lambda \frac{\partial f}{\partial \varepsilon} > 0 \), it follows that

\[ \frac{\partial f}{\partial \varepsilon} : \varepsilon^B : d\varepsilon > 0 \]

\[ (2.12) \]

For an isotropic \( \varepsilon^B \), and for Von Mises type deviatoric plasticity, we have from (2.12) the definition of a plastic process:

\[ (P) \quad f < 0 ; \quad \frac{\partial f}{\partial \varepsilon} : d\varepsilon > 0 \]

\[ (2.14) \]

(b) Elastic-Strain Hardening Plastic Material:

\[ (E): (1) \quad f < 0 \]

\[ (11) \quad f = 0 ; \quad d'f = \frac{\partial f}{\partial \varepsilon} : d\varepsilon \leq 0 \quad (2.6a) \]

As before, (2.6b) implies, for an elastically isotropic material,

\[ (E): (11) \quad f = 0 ; \quad \frac{\partial f}{\partial \varepsilon} : d\varepsilon \leq 0 \]

\[ (2.6b) \]

\[ (P): \quad f = 0 ; \quad d'f = \frac{\partial f}{\partial \varepsilon} : d\varepsilon > 0 \]

\[ (2.6c) \]

For an elastically isotropic solid, and deviatoric plasticity, (2.6c) implies as above,

\[ (P): \quad f = 0 ; \quad \frac{\partial f}{\partial \varepsilon} : d\varepsilon > 0 \]

\[ (2.6c) \]

Hence, in summary, for elastic-perfectly plastic or elastic-workhardening materials which are elastically isotropic, a unified set of loading/unloading conditions may be stated as:

\[ (E): (1) \quad f < 0 \]

\[ (11) \quad f = 0 ; \quad \frac{\partial f}{\partial \varepsilon} : d\varepsilon \leq 0 \quad \text{or} \quad \varepsilon : d\varepsilon \leq 0 \]

\[ (2.13a) \]

\[ (P): \quad f = 0 ; \quad \frac{\partial f}{\partial \varepsilon} : d\varepsilon > 0 \quad \text{or} \quad \varepsilon : d\varepsilon > 0 \]

\[ (2.13b) \]

Note that the normal \( \varepsilon \) in (2.13) is implied to have been evaluated at the current state of stress.

Consider, as an example, a case of combined isotropic/kinematic hardening, i.e. the case when the yield surface both expands and translates. Let the yield surface be considered, for simplicity, to be a Von Mises circular cylinder in the stress space, as:

\[ (q' - q') \cdot (q' - q') = R^2 = \frac{2}{3} \varepsilon^B \]

\[ (2.14) \]

where \( q' \) is the deviator of the back stress (the location of the center of the yield surface), \( R \) is the radius of the yield cylinder (which changes with the equivalent plastic strain), and \( \varepsilon \) is the yield stress in uniaxial tension which also depends on \( \varepsilon^B \). For simplicity, we shall consider the evolution equations:

\[ \text{Fed} \text{a Vol. J} \]
where \( K \) and \( C \) are constants. Now, during a plastic process,

\[
f' = 0; \quad df' = \frac{\partial f}{\partial \mathbf{g}} : d\mathbf{g} \geq 0 \quad \text{and} \quad dg^p = g \left( \frac{\partial f}{\partial \mathbf{g}} \right) : d\mathbf{g}.
\]

Due to nonzero plastic flow, the yield surface expands and translates as specified by (2.15) and (2.16), respectively. The consistency condition then states that:

\[
[(g' + dg') - (g + dg')][(g' + dg') - (g + dg')] = (R + dR)^2
\]

The use of (2.15 and 2.16) in (2.17) results in:

\[
dg^p = \frac{1}{R^2(4C + \frac{8}{3}K)} \frac{\partial f}{\partial \mathbf{g}} \left( \frac{\partial f}{\partial \mathbf{g}} : d\mathbf{g} \right)
\]

Hence, for a material that is elastically isotropic, during a plastic process:

\[
(P): \quad dg^p = 2\mu[2\chi'] - \frac{1}{R^2(4C + \frac{8}{3}K)} \frac{\partial f}{\partial \mathbf{g}} \left( \frac{\partial f}{\partial \mathbf{g}} : d\mathbf{g} \right)
\]

By taking the trace of both sides with \( \frac{\partial f}{\partial \mathbf{g}} \) and noting that \( (\frac{\partial f}{\partial \mathbf{g}} : d\mathbf{g} = (\frac{\partial f}{\partial \mathbf{g}} : d\mathbf{g}) \),

\[
dg^p : (g - g) = \frac{2\mu}{1 + (\frac{6\mu}{4C + \frac{8}{3}K})} [dg^p : (g - g)]
\]

Using (2.20) in (2.18), one obtains:

\[
dg^p = \frac{8\mu N (N dg)}{(4C + 8\mu + \frac{8K}{3})} \quad \text{when} \ N \rightarrow 0
\]

\[
= \frac{A N}{N} \quad \text{when} \ C, K \geq 0
\]

where \( N \) is the unit normal to the yield surface, and a plastic process occurs, clearly, when \( N \rightarrow 0 \). Also, from (2.19) it follows that:

\[
\text{Plastic Process (P)}:
\]

\[
dg^p = 2\mu \left( \frac{8\mu}{(8\mu + 4C + \frac{8K}{3})} (N \cdot dg) N \right)
\]
(E) (i) If \( f_n < 0 \)
Then \( d_\varepsilon = \frac{E}{D} \varepsilon_\varepsilon \) \hspace{1cm} (2.22f)

If \( f_{n+1} \leq 0 \) the entire process has been elastic. On the other hand, if \( f_{n+1} > 0 \), then find the factor \( m(<1) \) such that \( m \varepsilon_\varepsilon \) will bring the new stress state to the yield surface. Thus, the strain \( (1-m) \varepsilon_\varepsilon \) will now correspond to a plastic process.

(k) (ii) If \( f_n = 0 \); \( \varepsilon_\varepsilon < 0 \) \hspace{1cm} (2.22g)
Then \( d_\varepsilon = \frac{E}{D} \varepsilon_\varepsilon \) \hspace{1cm} (2.22h)

The plasticity algorithm based on (2.22a-h) is exact and easily implementable in the context of differential changes in strain, \( \varepsilon_\varepsilon \). However, in practical computational mechanics, one seeks an algorithm for incremental changes in plastic strain, \( \Delta \varepsilon \). Note that in Eq. (2.22), pertaining to a differential change in strain, all considerations are based on the normal \( N \) to the yield surface prior to the considered change, \( \varepsilon_\varepsilon \). Thus, in an approximate algorithm pertaining to an incremental change \( \Delta \varepsilon \), one may base considerations of elastic and plastic processes, analogous to those in (2.22), on the normal \( N \) to the yield surface at any intermediate point during the considered change \( \Delta \varepsilon \) or at the end of the strain increment \( \Delta \varepsilon \). The simplest algorithm will result when all these considerations are based on the normal \( N^* \) at the end of the strain increment \( \Delta \varepsilon \). We present the details for this algorithm below for a strain-hardening material, while that for a strain-perfect-elastic material follows by simple modifications.

(a) First consider a plastic process, and assume that \( g_n \) is on \( f_n = 0 \).

(i) Let the increment of strain be \( \Delta \varepsilon \) with deviatoric part, \( \Delta \varepsilon' \).

(ii) Let the "guess" for increment in deviatoric stress be:
\[ \Delta \sigma' = 2\mu \Delta \varepsilon' \]

(iii) Let the "guesses" for total Cauchy and back strain deviators, respectively, be:
\[ E_{\varepsilon_{n+1}} = \varepsilon'_{n+1} + 2\mu \Delta \varepsilon' \]
\[ E_{\varepsilon_n} = \varepsilon'_n + 2\mu \Delta \varepsilon' \]

(iv) Check if \( (E_{\varepsilon_{n+1}} - \varepsilon'_n);(E_{\varepsilon_n} - \varepsilon'_n) = E_{\varepsilon_{n+1}} < E_n \)
If \( E_{\varepsilon_{n+1}} \leq E_n \), the process has been elastic.

If \( E_{\varepsilon_{n+1}} > E_n \), the process has been plastic. (Note that this condition automatically implies that \( N \Delta \varepsilon > 0 \), where \( N \) is the normal at the beginning of the increment \( \Delta \varepsilon \).)

(v) If the process has been plastic, find the normal \( N^* \) to the yield surface at the end of the plastic process during which the strain has increased by \( \Delta \varepsilon \).

From Fig. 2, it may be seen that \( N^* \Delta \varepsilon = E_{\varepsilon_{n+1}} - E_n > 0 \)
(vii) The correct increment in stress in a plastic process, then, is:
\[ \Delta \sigma' = 2\mu (\Delta \varepsilon' - \Delta \varepsilon N^*) \]
\[ \Delta \sigma: I = (3\theta + 2\mu) (\Delta \varepsilon: I) \]
\[ g_{n+1} = g'_n + \frac{2}{3} \mu N^* \]
\[ g'_{n+1} = g'_n + C N^* \]

The above algorithm is illustrated in Fig. 2.
\[
N_\alpha = \frac{(c_1 + 2\mu\Delta e') - c_0}{\| (c_1 + 2\mu\Delta e') - c_0 \|} \\
\Delta e^P = \frac{8\mu}{(8\mu + 4\nu + 8K/3)} (N_\alpha^* - \Delta e') N_\alpha^* \\
\Delta e' = 2\mu[\Delta e' - \lambda \frac{\Delta \epsilon}{\delta}] \\
\Delta e' = (3\mu + 2\nu) (\Delta \epsilon/2) \\
R_{N+1} = R_{N} + \frac{2}{3} K A_{\alpha}^* \quad \Delta e_{N+1} = \Delta e_{N} + (\Delta e' / \Delta \epsilon) A_{\alpha}^* 
\]

While the algorithm in Fig. 2 pertains to \( \beta = 1 \), one may use \( \beta = \frac{1}{2} \) to lead to a "mid-point" algorithm.

Also, if \( \Delta e \) had been too large, the above algorithms may be implemented in \( M \) steps, wherein the strain-increment in each step is \( (\Delta e/M) \).

The above discussions are related to the basic concepts involved in a computational implementation (i.e., involving small but finite increments as opposed to differential changes) of any plasticity theory involving a yield surface and its translation and expansion.

The concept of constitutive modeling in rate-independent plasticity involves theoretical models for evolution laws which specify the yield surface translation and expansion such that experimentally observed phenomena in metal plasticity can be modeled as accurately as desired. Melan's, especially under cyclic loading, display phenomena such as: (i) strain-hardening; (ii) the Bauschinger effect (i.e., after a certain amount of plastic straining, the compressive elastic limit may be much different from that in tension); (iii) cyclic hardening and cross-hardening; and (iv) ratcheting, etc. A multitude of theories, all of which may be classified as theories involving internal variables, have been proposed to account for various observed phenomena of cyclic and nonproportional multiaxial loading. These include the "multiple-yield-surface" theories (Mroz, 1967, 1969; Dafalias and Popov, 1975, 1976; Krieg, 1975), the general internal variable theories (1981), the nonlinear kinematic hardening theories (Chaboche and Boussinier, 1982), and the endochronic (internal time) theory (Valanis, 1980). Recently, Watanabe and Atluri (1984a) derived a rate form of the endochronic theory, using the concepts of Valanis (1980). It has been shown that the rate relations of the endochronic theory (Watanabe and Atluri, 1984a) are entirely similar in structure to that of the earlier sketched classical plasticity theory and are easy to implement, thus removing the "mystery" surrounding the "endochronic" theory. The problem of modeling experimental data for cyclic plasticity and creep has been addressed in detail in (Watanabe and Atluri, 1984a). Later, Watanabe and Atluri (1984b) presented certain unifying concepts underlying the endochronic theory, general internal variable theories, and multiple-yield-surface theories of plasticity and creep. It has been shown (Watanabe and Atluri, 1984b) that by an appropriate choice of kernels in the endochronic theory (Watanabe and Atluri, 1984a; Valanis, 1980), the nonlinear kinematic hardening models of Chaboche, Mroz, and co-workers; the multiple-yield-surface theories of Mroz, Krieg, Dafalias, Popov, and co-workers; and the linear kinematic hardening models of Prager, etc. may be deduced as special cases of the rate form of endochronic theory.
where \[ p(o) = p(z = 0) \quad \text{and} \quad b(z) = \int_{0}^{z} \frac{\partial \sigma}{\partial z'} (z - z') \frac{dg'}{dz'} \text{dz'} \] (3.5a, b)

\[ 2\mu_0 = 2\mu_0 |1 + p(o)/\lambda| \quad \text{and} \quad dg = dg' - \frac{dg'}{2\mu} \] (3.5c, d)

Equations (3.3) and (3.4a) appear to obviate the need for the introduction of a "yield-surface" as in a classical plasticity theory. However, the differential deviatoric stress \( dg' \) as in (3.4b, c), depends nonlinearly on \( dg \) in contrast to the usual linear relation of a classical plasticity theory.

It can be shown (Watanabe and Atluri, 1984a, b) that, through appropriate choices for the kernel \( p(z) \) in (3.3), one may: (a) deduce a classical plasticity theory involving a yield surface, (b) deduce a linear relation between \( dg' \) and \( dp \) as in a classical plasticity theory, and (c) show that all the plasticity theories in prior literature, such as the multiple-yield-surface theories of Mroz, Dafalias and Popov; nonlinear kinematic hardening theories of Chaboche and co-workers; linear hardening theories of Prager and co-workers, may be considered as special cases of the present endochronic rate theory. Towards this objective, first assume that the kernel \( p(z) \) is of the form:

\[ p(z) = p_0 \delta(z) + p_1(z) \] (3.6)

where \( \delta(z) \) is a Dirac delta function at \( z = 0 \); and \( p_1(z) \) is a "smooth" function. The presence of \( p(z) \) in (3.6) leads, as shown by Watanabe and Atluri (1984a) to the classical notion of a yield surface. That this is so may be seen by using (3.6) in (3.3), which results in:

\[ \frac{dg}{dz} = \frac{d\sigma}{d\zeta} + g'(z) \] (3.7a)

\[ g'(z) = 2\mu_0 \int_{0}^{z} p_1(z - z') \frac{dg}{dz'} \text{dz'} \] (3.7b)

Thus, when plastic flow occurs, it is seen from (3.7a) that:

\[ \frac{dg}{dz} = \frac{(g' - g')}{(2\mu_0 p_0)} f(z) \] (3.8)

Eq. (3.8) is entirely analogous to the Von Mises classical plasticity theory with normality, i.e., \( d\sigma = -\lambda \nabla \), where \( \nabla = 3f/\delta g = (g' - g') \). Thus, \( g' \) in (3.7b) may be viewed as specifying the center of the yield surface; and in that sense, the kernel \( p_1(z) \) specifies the way in which the yield surface translates. Likewise, the function \( f(z) \) which enters into the definition of \( A \) clearly specifies the way in which the yield surface expands.

In the present theory (Watanabe and Atluri, 1984a), the loading and unloading conditions may be characterized as:

\[ \begin{align*}
(1): & \quad \|g' - g'\| < 2\mu_0 p_0 f(z) \\
(11): & \quad \|g' - g'\| < 2\mu_0 p_0 f(z) \quad \text{and} \quad \nabla g' = 0 \\
(2): & \quad \|g' - g'\| = 2\mu_0 p_0 f(z) \quad \text{and} \quad (g' - g') \cdot \delta g = 0 \\
(3): & \quad \|g' - g'\| > 0 \quad \text{and} \quad (g' - g') \cdot \delta g > 0
\end{align*} \] (3.9a, 3.9b, 3.9c, 3.9d)

An important attribute of the present theory, as seen from (3.9), is that the definitions of elastic and plastic processes involve, ab initio, the strain rate \( \delta g \), rather than \( dg \) as in classical theory (see Eqs. 2.5 and 2.6).

The specific choice of the kernel \( p_1(z) \) which renders the present endochronic theory general enough, so as to encompass the multiple-yield-surface theories of Mroz, Dafalias, and Popov; nonlinear kinematic hardening theories of Chaboche and co-workers; linear hardening theories of Prager and co-workers, may be considered as special cases of the present theory, is:

\[ p_1(z) = \int_{-\infty}^{\infty} p_{11} \exp(-a_1 z) \] (3.10)

With the above choices of kernels \( p(z) \), the rate form of endochronic theory as derived by Watanabe and Atluri (1984a, b) may be summarized as in Table 1.

In contrast, the other theories of plasticity may be summarized as in Table 2, from which it may be observed (see Watanabe and Atluri 1984b for a comprehensive discussion) that all the theories in literature may be viewed as special cases of the present theory summarized in Table 1.

Explicit expression of the present rate form of endochronic elastic-plastic stress/strain relations (Table 1) for the cases of plane-stress, plane-strain, and three dimensions, were given in Watanabe and Atluri (1984c).

The computational implementation of the present endochronic rate-theory (Table 1) for small but finite strain increments \( \delta \zeta \) is entirely analogous to that sketched in Fig. 2 (and the detailed algorithms presented in connection thereof) in Section 2 of the present paper. Examples of cyclically loaded cracked plates (Watanabe and Atluri, 1984c) and plates subject to nonproportional biaxial cyclic loading (Watanabe and Atluri 1985a) point to the superior constitutive modeling capability of the present endochronic rate theory in situations of cyclic plasticity.
Endochronic Theory: \([\cdot]^{(\cdot)}\) denotes a derivative of \((\cdot)\) with respect to Newtonian-time or a Newtonian-time-like external parameter such as external load.

\[
\text{tr}(\xi) = (2\mu + 3\lambda)\text{tr}(\varepsilon); \quad \text{where } \mu, \lambda \text{ are Lamé constants}
\]

\[
\dot{\varepsilon}' = 2\mu \varepsilon' - \frac{2\mu}{C(S_y^O)^2} \langle (\varepsilon' - \varepsilon')^2 \rangle \varepsilon' \Gamma; \quad \Gamma = 0 \text{ in } (E) \text{ and } \Gamma = 1 \text{ in } (P)
\]

\[
f(\xi) = (1 + 8\xi) (\text{linear}) \text{ or } f(\xi) = [a + (1 - a)\exp(-\gamma\xi)](\text{expon})
\]

\[
C = 1 + \rho_1(\alpha) = \frac{(\varepsilon' - \varepsilon')Y^*}{S_y^O}; \quad S_y^O = 2\mu_0\rho_0
\]

\[
p(z) = 2\mu_0\delta(z) + \rho_1(z) = \rho_0(\xi) + \sum_1^\infty \rho_1\exp(-\alpha_1z)
\]

\[
\dot{g}' = \xi^* g'(1); \quad \dot{\eta} = \xi^* \eta^* (1) = \sum_1^\infty (-\frac{\alpha_1}{2\mu_0}) g'(1)
\]

Kinematic Hardening:

Isotropic Hardening:

\[
\dot{\varepsilon}_y = S_y^O g'(\xi) \dot{\xi}^2 \quad \text{(linear } f) \quad \text{or} \quad 2\mu_0 D_1^2
\]

\[
\dot{\varepsilon}' = \xi^* \dot{g}'(1) = \xi^* \dot{g}'(1) = \frac{\alpha_1 g'(1)}{\mu_0 (\xi^* \xi^*)^2} \]

\[
\dot{\varepsilon}_y = \gamma(S_y^O - 2\gamma [a + (1 - a)\exp(-\gamma\xi)])(\xi^* \xi^*)^2 \quad \text{(saturated } f)
\]

Classification of Theories of Plasticity:

Classical Theories of Plasticity: \([\cdot]^{(\cdot)}\) denotes a derivative of \((\cdot)\) with respect to a Newtonian-time-like parameter.

\[
\text{(i) Prandtl-Reuss Isotropic Hardening}
\]

\[
\text{tr}(\xi) = (2\mu + 3\lambda)\text{tr}(\varepsilon);
\]

\[
\dot{\varepsilon}' = 2\mu \varepsilon' - \frac{3\mu}{C(S_y^P)^2} \langle (\varepsilon' - \varepsilon')^2 \rangle \varepsilon' \dot{\varepsilon}^P; \quad \dot{\varepsilon}^P = \text{eq. plastic strain}
\]

\[
\varepsilon^* = 2\mu \varepsilon^* - \frac{2\mu}{C(S_y^P)^2} \langle (\varepsilon^* - \varepsilon^*) \varepsilon^* \varepsilon^* \rangle \varepsilon^* \Gamma; \quad \Gamma = 0 \text{ in } (E) \text{ and } \Gamma = 1 \text{ in } (P)
\]

\[
\dot{\varepsilon}' = \xi^* \dot{g}'(1) = \xi^* \dot{g}'(1) = \frac{\alpha_1 g'(1)}{\mu_0 (\xi^* \xi^*)^2}
\]

\[
\dot{\varepsilon}_y = C(\xi) \varepsilon_y^* - d(\xi) \varepsilon_y^* \xi^* \]

\[
\dot{\varepsilon}' = \xi^* \dot{g}'(1) = \xi^* \dot{g}'(1) = \frac{\alpha_1 g'(1)}{\mu_0 (\xi^* \xi^*)^2}
\]

\[
\dot{\varepsilon}_y = b(Q - S_y)(\xi) \varepsilon_y^* \xi^* \quad \text{or} \; S_y = Q(1 - \exp(-b\xi)) \quad (b, Q \text{ are constants})
\]

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\]
restrict our attention here to infinitesimal deformations. For a
preliminary account of finite deformation problems in elasto-plasticity,
refer to the earlier work (Aturi, 1980; Murakawa and Aturi,
1984a; Reed and Aturi, 1983 a,b, 1984; etc.)

The finite element displacement method for elasto-plasticity, the trial
and test function spaces are identical. In the boundary-element method,
trial and test function spaces are distinctly different: the trial
functions for total displacement increments may be simple polynomials,
le test functions are chosen to be the singular (kevin) solutions for a
load in the linear elastic isotropic continuum. In the following,
integrating with the theme of this conference, we discuss only the finite
element method. For a discussion of the boundary element method
for elasto-plasticity (and for deformations are also considered), see
uri (1984 b). Let \( \Delta u \) be the trial functions for incremental
placements that satisfy the boundary conditions at \( s_0 \). Then the trial
placements for incremental strains are: \( \Delta e_{i,j} = \mu (\Delta u_{i,j} + \Delta u_{j,i})/2 \). For an
anisotropically elastic-plastic solid, the trial stresses are:

\[
\sigma_{i,j} = 2\mu \Delta e_{i,j} + \lambda \Delta e_{k,k} - 2\mu \Delta e_{k}(N)_{i,j} (4.1)
\]

where \( \mu \) is a constant (see Section 2); \( N \) is the unit normal to the yield
face prior to the imposition of strain increment \( \Delta s \); and, as discussed
below, \( \Gamma = 0 \) if (i) \( f_r < 0 \) and (ii) \( f_s = 0 \), and \( \Gamma = 0 \); and \( \Gamma = 1 \)
if \( f_r = 0 \) and \( \Gamma = 0 \). As stated in Section 2, one does not know a priori
the solution process, if \( \Gamma = 0 \) or. Thus, a finite element algorithm
must be classified as (a) initial strain/initial stress methods based on
linear elastic matrixes [i.e. assume \( \Gamma = 0 \) to start with] and (b) tangent
stress methods [i.e. assume \( \Gamma = 1 \) to start with, when \( f_r = 0 \)].

static equilibrium equations and boundary conditions are:

\[
\begin{align*}
\Delta e_{i,j} &= -f_i - (q_{i,j}) \quad (4.2) \\
\Delta e_{i,j} &= (N+1)_{i,j} - q_{i,j} \quad (4.3)
\end{align*}
\]

Here, \( (N+1)f_r \) and \( (N+1)f_s \) are body forces and surface tractions,
respectively, at the end of the increment, and \( q_{i,j} \) are stresses at the
beginning of the increment. The combined weak form of Eqs. (4.2) and (4.3)
easily be written as:

\[
\int \Delta e_{i,j} \Delta t_{i,j} \, dv = \int \Delta s_{i,j} \Delta t_{i,j} \, dv + \int \Delta s_{i,j} \Delta t_{i,j} \, dv = \int \Delta e_{i,j} \Delta t_{i,j} \, dv
\]

where \( \Delta e_{i,j} \) and \( \Delta t_{i,j} \) are the test functions (belonging to the same function space as trial
strains \( u_{i,j} \)).

Method 1: Assume \( f_s = 0 \) throughout at time of the current increment.
Eq. (4.4) may be written as:

\[
\int \Delta e_{i,j} \Delta t_{i,j} \, dv = \int \Delta s_{i,j} \Delta t_{i,j} \, dv
\]
We restrict our attention here to infinitesimal deformations. For a comprehensive account of finite deformation problems in elasto-plasticity, we refer the reader to our earlier work (Atluri, 1980; Murakawa and Atluri, 1979; Atluri, 1983a; Reed and Atluri, 1983a,b, 1984; etc.).

In the finite element displacement method for elasto-plasticity, the trial and test function spaces are identical. In the boundary element method, the trial and test function spaces are distinctly different: the trial functions for total displacement increments may be simple polynomials, while test functions are chosen to be the singular (Kelvin) solutions for a point load in the linear elastic isotropic continuum. In the following, in keeping with the theme of this conference, we discuss only the finite element method. For a discussion of the boundary element method for elasto-plasticity (where finite deformations are also considered), see Atluri (1984b). Let $\Delta u_i$ be the trial functions for incremental displacements that satisfy the boundary conditions at $S_n$. Then the trial functions for incremental strains are: $\Delta e_{ij} = \frac{1}{2} (\Delta u_i, u_j + \Delta u_j, u_i)$. For an elastically isotropic elastic-plastic solid, the trial stresses are:

$$\sigma_{ij} = 2\mu \Delta e_{ij} + \lambda \Delta e_{kk} - 2\mu \Delta e_{ij}(N_i \Delta e_{ij})$$  \hspace{1cm} (4.1)

where $\lambda$ is a constant (see Section 2); $N$ is the unit normal to the yield surface prior to the imposition of strain increment $\Delta e$; and, as discussed in Section 2, $\Gamma = 0$ if (1) $f_n < 0$ and (11) $f_n = 0$, and $\Gamma = 1$ if $f_n = 0$ and $N_i \Delta e_{ij} < 0$. As stated in Section 2, one does not know $\lambda$ a priori in the solution process, if $\Gamma = 0$ or 1. Thus, a finite element algorithm may proceed by assuming either (a) $\Gamma = 0$ or (b) $\Gamma = 1$. These methods may thus be classified as (a) initial strain/stress methods based on linear elastic matrices [i.e. assume $\Gamma = 0$ to start with] and (b) tangent stiffness methods [i.e. assume $\Gamma = 1$ to start with, when $f_n > 0$].

The static equilibrium equations and boundary conditions are:

$$\Delta e_{ij} = - f_i$$  \hspace{1cm} (4.2)

$$\Delta e_{ij} = (N+1) f_i - \sigma_{ij}$$  \hspace{1cm} (4.3)

where $(N+1) f_i$ and $(N+1) f_i$ are body forces and surface tractions, respectively, at the end of the increment; and $\sigma_{ij}$ are stresses at the beginning of the increment. The combined weak form of Eqs. (4.2) and (4.3) may easily be written as:

$$\int_V (\Delta e_{ij} \Delta e_{ij}) \, dv = \int_V (N+1) f_i \Delta e_{ij} \, dv + \int_S (N+1) f_i \Delta e_{ij} \, ds$$

where $\Delta e_{ij}$ are test functions (belonging to the same function space as trial functions $u_i$).

Method 1: Assume $\Gamma = 0$ throughout at beginning of the current increment. Then Eq. (4.4) may be written as:

$$\int_V K^{e}(1) \Delta e_{ij} \Delta e_{ij} \, dv = \int_V (N+1) f_i \Delta e_{ij} \, dv$$

The finite element counterpart of (4.5) is:

$$K^{e} \Delta e = f_{N+1} - B_{N}$$

where $K^{e}$ is the linear elastic stiffness, and $B_{N}$ are internal forces at the beginning of the increment. From (4.6), calculate $\Delta e$; then go to the constitutive algorithm as sketched in Fig. 2 (and the discussion thereof) to find $(1)^{\Delta e}$ (i.e., the first approximation for $\Delta e$) and thus $(1)^{\Delta e + 1}$. The internal forces corresponding to $(1)^{\Delta e + 1}$ will be labelled $(1)^{\Delta e + 1} B_{N+1}$. The iterative algorithm may then follow from:

$$K^{e} (1)^{\Delta e} = f_{N+1} - (1)^{\Delta e + 1} B_{N+1}$$

where the superscript (1) denotes the 1st iteration.

Method 2: Assume $\Gamma = 1$ if $f_n = 0$; and $\Gamma = 0$ if $f_n < 0$. Then Eq. (4.4) may be written as:

$$\int_V (\Delta e_{ij} \Delta e_{ij}) \, dv = \int_V (N+1) f_i \Delta e_{ij} \, dv + \int_S (N+1) f_i \Delta e_{ij} \, ds$$

The finite element counterpart of (4.8) is:

$$K^{t} \Delta e = f_{N+1} - B_{N}$$

From (4.9) find $\Delta e$; go to the constitutive algorithm of Fig. 2 to find $(1)^{\Delta e}$ and thus $(1)^{\Delta e + 1}$. The iterative algorithm to find $(1)^{\Delta e}$ may now be written:

$$K^{t} (1)^{\Delta e} = f_{N+1} - (1)^{\Delta e + 1} B_{N+1}$$

(Clearly, $(1)^{\Delta e + 1} B_{N+1} = B_{N}$)

The above two methods are commonly known as initial strain and tangent stiffness compatible displacement finite element methods, respectively. Other methods of finite element elasto-plasticity, based on complementary energy approaches and mixed variational principles are often times much more efficient than the usual displacement base approaches. For a discussion of these alternate approaches, the so-called hybrid/mixed approaches, see Atluri (1980) and Atluri, Gallagher, and Zienkiewicz (1983).

When significantly large plastic flow occurs, since metal plasticity is essentially deviatoric, the total deformation of the solid is essentially incompressible. This incompressibility plagues the numerical solutions for plane strain and three-dimensional elasto-plasticity. In the usual displacement finite element method, the above incompressibility poses a
severe constraint. On the other hand, the use of stress-based finite element approaches based on complementary energy alleviate these difficulties posed by large incompressible plastic flow (see Atulri, 1980). Moreover, since the constitutive algorithm, namely the determination of $\sigma$ from $\gamma$ depends on the current stress state $(\sigma - \sigma_0)$, it may be seen that there are additional reasons as to why stress-based finite element algorithms, that essentially lead to better approximations for stress states, are to be preferred. It is to be expected, thus, that such future research work will be aimed at assumed stress type finite element approaches in elasto-plasticity and inelastic solid response (see, for instance, Reed and Atluri, 1983a,b).

ACKNOWLEDGEMENTS

These notes and comments arose as a result of research studies supported by the National Aeronautics and Space Administration, Lewis Research Center, under grant NAS-3-346 and Office of Naval Research under contact N0014-73-C-0826. These supports, and the encouragement of Drs. C.C. Chamis and Y. Rajapakse are thankfully acknowledged. It is a pleasure to thank Ms. Joyce Webb for her assistance in the preparation of this paper.

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