Nonlinear Vibrations of a Hinged Beam Including Nonlinear Inertia Effects

This investigation treats the large amplitude transverse vibration of a hinged beam with no axial restraints and which has arbitrary initial conditions of motion. Nonlinear elasticity terms arising from moderately large curvatures, and nonlinear inertia terms arising from longitudinal and rotary inertia of the beam are included in the nonlinear equation of motion. Using a Galerkin variational method and a modal expansion, the problem is reduced to a system of coupled nonlinear ordinary differential equations which are solved for arbitrary initial conditions, using the perturbation procedure of multiple-time scales. The general response and frequency-amplitude relations are derived theoretically. Comparison with previously published results is made.

Introduction

The author’s interest in the present problem arose out of an earlier investigation of the nonlinear dynamic response of simple-supported cylindrical shells. In the author’s analysis [1, 2], where the simply supported conditions on the normal displacement were satisfied exactly, and the in-plane conditions of axially immovable hinges were satisfied on an average, the resulting nonlinear modulated equations contained only nonlinear elasticity terms and the nonlinearity was found to be always of the hardening type for arbitrary initial conditions. In a parallel analysis however, employing only two modes, Evenson [3] satisfied the continuity conditions for the circumferential in-plane displacement exactly by assuming nonlinear terms in the modal expansion for the normal displacement; these nonlinear terms, however, disobey the boundary conditions on the normal displacement, and further the shell was assumed to be free to move in the axial direction. The resulting modulated equations of Evenson [3] contained both nonlinear elasticity terms and nonlinear inertia terms, the reason for presence of the latter being traceable to the presence of nonlinear terms in the assumed mode shape. It should be pointed out that in both the author’s [1, 2] and Evenson’s [3] analysis, the effect of longitudinal inertia forces had been neglected. Evenson’s [3] results indicated that the nonlinearity can either be hardening or softening type depending on the ratio of circumferential and axial wavelengths of vibrational mode; thus pointing to the importance, as discussed by the author [1], of the interplay between the nonlinear elastic forces and nonlinear inertia forces. Thus it is of interest to consider the interaction of major nonlinearities (elastic and inertial) in a vibration problem in a systematic way.

The nonlinearities in a beam vibration can arise due to the following:

1. Moderately large curvatures.
2. Longitudinal elastic forces generated due to longitudinally immovable supports.
3. Longitudinal inertia forces.
4. Rotary inertia forces.

The first two can be classified as elasticity nonlinearities and the latter two as the inertia nonlinearities. Much of the earlier work (Eringen [4], Winowsky-Krieger [5], MacDonald [6], Burgreen [7], Wahn [8], Easley [9], Bennett and Easley [10], Tseng and Dugundji [11, 12], and Srinivasan [13]) on the nonlinear transverse vibration of beams has been concerned with simply supported or clamped beams whose ends are restrained from axial displacement. In the foregoing works, curvatures were restricted to be small; the effects of longitudinal and rotary inertia, and that of transverse shear, have been neglected; and the only nonlinearity considered is that due to the average longitudinal force.
generated due to the average midplane stretching strain induced when the supports are held a constant distance apart. The resulting equations, involving only nonlinear elasticity terms, have been solved; by MacDonald [6] for arbitrary initial conditions using an expansion of Jacobi elliptic functions of time; by Tseng and Dugundji [11, 12] using the method of harmonic balance to study the forced response in a single mode; by Bennett and Eisley [10] who studied the forced response in a three mode case; and by Srivinasaan [13] who used the Ritz technique in both space and time variables to study the steady-state forced response in two modes.

In the present analysis, the beam is considered as simply supported and one end of the beam (x = l) is assumed to be free to move in the axial direction. Thus, in contrast to the earlier mentioned works, in the present paper, the effects of large curvature, longitudinal inertia, and rotary inertia are included, while the effects of midplane stretching (obtained if the supports are fixed a constant distance apart), as well as transverse shear deformation, are ignored. The resulting partial-differential equation for the only independent variable w (transverse displacement) is reduced to a system of N (arbitrary) coupled nonlinear ordinary differential equations, which are solved for arbitrary initial conditions by using the perturbation method of multiple time scales. Analytical results are obtained for the general nonlinear response and amplitude-frequency relations. An example and comparison of results are presented.

### Problem Formulation

The beam of length l is considered to be pinned at one end (x = 0) and pinned-sliding at the other end (x = l) where z is the length measured along the deformed beam. Referring to the Nomenclature, the dynamic equations of equilibrium of the beam can be written as

$$\frac{\partial M}{\partial x} = Q + \rho \frac{\partial^2 w}{\partial t^2}$$  

(1)

$$+ \frac{\partial N}{\partial x} = -\rho \frac{\partial^2 w}{\partial x^2}$$  

(2)

$$\frac{\partial}{\partial x} (N \varphi) + \frac{\partial Q}{\partial x} + \frac{\partial \varphi}{\partial t} = 0$$  

(3)

Since the beam is assumed to be free to move axially, and the midplane stretching strains can be considered approximately to be zero, it has already been shown by Novozhilov [14], that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} \left[ 1 - \left( \frac{\partial w}{\partial x} \right)^2 \right]^{1/2} = \frac{\partial^2 w}{\partial x^2} \left[ 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right]$$  

(4)

and that the resulting axial displacement due to large transverse displacement is given by

$$u(x) = -\frac{1}{2} \int_0^x \left( \frac{\partial w}{\partial x} \right)^2 \, dx$$  

(5)

Substituting equation (5) into equation (2) and with use of the impulse conditions u(0) = 0 and N(l) = 0, one can integrate equation (2) from x to l, to obtain

$$N(x, t) = -\frac{1}{2} \int_0^x \left( \frac{\partial w}{\partial x} \right)^2 \, dx$$  

(6)

Also, for the beam-bending theory neglecting transverse shear deformation,

$$M(x, t) = E I \frac{\partial^2 w}{\partial x^2}$$  

(7)

by virtue of equation (4).

Equations (1) and (3) are combined by elimination of Q to obtain

$$\frac{\partial}{\partial x} (N \varphi) + \frac{\partial^3 w}{\partial x^2 \partial t} - \rho \frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi}{\partial t} = 0$$  

(8)

Using equations (4) and (7) into equation (8), one obtains the following equation for w only, which contains all the principal nonlinear terms up to third order, for the present problem:

$$\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^4 w}{\partial x^4} + 3 \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial t^2}$$  

$$- \rho \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial t^2} - 2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + 4 \frac{\partial^4 w}{\partial x^4} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + 2 \frac{\partial^4 w}{\partial x^4} \frac{\partial^2 w}{\partial t^2}$$  

$$+ \frac{\partial}{\partial x} \left( N \frac{\partial w}{\partial x} \right) + \frac{\partial \varphi}{\partial t} = 0$$  

(9)

where N is given by equation (6).

Using the nondimensional variables,

$$\tilde{w} = w / h, \quad \tilde{x} = x / l, \quad \tilde{\varphi} = \varphi / h, \quad \tau = t \left( \frac{EI}{h^4} \right)^{1/4}, \quad N = \frac{N}{EI h^4}$$  

Equation (9) can be transformed as

$$\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} + \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} + \frac{1}{2} \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} + 2 \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^2 \partial \tau} + \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} \frac{\partial^2 \tilde{w}}{\partial \tau^2}$$  

$$- \frac{\rho \partial^2 \tilde{w}}{\partial \tilde{x}^2} \frac{\partial^2 \tilde{w}}{\partial \tau^2} - \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^2 \partial \tau^2} + 4 \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} + 2 \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^2 \partial \tau^2} + 2 \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} \frac{\partial^2 \tilde{w}}{\partial \tau^2}$$  

$$+ \frac{\partial}{\partial \tilde{x}} \left( N \frac{\partial \tilde{w}}{\partial \tilde{x}} \right) + \frac{\partial \tilde{\varphi}}{\partial \tau} = 0$$  

(10)

where

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>w</td>
<td>transverse displacement of beam</td>
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<tr>
<td>h</td>
<td>thickness of beam</td>
</tr>
<tr>
<td>L</td>
<td>length of beam</td>
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<tr>
<td>ρ</td>
<td>mass/unit length</td>
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<td>E</td>
<td>Young’s modulus</td>
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<tr>
<td>N</td>
<td>axial force positive in compression</td>
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<tr>
<td>M</td>
<td>bending moment along beam</td>
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<tr>
<td>Q</td>
<td>transverse shear resultant</td>
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<tr>
<td>b_0</td>
<td>radius of gyration of a section of rod about the axis through the center of gravity perpendicular to plane of motion</td>
</tr>
<tr>
<td>I</td>
<td>moment of inertia of beam cross section</td>
</tr>
<tr>
<td>ϕ</td>
<td>rotation of beam</td>
</tr>
<tr>
<td>c_{ij}, d_{ij}</td>
<td>coefficients in Galerkin system of equations, listed in the Appendix</td>
</tr>
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N = nondimensional modal amplitude
ε = small parameter that characterizes the amplitude conditions
qi = component of q̃i, multiplied by ε^n
δ = h/L
τ = (EI/h^4)^{1/4}, nondimensional time

Transactions of the ASME
\[ N = - \int_{\xi}^{1} \left[ \frac{\partial^2 \psi}{\partial \xi^2} \right] \psi \, d\psi. \]

A modal expansion for \( \bar{\psi} \) can be assumed as
\[
\bar{\psi} = \sum_{i=1}^{m} q_i \sin k\pi \xi
\]
which satisfies the simply supported boundary conditions,
\[
\bar{\psi} = \frac{\partial \bar{\psi}}{\partial \xi^2} = 0 \text{ at } \xi = 0 \text{ and } \xi = 1
\]
Substituting equation (11) into equation (10), one obtains a single nonlinear ordinary differential equation for the variables \( q_i(k = 1, m) \). From this equation, using the Galerkin technique, and weighing in turn each of the functions \( \sin k\pi \xi (i = 1, m) \), and integrating over the length of the beam, a system of \( m \) ordinary coupled nonlinear differential equations in time can be obtained thus
\[
d\frac{dq_i}{dt^2} + \Omega_i \frac{dq_i}{dt} + \sum \sum b_{kjk} \frac{dq_k}{dt} - \sum \sum c_{ijk} \frac{dq_j}{dt} = 0
\]
Explicit values of the coefficients \( \Omega_i, b_{kjk}, c_{ijk}, d_{ixi} \) for the modal expansion given in equation (11), are given in the Appendix in terms of the nondimensional parameters already defined. It can be shown that equation (13) is correct to the order \( \frac{\partial^2 \bar{\psi}}{\partial \xi^2} \) in the principal nonlinear terms, because of the inherent assumptions involved in equations (4), (5), and (7). The first group of nonlinear terms appearing in equation (13) are static in origin and can be classified as “nonlinear elasticity” terms and, the second and third group of nonlinear terms can be classified as “nonlinear inertia” terms since they are dynamic in origin.

Solution of Problem
The system of equation (13), corresponding to the free transverse vibration of the beam, can now be solved for any given initial conditions of the type, \(^{2}\)
\[
q_i = e^{i\omega_0} \xi; \quad \frac{\partial q_i}{\partial \tau} = e^{i\omega_0} \xi \text{ at } \tau = 0
\]
where \( \xi \) is an arbitrary small parameter that defines the amplitudes of the initial conditions. Correspondingly, one can define for the solution of equation (13), that
\[
q_i(\tau) = e^{i\omega_0 \tau}
\]
Using equation (15), equation (13) can be reduced to
\[
d\frac{dq_i}{dt^2} + \Omega_i \frac{dq_i}{dt} + \sum \sum b_{kjk} \frac{dq_k}{dt} = 0
\]
Equations of the previous type can be solved elegantly using the perturbation method of multiple-time scales which is discussed for single-degree-of-freedom systems by Kevorkian [15] and Nayfeh [16].

One can define multiple-time scales \( \tau_b, \tau_1, \ldots, \tau_m \) accordingly as

\[
\tau_m = (e)^{m\tau}
\]
One can now assume that there exists a uniformly valid asymptotic solution for \( \bar{q}_i \) of the form
\[
\bar{q}_i = \sum_{m=0}^{M} e^{m\tau_0} (\tau, \tau_1, \ldots, \tau_M) + O(e^{M+1})
\]
where \( \tau_0 \) are now functions of the independent time scales \( \tau_b, \tau_1, \ldots, \tau_M \) etc. From equation (17) it can be seen that
\[
\frac{d}{d\tau} = \sum_{m=0}^{M} e^{m\tau_0} \frac{\partial}{\partial \tau_0} \quad \text{and} \quad \frac{d}{d\tau} = \sum_{m=0}^{M} e^{m\tau_0} \frac{\partial}{\partial \tau_0}
\]
Substituting equations (18) and (19) into equation (16), and identifying terms multiplied by equal powers of \( \epsilon \), one can obtain the following system of equations:

Terms with \( \epsilon^0 \)
\[
\frac{\partial \bar{q}_i}{\partial \tau_0} + \Omega_i \bar{q}_i = 0
\]

Terms with \( \epsilon^1 \)
\[
\frac{\partial^2 \bar{q}_i}{\partial \tau_0^2} + \Omega_i \frac{\partial \bar{q}_i}{\partial \tau_0} = -2 \Omega_i \frac{\partial \bar{q}_i}{\partial \tau_0} - \sum \sum b_{kjk} \frac{\partial \bar{q}_j}{\partial \tau_0} - \sum \sum c_{ijk} \frac{\partial \bar{q}_k}{\partial \tau_0} + \Omega_i \frac{\partial \bar{q}_i}{\partial \tau_0}
\]
Similar boundary conditions, equation (14), can likewise be transformed as
\[
\bar{q}_0 = \alpha_i; \quad \frac{\partial \bar{q}_0}{\partial \tau_0} = \beta_i \text{ for } \tau_0 = 0
\]
\[
\bar{q}_0 = \bar{q}_i = 0 \quad \frac{\partial \bar{q}_0}{\partial \tau_0} + \frac{\partial \bar{q}_0}{\partial \tau_0} = 0 \text{ for } \tau_0 = 0
\]

Solution for Zeroth-Order System
Equation (20) has the simple solution,
\[
\bar{q}_0 = A_i (\tau, \tau, \ldots, \lambda \tau, \lambda \tau^2, \ldots) e^{-\lambda \tau, \tau^2, \ldots} \]
where \( \lambda = \sqrt{-1}, A_i \) is a complex quantity that is a function of the time scales \( \tau, \tau^2, \ldots, \) etc; and \( A_i^* \) is the complex conjugate of \( A_i \). \( A_i \) and \( A_i^* \) can be determined from the initial conditions in equation (22).

Solution for First-Order System
After substituting for \( \bar{q}_0 \) from equation (24), the first-order system, equation (21) can now be written as
\[
\frac{\partial \bar{q}_i}{\partial \tau_0} + \Omega_i \frac{\partial \bar{q}_i}{\partial \tau_0} = -2 \Omega_i \frac{\partial A_i}{\partial \tau_0} e^{\lambda \tau, \tau_0^2} + 2 \Omega_i \frac{\partial A_i}{\partial \tau_0} e^{-\lambda \tau, \tau_0^2} - \sum \sum b_{kjk} \frac{\partial \bar{q}_j}{\partial \tau_0} - \sum \sum c_{ijk} \frac{\partial \bar{q}_k}{\partial \tau_0} + \Omega_i \frac{\partial \bar{q}_i}{\partial \tau_0}
\]
In solving equation (25), the terms on the right-hand side which vary with frequency \( \Omega \) must be suppressed; otherwise, these would lead to spurious resonance in the solution and hence destroy the uniformity as assumed in equation (17). Thus, group-
ing terms on the right-hand side which vary with frequency $\Omega$, equation (25) can be written as

$$
\frac{\partial^2 \Omega_{ij}}{\partial \tau^2} + \Omega_{ij} \frac{\partial \Omega_{ij}}{\partial \tau} = 
- \left(2\Omega_i \frac{\partial A_i}{\partial \tau} + A_i \sum_l \gamma_l A_l A_i^* \right) e^{\lambda \Omega_{ij}} 
+ \left(2\lambda \Omega_i \frac{\partial A_i}{\partial \tau} + A_i \sum_l \gamma_l A_l A_i^* \right) e^{-\lambda \Omega_{ij}} 
+ \sum_{l \neq r} P_{e} e^{\lambda \Omega_{ij}} + \sum_{l \neq r} P_{e} e^{-\lambda \Omega_{ij}} 
$$

(26)

In the foregoing, $\Omega_{ij}$ stands for the combinations

$$
\Omega_{ij} = \pm \Omega_i \pm \Omega_j \pm \Omega_k 
$$

such that $\Omega_{ij} = \pm \Omega_j$. The explicit expressions for $P_{e}$ are lengthy and hence are not recorded here. After some manipulation, the coefficients $\gamma_l$ associated with terms varying with frequency $\Omega_i$ in equation (26) can be shown to be

$$
\gamma_i = (3b_{iii} + c_{iii} \Omega_i^2 - 3d_{iii} \Omega_i^2) \quad \text{for} \quad l = i 
$$

$$
-2(b_{iii} + b_{idi} + b_{idi} - \Omega_i^2 (d_{iii} + d_{idi} + 2a_{iii} \Omega_i^2)) \quad \text{for} \quad l \neq i 
$$

(28)

Thus spurious resonance in the solution for $\Omega_{ij}$ can be avoided by setting

$$
2\Omega_i \frac{\partial A_i}{\partial \tau} + A_i \sum_l \gamma_l A_l A_i^* = 0 
$$

(29)

The system of equation (29) can easily be solved by letting

$$
A_i = \psi_i e^{\lambda \tau}, 
$$

(30)

where $\psi_i$ is a real quantity. Substituting equation (30) into equation (29) and separating real and imaginary parts, one obtains

$$
\frac{\partial \psi_i}{\partial \tau} = 0 \quad \text{and} \quad \frac{\partial \theta_i}{\partial \tau} = \frac{1}{2\Omega_i} \sum_l \gamma_l \psi_i \psi_l 
$$

(31)

From the first of equation (31), $\psi_i$ is independent of $\tau_i$ and hence we can write

$$
\psi_i = \psi_i(\tau_i, \tau_1 \ldots \tau_n) 
$$

(32)

and

$$
\theta_i = \frac{1}{2\Omega_i} \left( \sum_{l=0}^{N} \gamma_l \psi_l^2 \right) \tau_i + \theta_i \Omega_i (\tau_i \ldots \tau_m) 
$$

(33)

Using equations (32) and (33), equation (30) can be written as

$$
A_i(\tau_i, \tau_1 \ldots \tau_n) = \tilde{A}_i \exp \left[ \frac{1}{2\Omega_i} \sum_{l=0}^{N} \gamma_l \tilde{A}_l \tilde{A}_i^* \right] \tau_i \tau_i 
$$

(34)

$$
\tilde{A}_t(\tau_i, \tau_1 \ldots \tau_m) = \tilde{A}_t(\tau_i, \tau_1 \ldots \tau_m) \exp \left[ \lambda \theta_i (\tau_i \ldots \tau_m) \right] 
$$

(35)

Finally, one can now solve equation (26) as

$$
\dot{\theta}_{ij} = B_i(\tau_i \ldots \tau_m) e^{\lambda \Omega_{ii}} + B_i^* e^{-\lambda \Omega_{ii}} 
+ \sum_{r \neq i} \frac{P_{e} e^{\lambda \Omega_{ij}} + D_{e} e^{-\lambda \Omega_{ij}}}{\Omega_{ji}^2 - \Omega_{ii}^2} 
$$

(36)

where $B_i$ and $B_i^*$ are determined from the initial conditions,

$$
\dot{\theta}_{ij} = 0; \quad \frac{\partial \theta_{ij}}{\partial \tau} = \frac{\partial \theta_{ij}}{\partial \tau} \quad \text{for} \quad \tau_m = 0 
$$

(37)

Using equations (15), (17), (18), (24), (34), and (36), the solution for $\theta_i(\tau)$, correct to the order $\eps^{1/4}$ can be written as

$$
\dot{\theta}(\tau) = \eps^{1/4} \left[ \tilde{A}_i \tilde{A}_i^* e^{-\lambda \Omega_{ii}} + \frac{\gamma_i}{\Omega_{ii}^2} \right] 
+ \eps^{1/4} \left[ B_i e^{\lambda \Omega_{ii}} + B_i^* e^{-\lambda \Omega_{ii}} + \sum_{r \neq i} \frac{P_{e} e^{\lambda \Omega_{ij}} + D_{e} e^{-\lambda \Omega_{ij}}}{\Omega_{ji}^2 - \Omega_{ii}^2} \right] 
+ 0(\eps^{1/4}) 
$$

(38)

where $\hat{\theta}(\tau_i, \tau_1 \ldots \tau_m)$ and $B_i(\tau_i, \ldots \tau_m)$ are determined from initial conditions (22), (23), and (37); and where

$$
\tilde{\Omega}_i = \Omega_i \left[ 1 + \frac{\eps}{2\Omega_i} \sum_{k=1}^{N} \gamma_k \hat{A}_k \hat{A}_i^* \right] 
$$

(39)

Equation (38) indicates that, because of the nonlinear coupling between the modes, even though the initial conditions correspond only to a single mode, other modes would also be excited. Further, it can be seen that if the principal response of the excited mode is of order $0(\eps^{1/4})$, the other modes are excited to order $0(\eps^{1/2})$. Equation (39) on the other hand, indicates the effect of coupling between modes on the frequencies of natural oscillation of each mode. The nature of nonlinearity (whether hard spring or soft spring) depends on whether the quantity $\sum \gamma_i \hat{A}_i \hat{A}_i^*$ is positive or negative. From equation (28), it can be seen that $\gamma_i$ depends on the nonlinear coefficients $b_{ii}$ (due to elasticity), and $c_{ii}$ and $d_{ii}$ (due to inertia). Therefore, the sign of $\gamma_i$ depends on the relative magnitudes of the nonlinear elasticity and the nonlinear inertia coefficients, respectively, thus pointing to the interplay between the different types of nonlinearities. From the Appendix, it can be seen that each of these coefficients depends on the nondimensional properties of the beam as well as the mode numbers under consideration. Specific examples to illustrate this are discussed in the following.

Example and Discussion

Since the coefficients of nonlinearity, as given in the Appendix, depend on the slenderness ratio $(k/L)$, and the mode numbers, the examples given here are chosen to study the effect of these parameters on the nonlinear dynamic behavior of the beam. A beam of rectangular cross section with constant thickness is considered.

First, consider the beam excited in the first mode with the initial conditions,

$$
\psi(\xi, \tau) = 0; \quad \frac{\partial \psi}{\partial \tau} = \eps^{1/2} \Omega_i \sin \pi \xi \quad \text{at} \quad \tau = 0 
$$

(40)

where $\Omega_i$ is the linear natural frequency of the first mode. The corresponding initial conditions for the generalized coordinates are

$$
\dot{q}_i = 0 (i = 1, 2, \ldots) \quad \frac{\partial \theta_i}{\partial \tau} = \eps^{1/2} \Omega_i \beta_i \quad \frac{\partial \delta_i}{\partial \tau} = 0 (i = 2, 3, \ldots) 
$$

(41)

From equations (24) and (34), the zeroth-order solution, using initial conditions (41), can be written

$$
\ddot{q}_i = \beta_i \sin \Omega_i \tau \quad \text{and} \quad \dot{q}_i = 0 \quad k > 1 
$$

(42)

where

$$
\Omega_i = \Omega_i \left[ 1 + \frac{\eps^2 \beta_i}{2\Omega_i^2} \right] 
$$

(43)

From the Appendix and equation (28), it can be seen that

$$
\Omega_i = \pi \sqrt{\left[ 1 + \pi \left( \frac{k \Omega_i^2}{L^4} \right) \right]} 
$$

(44)

and

$$
\gamma_i = 3b_{ii} + c_{ii} \Omega_i^2 - 3d_{ii} \Omega_i^2 
$$

(45)

The coefficient $\gamma_i$ can be computed numerically from the values of coefficients $b_{ii}$, etc., given in the Appendix, but for purposes of discussion it has been evaluated analytically, as

Transactions of the ASME
\[ \gamma_1 = \frac{\delta^4}{8\Omega_2^2} \left[ \frac{3\pi^2}{64} - \frac{\pi^4}{8(1 + \delta \pi^2)} \left( 1 + \frac{3}{8\pi^2} \right) - \frac{3\delta \pi^4}{64(1 + \delta \pi^2)} \right] \] (46)

From equation (46), it is evident that the nature of nonlinearity is governed by the sign of the right-hand side of equation (46) and the magnitude of nonlinearity is roughly proportional to \( \delta^4 \). The ratio of nonlinear frequency to linear frequency as given by equations (45) and (46) is plotted in Fig. 1, which shows that the nonlinearity is of the softening type (frequency decreases with amplitude). This result is in contrast to the purely hardening type of nonlinearity predicted, for simply supported beams with axially immovable hinges, by Winowsky-Krieger [5] and Srinivasan [13], who considered only the nonlinearity arising from the tension developed in the beam due to immovable hinges. Also, the first-order solution \( \tilde{\varphi}_m \), as given by equation (36), and initial conditions (37), can be found as

\[ \tilde{\varphi}_m = \frac{\varepsilon \pi}{4\Omega_1} - \frac{3}{4(\Omega_2^2 - 9\Omega_1^2)} \Omega_1 \left[ b_{1111} - \Omega_1^2 \Omega_2 + \Omega_1 \Omega_4 \varepsilon \sin \Omega_1 \tau \right] \sin \Omega_1 \tau + \frac{1}{4(\Omega_2^2 - 9\Omega_1^2)} \left[ b_{11} - \Omega_1 \varepsilon \sin \Omega_1 \tau \right] \sin \Omega_1 \tau \] (47a)

\[ \tilde{\varphi}_m = \frac{\varepsilon \pi}{4\Omega_1} - \frac{3}{4(\Omega_2^2 - 9\Omega_1^2)} \Omega_1 \left[ b_{1111} - \Omega_1^2 \Omega_2 + \Omega_1 \Omega_4 \varepsilon \sin \Omega_1 \tau \right] \sin \Omega_1 \tau + \frac{1}{4(\Omega_2^2 - 9\Omega_1^2)} \left[ b_{11} - \Omega_1 \varepsilon \sin \Omega_1 \tau \right] \sin \Omega_1 \tau + \left[ b_{1111} - \Omega_1^2 \Omega_2 + \Omega_1 \Omega_4 \varepsilon \sin \Omega_1 \tau \right] \sin \Omega_1 \tau \] (47b)

Thus, from equations (38), (42), (47a), and (47b), the response of the beam subjected to initial conditions (46) is given by

\[ \varphi(\xi, \tau) = e^{\frac{\varepsilon}{2}} \beta \sin \pi \xi \sin \Omega_1 \tau \]

\[ + e^{\frac{\varepsilon}{2}} \sum_{i=1}^{M} q_i \sin i\pi \xi + 0(e^{\frac{\varepsilon}{2}}) \] (48)

Thus it can be seen from equation (47) that corresponding to an initial excitation in the first mode of order \( e^{\frac{\varepsilon}{2}} \), the response consists of the first mode with amplitude \( 0(e^{\frac{\varepsilon}{2}}) \) and frequency \( \Omega_1 \) modified by the nonlinear effects as given in equation (45); the other modes are excited with amplitudes of order \( 0(e^{\frac{\varepsilon}{2}}) \) because of the third-order nonlinear terms present in the equation of motion and equations (47a, b) also indicate the presence of superharmonics of order 3 in the response.

As a second example, consider the beam excited initially in the third mode. Following the procedure as just indicated, the first-order solution can be written as

\[ \tilde{q}_{30} = \beta \sin \Omega_3 \tau \quad \text{and} \quad \tilde{q}_{30} = 0 \quad i \neq 3 \] (49)

where

\[ \Omega_3 = \Omega_1 \left[ 1 + \frac{3}{8\pi^2} \right] \] (50)

and

\[ \gamma_3 = 3b_{1111} - \Omega_1^2 \Omega_2 + \Omega_1 \Omega_4 \varepsilon \sin \Omega_1 \tau \] (51)

It can be shown from the Appendix,

\[ \gamma_3 = \frac{\delta^4}{8\Omega_2^2} \left[ \frac{3(3\pi)^4}{64} - \frac{(3\pi)^4}{8(1 + \delta \pi^2)} \right] \sin \Omega_1 \tau \] (52)

The ratio of \( \Omega_3/\Omega_1 \) is plotted in Fig. 2 for \( \delta = 0.01 \), which shows that the nonlinearity is of softening type, and a comparison with Fig. 1 reveals that the degree of softening is much more pronounced than in the first mode case, for the same value of \( \delta \).

The degree of softening in the third mode also varies roughly as \( \delta^4 \) as can be seen from equation (53).

However, it should be pointed out that the simplified beam equation (1) of the engineering beam theory is valid when the ratio

\[ \frac{1}{\text{Wavelength}} \ll 1 \]
(Non-dimensional amplitude) \( \times (\hat{v}) \times (i) \ll 1 \)

where \( i \) is the mode number.

**Summary and Conclusions**

In contrast to most of the work on large amplitude beam vibration, wherein the only non-linearity considered was the effect of average midplane stretching strain (due to large \( \psi \)), induced when the supports are held a constant distance apart, in the present analysis the effects of large curvature, longitudinal inertia, and rotary inertia are included while the effect of midplane stretching is excluded, and one end of the beam is assumed to be free to move longitudinally. Calculated results for the present problem show that the predominant non-linearity is that due to nonlinear longitudinal inertia which is of softening type. This is in contrast to the earlier analyses where a hardening non-linearity was predicted when the only non-linearities considered were as mentioned in the foregoing. Thus, in practical structures where the end conditions are in between axially movable and axially immovable conditions, the interplay between the non-linear elastic forces and nonlinear inertia forces is then of importance.

Finally, it should be mentioned that the perturbation method given for the multiple mode response is general and is applicable to other vibrating systems.

**References**


**APPENDIX**

The coefficients in the Galerkin system of equations (13) are given here in terms of the non-dimensional properties of the beam, defined already in the text.

\[
\Omega = \frac{(\pi \alpha)^4}{(1 + (\pi \alpha)\kappa)}
\]

where

\[
b_{ijkl} = \frac{\delta_{ijkl}}{[1 + (\pi \alpha)\kappa]}
\]

\[
c_{ijkl} = \frac{2\kappa}{[1 + (\pi \alpha)\kappa]}
\]

\[
d_{ijkl} = \frac{2\kappa}{[1 + (\pi \alpha)\kappa]}
\]

where

\[
f(\xi) = \cos (\pi \alpha \xi) \int_0^1 \cos (\kappa \eta) \cos \theta d \eta
\]

\[
+ \pi \sin (\pi \alpha \xi) \int_0^1 \sin (\kappa \eta) \cos \theta d \eta
\]

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