On Some New General and Complementary Energy Theorems for the Rate Problems in Finite Strain, Classical Elastoplasticity

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ABSTRACT

General variational theorems for the rate problem of classical elastoplasticity at finite strains, in both Updated Lagrangian (UL) and Total Lagrangian (TL) rate forms, and in terms of alternate measures of stress-rate and conjugate strain-rates, are critically studied from the point of view of their application. Attention is primarily focused on the derivation of consistent complementary energy rate principles which could form the basis of consistent and rational assumed stress-type finite element methods, and two such principles, in both UL and TL forms, are newly stated. Systematic procedures to exploit these new principles in the context of a finite element method are discussed. Also discussed are certain general modified variational theorems which permit an accurate numerical treatment of near incompressible behavior at large plastic strains.
I. INTRODUCTION

The advent of high speed-digital computers and powerful numerical methods, such as the finite element methods, in the past two decades or so have greatly expanded the scope of application of nonlinear theories of solid continua to practical problems in engineering. In the formulation of such numerical methods as the finite element methods for problems of nonlinear solid mechanics, variational theorems (and their generalizations to account for discontinuities at interelement boundaries in a finite element assembly) have played a central role. (See, for instance, Refs. 2, 3, 28, and 43 for discussions of finite element formulations in nonlinear elasticity.)

Rigorous and consistent formulations for numerical analysis of large strain elastoplastic problems have become necessary due to the increased interest in recent years in analyzing problems such as metal-forming processes, ductile fracture initiation, and stable crack growth in cracked bodies. Indeed, several such formulations and their applications have appeared in the recent literature. Among these can be cited the works of Hibbit et al. [9] who use a Total Lagrangian (TL) rate formulation (wherein a fixed reference frame is used); Needleman [26], Needleman and Tvergaard [27], and Hutchinson [14] who also use a TL rate formulation but with convected coordinates; Yamada et al. [44] who use an Updated Lagrangian (UL) rate formulation (wherein the current configuration is used as a reference for the subsequent step); Osias [33] who also uses an UL scheme which, due to the use of an elastic-plastic rate constitutive law that does not admit to a potential, leads to nonsymmetric stiffness matrices through a Galerkin scheme; McMeeking and Rice [18] who also use a UL scheme which, through the use of a rate constitutive law with a potential, leads to symmetric stiffnesses; and Nemat-Nasser and Taya [29] whose formulation represents a modification of that of McMeeking and Rice [18] to improve the accuracy in the case of large deformation of compressible materials. All the above cited works employ a classical rate-independent elastoplastic theory as generalized by Hill [10]. It should also be noted that all the above finite element rate formulations are based on the principle of virtual work in rate form as first stated by Hill [10], and are therefore based on assumed displacements that are compatible at interelement boundaries.

To the best of the author's knowledge, no studies concerning the convergence of the assumed displacement finite element methods of the above cited type for the rate problems of classical elastoplasticity exist in the literature. Even in the somewhat simpler problem of finite elasticity, studies of convergence of finite element methods based on potential energy principles are just beginning to emerge [30]. From this standpoint as well as from that of possibly studying solution bounds, it is of interest to consider consistent formulations of numerical (finite element) methods based on complementary energy principles for the rate problem of finite strain elastoplasticity.

Another important question in numerical schemes for elastic-plastic flow is how to deal with the effectively incompressible behavior at large strain. It is well known that numerical schemes based directly on the principle of virtual work fail in the limit of incompressibility unless the mean stress is introduced as an additional variable in the formulation. Such formulations, which are essentially variations of the well-known Hellinger [7]-Reissner [36] theorem, were introduced for nearly or precisely incompressible linear elastic materials by Herrmann [8], Taylor et al. [39], and Key [15]. To improve the numerical accuracy in the near-incompressible case, Nagtegaal et al. [24, Appendix 2] modified their UL rate potential energy formulation for elastoplasticity in a way analogous to that of Key [15], except that, instead of the mean pressure as in Key, they used the dilatational strain rate as an independent variable. As a consequence, even though the formulation of Key is valid for both nearly and precisely incompressible cases, the formulation of Nagtegaal et al. ceases to be valid in the case of precise incompressibility.

On the other hand, the inherent nature of the complementary energy principle (with assumed stresses as variables) makes it much easier to treat situations of near or precise incompressibility when finite element schemes based on a complementary energy principle are used. The works of Tong [38] and Pian and Lee [34] in solving problems of linear elastic near-incompressible solids, and that of Murakawa and Atluri [22] in solving finite strain problems of incompressible nonlinear elastic solids (all of which are based on appropriate complementary energy principles), tend to support the above view. Moreover, as is well known, better solutions for stresses are obtained from numerical schemes based on complementary energy principles than from those based on potential energy principles (wherein the stress solution is obtained by differentiation of the numerical solution for displacements, which results in a loss of accuracy for stresses). Finally, as noted by Masur and Popelar [17], the complementary energy approach holds considerable promise for applications to buckling problems wherein an approximation to the stress state before buckling is often possible even when displacements remain unknown.

For the above reasons, special emphasis is placed in the present paper on the study of the existence of consistent complementary energy rate theorems for the rate problem of finite strain classical, rate-independent, elastoplasticity. Both types of formulations, viz., the Total Lagrangian as well as the Updated Lagrangian, are considered. In this process two new consistent rate complementary energy principles for the rate problem of finite strain classical elastoplasticity are found. Systematic procedures to exploit
these complementary rate principles in the context of assumed stress finite element methods are also discussed. In addition, a critical evaluation of the relative effectiveness of general rate principles, in both TL and UL rate forms and in terms of alternate stress-rate and conjugate strain-rate measures for application to numerical analysis of finite strain elastoplasticity problems, is made. Also included in the present paper are certain general modified variational theorems which are of significance in the numerical treatment of near incompressible behavior at large magnitudes of plastic strain.

II. PRELIMINARIES

For simplicity we refer all configurations of the body to a fixed rectangular Cartesian coordinate system. We adopt the notation: a boldface symbol denotes a vector; a boldface italic symbol denotes a second-order tensor; a \( A \cdot B \) implies that \( A \) is a matrix; \( A: B \) denotes the product of two tensors such that \((A \cdot B)_{ij} = A_{ik}B_{kj}\); \((A: B) = \text{trace} (A^T \cdot B) = A_{ij}B_{ij}\); and \( u \cdot t = u_i t_i \).

A particle in the undeformed body has a position vector \( x = (x_1 e_1) \) where \( e \) are unit Cartesian bases. We adopt the gradient notation \( \nabla x = (e_\alpha \partial / \partial x_\alpha) \) in the undeformed configuration \( C_0 \). The position vector of the particle in the current deformed state, say \( C_N \), is \( y = (y_1 e_1) \). We also use the notation that \( \nabla y = (e_\alpha \partial / \partial y_\alpha) \). The gradient of \( y \) is the tensor \( F \), i.e., \( F = (\nabla y)^T \) \((F_{i\alpha} = y_{i,\alpha} = \partial y_i / \partial x_\alpha)\). We also note that the base vectors of the convected coordinates \( y_\alpha \) in the current deformed configuration \( C_N \) are given by \( e_\alpha = y_{\alpha i} e_i \).

The nonsingular tensor \( F \) is considered to have the polar decomposition \( F = \alpha \cdot (I + h) \), where \( (I + h) \) is a symmetric positive definite tensor called the stretch tensor, \( I \) is the identity tensor, and \( \alpha \) is an orthogonal rotation tensor such that \( \alpha^T = \alpha^{-1} \). The deformation tensor \( G \) is defined by \( G = F^T \cdot F \equiv (I + h)^2 \). The Green-Lagrange strain tensor is defined by \( \varepsilon = \frac{1}{2}(G - I) = \frac{1}{2}(e + e^T + e^T \cdot e) \) where \( e \) is the gradient of the displacement vector \( u \equiv (y - x) \), i.e., \( e = (\nabla y)^T \) such that \( e_{\alpha i} = u_{\alpha i} \).

As shown, for instance, by Truesdell and Noll [41] and Fraeijs de Veubeke [6], the linear momentum balance (LMB) equation, the angular momentum balance (AMB) equation, the traction boundary condition (TBC), and the displacement boundary condition (DBC) are as follows.

**LMB:**

\[
\nabla y \cdot \tau + \rho^N B = 0
\]

\[
\nabla y \cdot (s \cdot F^T) + \rho^0 B = 0
\]

\[
\nabla y \cdot t + \rho^0 B = 0
\]

where \( B \) is the body force per unit mass, and \( \rho^N \) and \( \rho^0 \) are the mass densities per unit volume in \( C_N \) and \( C_0 \), respectively.

**AMB:**

\[
\tau = \tau^T
\]

\[
s = s^T
\]

\[
F \cdot t = t^T \cdot F^T, \quad (h + I) \cdot t \cdot \alpha = \text{symmetric}
\]

**TBC:**

\[
i = n \cdot t = n \cdot (s \cdot F^T)
\]

Finally we introduce a stress measure which is labeled by Fraeijs de Veubeke [6] as the Jaumann stress, \( r \), through the relation,

\[
r = \frac{1}{2}(t \cdot \alpha + \alpha^T \cdot t^T)
\]

\[
r = \frac{1}{2}[s \cdot (I + h) + (I + h) \cdot s]
\]

The stress tensors \( \tau, \alpha, s, \) and \( r \) are symmetric, while \( t \) is unsymmetric.
where \( \mathbf{n} \) is a unit normal to the surface \( S_{o0} \) in configuration \( C_0 \) with prescribed tractions \( \mathbf{t} \) per unit area.

DBC:

\[
\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad S_{o0} \tag{12c}
\]

where \( S_{o0} \) is the surface in \( C_0 \) on which displacements are prescribed to be \( \mathbf{u}_0 \). It is also possible to have a mixed-mixed problem wherein, at a point on the surface of the body in \( C_0 \), certain components of tractions and the conjugate components of displacements may be simultaneously prescribed.

In connection with the rate formulations of classical elastoplasticity, the requirements for a suitable stress rate are now well recognized to imply that the stress rate vanishes when the solid continuum undergoes a rigid motion alone and when the stress tensor is referred to a coordinate system undergoing the same motion; and that the invariants of the stress tensor are stationary when the stress rate vanishes. The questions of objective stress rates and their use in classical rate theories of plasticity have been discussed by several authors: Oldroyd [31, 32], Truesdell [40], Cotter and Rivlin [4], Prager [35], Sedov [37], Masur [16], Naghdi and Wainwright [23], and Hill [11, 12].

III. FINITE STRAIN ELASTIC-PLASTIC ANALYSIS: RATE VARIATIONAL PRINCIPLES IN UPDATED LAGRANGIAN (UL) FORMULATION

In the UL formulation we refer the solution variables (displacements, strains, and stresses) in the state \( C_{N+1} \) to the configuration of the body in the immediately preceding state \( C_N \), which is presumed to be known. Let \( y_i^N \) be the current (in state \( C_N \)) Cartesian spatial coordinates of a particle to be used as a reference system for the current increment from \( C_N \) to \( C_{N+1} \). Let \( \mathbf{r}_N \) be the true stress in \( C_N \). Thus the UL rate formulation leads to an initial stress problem without initial displacements.

Let \( \dot{s}, \dot{t}, \text{and } \dot{r} \) represent the appropriate stress rates referred to the current configuration; that is, \( \dot{s} = s_{N+1}^N - \mathbf{r}_N \); \( \dot{t} = t_{N+1}^N - \mathbf{r}_N \), etc., where \( s_{N+1}^N \) is the second Piola-Kirchhoff stress in state \( C_{N+1} \) as referred to the configuration \( C_N \) (i.e., measured per unit area in \( C_N \)), etc. Let \( \mathbf{V}_N \) represent the gradient operator in the current coordinates and \( \mathbf{u} \) be the rate of deformation from the current configuration. We also define the rate of displacement gradient \( \dot{\mathbf{u}} \equiv (\mathbf{V}_N \dot{\mathbf{u}})^T = \dot{\mathbf{u}} + \dot{\omega} \), where \( \dot{\mathbf{u}}_{ij} = \frac{1}{2} (\partial \dot{u}_i / \partial y_j^N + \partial \dot{u}_j / \partial y_i^N) \) is the UL strain rate and \( \dot{\omega} \equiv \dot{\omega}_{ij} = \frac{1}{2} (\partial \dot{u}_i / \partial y_j^N - \partial \dot{u}_j / \partial y_i^N) \) is the spin rate.

A. Rate Potentials

Let \( \dot{\mathbf{a}}^* \) denote the corotational rate (or “Jaumann rate”) of the Kirchhoff stress, \( \mathbf{a} \). Then

\[
\dot{s} = \dot{\mathbf{a}}^* - \dot{\mathbf{a}} \cdot \mathbf{r}_N - \mathbf{r}_N \cdot \dot{\mathbf{a}} \quad \text{and} \quad \dot{t} = \dot{\mathbf{a}}^* - \dot{\mathbf{a}} \cdot \mathbf{r}_N - \mathbf{r}_N \cdot \dot{\mathbf{a}} \tag{13, 14}
\]

Considering a classical elastoplastic theory, it has been noted by Hill [11, 12] that a rate potential \( V \) exists such that

\[
\dot{\mathbf{a}}^* = \partial V / \partial \dot{\mathbf{a}} \tag{15}
\]

where [11]

\[
\dot{V} = \frac{1}{2} \mathbf{L} \mathbf{e}_{i} \mathbf{e}_{k} \mathbf{e}_{l} - \frac{\mathbf{a}}{g} (\lambda_{ij} \mathbf{e}_{il})^2 \tag{16}
\]

which yields a bilinear constitutive law through Eq. (15). Following Hill [11], we note that in Eq. (16) \( \mathbf{L} \) is a tensor of instantaneous elastic moduli assumed to be positive definite and symmetric under \( ij \leftrightarrow kl \) interchange; \( \alpha = 1 \) or 0 according to whether \( \lambda_{ij} \mathbf{e}_{il} \) is positive or negative; \( \lambda_{ij} \) is a tensor normal to the hyperplane interface between elastic and plastic domain the rate space; while \( g \) is a scalar related to the measure of rate of hardening due to plastic deformation. Prandtl-Reuss-type rate equations of the form of Eq. (15) for classical isotropically hardening materials have been used by several authors: Hutchinson [14], McMeeking and Rice [18], and Nemat-Nasser and Taya [29].

From Eqs. (13) and (15) it is seen that if a rate potential \( V \) exists for \( \dot{\mathbf{a}}^* \), then a potential \( W \) also exists for \( \dot{s} \) such that

\[
\dot{s} = \partial W / \partial \dot{\mathbf{a}} \tag{17}
\]

where

\[
W = \dot{V} - \mathbf{r}_N : (\dot{\mathbf{a}} \otimes \dot{\mathbf{a}}) \tag{18}
\]

Likewise, a rate potential \( \dot{U} \) for \( \dot{t} \) exists such that

\[
\dot{t} = \partial \dot{U} / \partial \dot{\mathbf{a}}^T \tag{19}
\]
Further, from the polar-decomposition theorem,
\[
F_N^{t+1} \equiv (\dot{\gamma}_N^{t+1})^T = \alpha_{N+1}^{t+1} \cdot (I + \dot{h})
\]
(22)
where \(F_N^{t+1}\) and \(\alpha_{N+1}^{t+1}\) are the deformation gradient and rotation tensors, respectively, in \(C_{N+1}\) as referred to \(C_N\), and \(\dot{h}\) is the UL rate of stretch. Writing \(\alpha_{N+1}^{t+1} = I + \dot{\alpha}\) (where \(\dot{\alpha}\) is the UL rate of rotation), it follows from Eq. (22) that
\[
\dot{\gamma} = \dot{\alpha} + \dot{h}
\]
(23)
or, as expected, \(\dot{h} \equiv \dot{\alpha}\) and \(\dot{\alpha} \equiv \dot{\omega}\). From Eqs. (4), (5), and (13),
\[
\dot{r} = \frac{1}{2}[\dot{t} + \dot{t}^T + \dot{\gamma} \cdot \dot{\alpha} + \dot{\alpha}^T \dot{\gamma}^T] - \gamma^T \dot{r}
\]
(24a)
\[
\equiv \frac{1}{2}[\dot{t} + \dot{t}^T + \dot{\gamma} \cdot \dot{\omega} + \dot{\omega}^T \dot{\gamma}^T]
\]
(24b)
\[
= \dot{s} + \frac{1}{2}(\dot{\gamma} \cdot \dot{e} + \dot{e} \cdot \dot{\gamma})
\]
(25)
\[
= \dot{\sigma} + \frac{1}{2}(\dot{\gamma} \cdot \gamma + \gamma \cdot \dot{\gamma})
\]
(26)
Thus if \(\dot{\gamma}\) is a rate potential for \(\dot{\sigma}\), it follows from Eq. (26) that there exists rate potential \(\dot{Q}\) for \(\dot{r}\) such that
\[
\dot{r} = \partial \dot{Q}/\partial \dot{h} \equiv \partial \dot{Q}/\partial \dot{\gamma}
\]
(27)
where
\[
\dot{Q} = \dot{V} - \frac{1}{2} \dot{\gamma} : (\dot{e} \cdot \dot{e})
\]
(28)
The results given in Eqs. (27) and (28) are useful, as shown later on, in formulating consistent complementary energy rate principles for the rate theory of finite strain plasticity.

### B. Field Equations and Boundary Conditions in UL Rate Form

1. In Terms of \(\dot{s}, \dot{e}, \text{ and } \dot{u}\). The linear momentum balance equation for \(s_N^{t+1}\) in UL coordinates (see Eq. 7) are
\[
V^N \cdot [s_N^{t+1} \cdot (F_N^{t+1})^T] + \rho^N B^{t+1} = 0
\]
(29)
where \(s_N^{t+1} = \gamma + \dot{s}\) and where
\[
V^N \cdot \gamma + \rho^N B = 0
\]
(30)
Hence
\[
\text{ (LMB)} \quad V^N \cdot [\dot{s} + \dot{\gamma} \cdot (V^N \dot{u})] + \rho^N B = 0
\]
(31)
\[
\text{ (AMB)} \quad \dot{s} = \dot{\gamma}
\]
(32)
\[
\text{ (Compatibility)} \quad \dot{\gamma} = \frac{1}{2}(\dot{e} + \dot{e}^T) \equiv \frac{1}{2}(V^N \dot{u}) + (V^N \dot{u})^T
\]
(33)
\[
\text{ (TBC)} \quad n^* \cdot [\dot{s} + \dot{\gamma} \cdot (V^N \dot{u})] \equiv \dot{t} = \dot{t} \text{ on } S_{\alpha N}
\]
(34)
\[
\text{ (DBC)} \quad \dot{u} = \dot{u} \text{ on } S_{u N}
\]
(35)
where \(\rho^N\) is the mass density in \(C_N\), \(B\) are rate of body forces/unit mas, \(S_{\alpha N}\) and \(S_{u N}\) are appropriate segments of the boundary of the solid in \(C_N\), and \(n^*\) is a unit normal to the boundary of the solid in \(C_N\).

2. In Terms of \(\dot{i}, \dot{e}, \text{ and } \dot{u}\). In a manner analogous to above, the field equations can be shown to be
\[
\text{ (LMB)} \quad V^N \cdot \dot{i} + \rho^N B = 0
\]
(36)
\[
\text{ (AMB)} \quad (V^N \dot{u})^T \cdot \dot{\gamma} + \dot{t} = \dot{t} + \dot{\gamma} \cdot (V^N \dot{u})
\]
(37)
or, equivalently,
\[
\text{ (AMB)} \quad \dot{\gamma} \cdot \dot{\gamma} + \dot{h} \cdot \dot{\gamma} + \dot{t} = \dot{t} + \dot{\gamma} \cdot \dot{h} + \dot{\gamma} \cdot \dot{\gamma}
\]
(38)
or
\[
\text{ (Compatibility)} \quad \dot{e} = (V^N \dot{u})^T
\]
(39)
or, equivalently,

\[(V^N \ddot{u})^T = \dot{x} + h \quad (40)\]

(TBC)

\[n^* \cdot \hat{t} = \dot{t} \text{ at } S_{SN} \quad (41)\]

(DBC)

Same as in Eq. (35)

C. General Variational Principles in UL Rate Form

1. In Terms of \( \dot{s}, \ddot{e}, \text{ and } \dot{u} \). Using a virtual work principle as applicable to an initial stress problem and following the procedure outlined by Washizu [43], we obtain a general UL rate principle for elastic-plastic problems which is analogous to the well-known Hu [13]-Washizu [42] principle of linear elasticity. This general rate principle\(^1\) governing Eqs. (31–35) and (17) can be stated as the condition of stationarity of the functional

\[
\pi_{hW}^*(\dot{u}, \dot{e}, \dot{u}) = \int_{V_N} \{ W(\dot{u}) - \rho^* \dot{B} \cdot \dot{u} + \frac{1}{2} \tau^N : ([V^N \ddot{u})^T (V^N \ddot{u})]\}
- \dot{s} : \left[ \dot{e} - \frac{1}{2} ((V^N \ddot{u}) + (V^N \ddot{u})^T) \right] dv - \int_{S_{SN}} \hat{t} \cdot \ddot{u} \, ds \\
- \int_{S_{SN}} \hat{t} \cdot (\ddot{u} - \ddot{u}) \, ds \quad (42)
\]

where \( \dot{W} \) is a rate potential for \( \dot{s} \) as defined through Eqs. (18) and (16). The above rate variational principle governs the rate variables from \( C_N \) to \( C_{N+1} \).

The satisfaction of the fully nonlinear field equations in \( C_N \) in a numerical solution method such as the finite element method, must be checked at each step; and these checks can be performed based on a variational principle governing the nonlinear field equations at \( C_N \). The details of corrective iterative procedures at the end of each increment, such as the “equilibrium check” and “compatibility mismatch check,” can be found, for instance, in the thesis by Murakawa [19].

We now consider certain special cases of the general rate principle given through Eq. (42). If Eqs. (17), (33), and (35) are met \( \text{a priori} \), one can reduce

\(^1\)The general variational principles stated in Eq. (42) or in Eqs. (53), (69), (82), (85), (114), (119), and (127) can be modified appropriately through the method of Lagrange multipliers, as discussed in Atluri and Murakawa [3], to account for discontinuities at interelement boundaries when these principles are applied to a finite element assembly.

\[
\pi_{hW}^*(\dot{u}, \ddot{e}, \dot{u}) = \int_{V_N} \{ W(\dot{u}) - \rho^* \dot{B} \cdot \dot{u} + \frac{1}{2} \tau^N : ([V^N \ddot{u})^T (V^N \ddot{u})]\}
- \dot{s} : \left[ \dot{e} - \frac{1}{2} ((V^N \ddot{u}) + (V^N \ddot{u})^T) \right] dv - \int_{S_{SN}} \hat{t} \cdot \ddot{u} \, ds \\
- \int_{S_{SN}} \hat{t} \cdot (\ddot{u} - \ddot{u}) \, ds \quad (43)
\]

The principle \( \delta \pi_{hW}^* = 0 \) leads to Eqs. (31), (32), and (34) and is equivalent to a principle stated originally by Hill [10].

By inverting the relation in Eq. (17) to express \( \dot{e} \) in terms of \( \dot{s} \), one can achieve a contact transformation,

\[
\dot{W} - \dot{s} = -S^*(\dot{s}) \quad (44)
\]

Using Eq. (44), one can eliminate \( \dot{e} \) from Eq. (42) to derive a Hellinger [7]-Reissner [36] type UL rate principle, with the associated functional

\[
\pi_{hW}^*(\dot{u}, \ddot{e}) = \int_{V_N} \{ -S^*(\dot{s}) - \rho^* \dot{B} \cdot \dot{u} + \frac{1}{2} \tau^N : ([V^N \ddot{u})^T (V^N \ddot{u})]\}
+ \frac{1}{2} \dot{s} : ([V^N \ddot{u}) + (V^N \ddot{u})^T]\right] dv - \int_{S_{SN}} \hat{t} \cdot \ddot{u} \, ds - \int_{S_{SN}} \hat{t} \cdot (\ddot{u} - \ddot{u}) \, ds \\
(45)
\]

Based on the arguments presented earlier for the linear elastic case (see, e.g., Ref. 2), it can be seen that in a finite element application of the principle stated through Eq. (45) one needs to assume over each finite element an arbitrary, symmetric, and differentiable stress-rate field \( \dot{s} \) and a differentiable \( \dot{u} \) that is also inherently compatible at the interelement boundaries.

In deriving a complementary energy rate principle from Eq. (45), we first note that in the present UL rate formulation the LMB conditions Eq. (31) are linear in \( \dot{s} \) and, unlike Eq. (7), do not involve coupling of \( \dot{s} \) with displacement gradients. Thus it becomes possible to satisfy both the LMB and AMB conditions, Eqs. (31) and (32), respectively, \( \text{a priori} \), by choosing a symmetric \( \dot{s} \) such that

\[
\dot{s} = \text{curl} \text{ curl } A + \dot{s}^p \quad (46)
\]

where \( A \) is the symmetric Maxwell-Morera-Beltrami second-order stress function tensor for a general three-dimensional case. In Eq. (46) \( \text{curl } A \) is defined such that \( (\text{curl } A)_{ij} = e_{ijk} A_{jk,p} \); (curl curl \( A \))_{ij} = e_{imn} e_{pq} A_{mp,nq}; e_{ijk} is the
alternating tensor; and \(\dot{s}_p\) is any symmetric particular solution such that

\[
\mathbf{V}^N \cdot \dot{s}_p = -\rho^N \mathbf{B} - \mathbf{V}^N \cdot \dot{\tau}^N [\mathbf{V}^N \mathbf{u}]
\]  

(47)

A simple way of satisfying Eq. (47) is to assume particular solutions

\[
\begin{align*}
\dot{s}_i &= \int_{\rho^N} \left[ -\rho^N \dot{B}_i - (\tau_i^* \dot{u}_i); \right] dy^N, \\
&= \text{no sum on } i, \ i = 1, 2, 3 \quad (48) \\
\dot{s}_j &= 0, \quad i \neq j
\end{align*}
\]  

(49)

where \((i, i)\) indicates \(\partial(i)/\partial y^N\). However, if the above assumptions are used in an assumed-stress-type numerical scheme (discussed further below), the question of completeness of the chosen stresses, that is, the numerical effect of the exclusion of the influence of displacement rates on the chosen shear-stress-rate field, as in Eq. (49), remains to be answered. Such effects can only be understood, in general, from a detailed mathematical study of the convergence of the method, which is not pursued in the present paper and remains an open question. Assuming that the satisfaction of the LMB and AMB conditions in the manner of Eqs. (48) and (49) is “satisfactory,” and further if the TBC condition is also met \textit{a priori}, one can reduce Eq. (45) to a functional associated with the complementary energy principle

\[
\pi_c^* (\mathbf{u}, \dot{s}) = \sum_{m} \int_{V_N} \left\{ -S^* (\dot{s}) - \frac{1}{2} \dot{\tau}^N [\mathbf{V}^N \mathbf{u}] \cdot [\mathbf{V}^N \mathbf{u}]^T \right\} dV + \int_{S_{\text{boundary}}} \mathbf{t} \cdot \dot{\mathbf{u}} dS
\]  

(50)

In a finite element application, \(V_N\) can be subdivided into \(M\) subdomains, \(V_{mN} (m = 1, \ldots, M)\), each with a boundary \(\partial V_{mN}\). In general it is seen that

\[
\partial V_{mN} = S_{mn} + S_{m0} + P_{mN},
\]

where \(S_{mn}, S_{m0}\) are, respectively, the portions of \(\partial V_{mN}\) where tractions and displacements are prescribed, and \(P_{mN}\) is that portion of \(\partial V_{mN}\) which is common to an adjacent element (interelement boundary). It is to be noted that in the finite element application of Eq. (50) the candidate stresses \(\dot{s}\) should not only satisfy the LMB and AMB conditions, Eqs. (31) and (32), but also satisfy the interelement traction reciprocity condition,\(^2\)

\[
i^+ + i^- = 0
\]

(52)

where \(S_{mn}\) is defined through Eq. 34 at \(P_{mN} a \text{ priori}\). One can introduce this interelement condition directly as a condition of constraint into Eq. (50) in order to preserve a wide choice of candidate stress-rates \(\dot{s}\) by letting

\[
\dot{s} = \text{curl curl } A - \dot{\tau}^N [\mathbf{V}^N \mathbf{u}] + \dot{s}_p
\]  

(52)

where \(\dot{s}_p\) satisfies \(\mathbf{V}^N \cdot \dot{s}_p = -\rho^N \mathbf{B}\). However, the \(\dot{s}\) so chosen then ceases to be symmetric, and thus the AMB condition must be introduced as a constraint condition into the associated complementary energy functional of the type given in Eq. (50) through additional Lagrange multipliers. It therefore appears that a rate complementary energy principle in UL form based on \(\dot{s}\) may
not be consistent and practically useful in the analysis of finite strain plasticity problems.

2. In Terms of \(i, \dot{e}, \text{and } \ddot{u}\). It can be shown similarly that Eqs. (36), (37), (39), (41), (35), and (19) follow as the Euler equations and natural boundary conditions corresponding to the stationarity of the functional

\[
\pi_{RW}^2(\dot{u}, \dot{e}, \ddot{u}) = \int_V \left\{ \dot{U}(\dot{e}) - \rho \nabla \dot{B} : \ddot{u} + i^T : [(\nabla^\prime \dot{u})^T - \dot{e}] \right\} dV
- \int_{S_{nn}} \dot{u} \cdot ds - \int_{S_{nn}} \dot{i} \cdot (\ddot{u} - \dot{u}) ds
\]  

(53)

where \(\dot{U}\) is the rate potential for \(i\) as defined through Eqs. (20) and (21). We now consider certain special cases of the above principle. If Eqs. (19), (39), and (35) are met \textit{a priori}, one can eliminate \(\dot{e}\) and \(i\) as variables from Eq. (53) and derive a rate functional governing the rate potential energy principle

\[
\pi_p^*(\dot{u}) = \int_V \left\{ \dot{U}(\dot{u}) - \rho \nabla \dot{B} : \dot{u} \right\} dV - \int_{S_{nn}} \dot{i} \cdot ds
\]  

(54)

This rate variational principle was first stated by Hill [10] and has been widely used in finite element applications to elastic-plastic problems. (See, for instance, Refs. 18, 26, and 29.)

It is interesting to note that both the LMB and AMB conditions, Eqs. (36) and (37), respectively, as well as the TBC, Eq. (41), must follow from the principle \(\delta \pi_p^*(\delta \dot{u}) = 0\). It is shown below that the AMB condition is inherently embedded in the special structure for \(\dot{U}\). That is, from Eqs. (21),

\[
i = \frac{\partial \dot{U}}{\partial \dot{e}^T} = \frac{\partial \dot{W}}{\partial \dot{e}^T} + \frac{1}{2} \nabla^\prime (\dot{e}^T \cdot \dot{e})
\]  

(55a)

\[
= \frac{\partial \dot{W}}{\partial \dot{e}^T} + \tau^N \cdot \dot{e}^T
\]  

(55b)

wherein the definition of \(\dot{s}\) from Eq. (17) has been used. Substituting for \(i\) from Eq. (55b) into the AMB condition Eq. (37), it is seen that the AMB condition is inherently met. This is due to the special structure for \(\dot{U}\) as given through Eq. (21). Conversely, it is also seen that if, instead of Eqs. (20) and (21), an arbitrary \(\dot{U}\) is postulated as a function of \(\dot{e}\), then the principle based on the functional in Eq. (54) ceases to be valid, since the AMB condition ceases either to be built into the structure of \(\dot{U}\) or to follow unambiguously as an Euler equation from the vanishing of the first variation of the said functional. This confirms the well-established conclusion that the nine components of \(\dot{e}\) cannot be used as arbitrary independent variables in an energy formulation. By the same token, a formal inversion of the bilinear relation

\[
\dot{U} = i^T : \dot{e} = -E^*(i)
\]  

(57)

and to the functional

\[
\dot{U} - i^T : \dot{e} = -E^*(i)
\]  

(58)

In fact, the above functional, as the basis for a Hellinger-Reissner-type variational principle, was used by Neale [25]. However, the validity of such a principle needs closer examination. If \(\delta \pi_{hr}^* = 0\), with \(\pi_{hr}^*\) as given in Eq. (59), is a valid Hellinger-Reissner-type rate principle, we note that the corresponding Euler equations and natural boundary conditions must be: (1) the LMB condition, Eq. (36); (2) the AMB condition, Eq. (37); (3) the compatibility condition, Eq. (39), (4) TBC, Eq. (41); and (5) the DBC, Eq. (35).

It is seen upon examining Eqs. (17) to (25) of Neale's [25] development that the AMB condition, Eq. (37) of the present paper, is in fact not an Euler equation of the principle \(\delta i = 0\) of Neale's [25] work, which is identical to \(\delta \pi_{hr}^* = 0\) of the present article). Thus, if the AMB condition is to be satisfied, then, for the validity of the principle, this condition must be embedded in the special structure, if any, for the complementary energy density function \(E^*(i)\) of Eq. (57). To examine this possibility, consider the form of \(\dot{U}\) as given from Eqs. (20) and (16):

\[
2\dot{U} = \dot{\epsilon}_{ik} \dot{e}_{ik} + \dot{\epsilon}_{ik} \dot{e}_{ik} - 2\frac{\alpha}{g} (\lambda_{ik} \dot{e}_{ik})^2 - 2\tau^N \dot{e}_{ik} \dot{e}_{ij} + \tau^N \dot{e}_{ik} \dot{e}_{ik}
\]  

(60)
The stress rate \( i \) as derived from the above is
\[
2i_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = \left( L_{ijkl} - \frac{2}{g} \lambda_{ij} \lambda_{kl} \right) \left( \varepsilon_{ik} + \varepsilon_{jk} - \varepsilon_{ij} \right) - \tau_{ij}^{\alpha} = (\alpha - \varepsilon_{ij} \lambda_{ij}) \varepsilon_{ij} - \varepsilon_{ij} \lambda_{ij} \lambda_{jk} - \varepsilon_{ij} \lambda_{ij} \lambda_{kl}
\]
(61)
where \( L_{ijkl} \), \( \alpha \), and \( \lambda_{ij} \) are as defined before. The constitutive law, Eq. (61), is of the bilinear type for \( i_{ij} \) in terms of \( \varepsilon_{ij} \). The inversion of this relation in closed form to express \( \varepsilon_{ij} \) in terms of \( i_{ij} \) appears to be in general impossible. In fact, Eq. (61) is of the form
\[
i_{ij} = *L_{jik} \delta_{kn}
\]
whose only symmetry property is
\[
*L_{jik} = *L_{kin}
\]
(62)
Even though an analytical formulation of the inverse of the matrix \((*L)\) appears impossible, one may perhaps numerically invert \( L \); that is,
\[
\varepsilon_{ij} = L^{-1}_{ijkl} k_{lk}
\]
(63)
where, in general, \(*L_{ijkl} = *L_{klji} \). Using Eq. (64) a contact transformation can then be made to find \( \mathcal{E}^*(t) \) such that
\[
\partial \mathcal{E}^*/\partial \varepsilon_{ij} = \varepsilon_{ij} = *L_{jik} l_{lk}
\]
(65)
If the AMB condition Eq. (37) is inherent in the structure of \( \mathcal{E}^*(t) \), then the condition
\[
\varepsilon_{ij} \varepsilon_{jk}^{\alpha} + \varepsilon_{ik} = \text{symmetric}
\]
(66)
must be identically satisfied when \( \varepsilon_{ij} \) is expressed in terms of \( i_{mn} \) through Eq. (65), or
\[
*L_{ijkl} i_{mn} \varepsilon_{jk}^{\alpha} + \varepsilon_{ik} = \text{symmetric}
\]
(67)
Neither of the two terms is by itself symmetric under \( i \leftrightarrow k \) interchanged, nor is one term the transpose of the other. Moreover, the first term cannot be

\[\text{expressed as the sum of a symmetric tensor and the transpose of the second term since \(*L_{ijkl}^\text{sym} \) cannot be found analytically; Eq. (67) can therefore not be verified, and the Hellinger-Reissner-type principle based on Eq. (59) appears to be of uncertain practical value. The same applies to the complementary energy rate principle as stated by Hill [10] (which can be derived formally from Eq. 59 by requiring \( t \) to satisfy the LMB condition, Eq. 36, and the TBC, Eq. 41, \textit{a priori}).}

3. In Terms of \( \hat{r}(t), \hat{\omega} \); \( \lambda, \hat{u} \). To seek alternative ways to avoid the above discussed difficulties in formulating a consistent complementary energy rate principle and the Hellinger-Reissner-type rate principle, we transform the general variational principle associated with Eq. (53) into one involving \( r, \omega, h, \) and \( \hat{u} \) as variables. First, by comparing Eqs. (20) and (28) we note that
\[
U = \frac{1}{2} \tau^N : (\varepsilon : \varepsilon) - \tau^N : (\dot{\varepsilon} : \varepsilon) + \frac{1}{2} \tau^N : (\dot{\varepsilon}^T : \dot{\varepsilon})
\]
(68)
Using (68) to express \( U \) in terms of \( \hat{Q} \) (which is a function of \( h \equiv \omega \) and writing \( \varepsilon = h + \omega, \) we rewrite Eq. (53) in the form
\[
\text{(69)}
\]
and hence
\[
\delta \pi^N_{\text{sym}}(\delta \omega; \delta h; \delta \omega; \delta \omega) = \int_{\Omega_N} \left[ \frac{\partial \hat{Q}}{\partial h} - \frac{1}{2} \tau^N : (\dot{\varepsilon} : \dot{\omega} + \dot{\omega}^T : \dot{\varepsilon}) : \dot{\omega} + [(\nabla \omega)^T : \dot{\omega} - \dot{\omega} : \dot{\omega}] : \dot{\omega} \right] dV
\]
(70)
Since \( \dot{z} \) and \( \delta \dot{z} \) are skew-symmetric tensors, it follows that the Euler equations and natural boundary conditions corresponding to Eq. (70) are: (1) the constitutive law, Eq. (27); (2) the LMB condition, Eq. (36); (3) the AMB condition, Eq. (38); (4) the compatibility condition, Eq. (40); (5) the TBC, Eq. (41); and (6) the DBC, Eq. (35).

One can now invert (if only numerically) Eq. (27) and achieve the contact transformation,

\[
\dot{Q} = \frac{1}{2} [i + i^T + \tau^N \cdot \dot{z} + \dot{z}^T \cdot \tau^N] ; h = -\dot{R}^*(i) (71)
\]

Substituting Eq. (71) in Eq. (69), we derive a functional involving only \( \dot{u} \) and \( \phi \) (and hence \( i \) and \( \dot{a} \)) as variables, corresponding to a Hellinger-Reissner-type principle which has as its Euler equations and natural boundary conditions, Eqs. (36), (38), (40), (41), and (35). If, in addition, one assumes that the LMB condition and TBC for \( i \), Eqs. (36) and (41), are satisfied a priori, one can eliminate \( h \) and \( \dot{u} \) as variables from the functional in Eq. (69) and thus obtain the complementary energy functional

\[
\pi_c^t(\dot{a}, i, \dot{u}) = \int_{V_N} \left\{ -\dot{R}^*(\phi) + \frac{1}{2} \tau^N \cdot (\dot{a}^T \cdot \dot{a}) - i^T \cdot \dot{a} \right\} dV + \int_{S_{in}} (n^* \cdot i) \cdot \dot{u} dS (73)
\]

In the above, the definition of \( \phi \) [\( = \frac{1}{2} (i + i^T + \tau^N \cdot \dot{z} + \dot{z}^T \cdot \tau^N) \)] is implied, and the spin-rate field \( \dot{a} \) is required to be skew-symmetric. The variational equation \( \delta \pi_c^t = 0 \) for constrained \( \delta i \) (which must obey the constraint \( \tau^N \cdot i = 0 \) in \( V_N \) and \( n^* \cdot \delta i = 0 \) at \( S_{in} \)) and for constrained \( \delta \dot{a} \) (which is required to be skew-symmetric) leads to

\[
\delta \pi_c^t(\dot{a}, i, \dot{u}) = 0 = \int \left\{ \left( n^* \cdot \delta \dot{a} \right) - \frac{\partial \dot{R}^*}{\partial \phi} \right\} dV + \int_{S_{in}} (n^* \cdot \delta i) \cdot (\dot{u} - \dot{u}) dS (74)
\]

Noting that, by definition \( \delta \dot{R}^* / \delta \phi = h \), it is seen that Eq. (74) implies as its Euler equations and natural boundary conditions (1) the compatibility condition, Eq. (40); (2) the AMB condition, Eq. (38); and (3) the DBC, Eq. (35).

Equation (73) therefore forms the basis of an entirely consistent and practically useful rate complementary energy theorem for the UL rate formulation of finite strain plasticity analysis methods in the sense that: (1) the admissible \( i \) is required to satisfy, a priori, only the uncoupled, linear LMB equation, Eq. (36), and TBC, Eq. (41), which can be met easily by setting \( i = \tau^N \cdot \Psi + i^p \), where \( \Psi \) are first-order stress functions (once differentiable) and \( i^p \) is any particular solution such that \( \tau^N \cdot i^p = -\rho^i \cdot \dot{B} \); (2) the AMB conditions, the compatibility condition, and the DBC follow unambiguously as Euler equations.

In a finite element application of the complementary rate principle as stated through Eq. (73), the assumed stress-rate field \( i \) must not only satisfy the LMB condition (Eq. 36) within each element but must also satisfy the traction reciprocity relation at the interelement boundary, viz., \( (n^* \cdot i)^+ + (n^* \cdot i)^- \) at \( \rho_{mn} \) (where + and - , respectively, indicate the two sides of \( \rho_{mn} \) in the limit that \( \rho_{mn} \) is approached). This may, in general, pose a severe restriction on the choice of \( i \) within each element, especially when the element is of an arbitrary curved geometry. In such a case it may be preferable to include this interelement traction reciprocity condition as a constraint condition directly into the functional in Eq. (73). The associated Lagrange multipliers turn out to be the interelement boundary displacements. The modified complementary energy rate principle for an assembly of a finite number of elements then becomes the stationarity condition of the functional

\[
\pi_{MC}^t(\dot{a}, i, \dot{u}_p) = \sum_{m} \left\{ \int_{V_{mn}} \left\{ -\dot{R}^*(\phi) + \frac{1}{2} \tau^N \cdot (\dot{a}^T \cdot \dot{a}) - i^T \cdot \dot{a} \right\} dV + \int_{S_{mn}} (n^* \cdot i) \cdot \dot{u}_p dS \right\} (75)
\]

In the above functional, \( \dot{a} \) and \( i \) are chosen independently within each element in terms of undetermined parameters, whereas \( \dot{u}_p \) are chosen in terms of displacements at nodes of a finite element and hence \( \dot{u}_p \) are common to elements sharing a common boundary. Thus the undetermined parameters in the field functions for \( \dot{a} \) and \( i \) can be eliminated at the element level and expressed in terms of the generalized nodal displacement coordinates. The finite element method based on Eq. (75) thus results eventually in a standard stiffness matrix procedure (see Refs. 21 and 22, for instance, for details of finite element application of the complementary energy rate principles in finite elasticity). Alternatively, the interelement traction reciprocity can be satisfied a priori by an appropriate choice of the first-order stress functions \( \Psi \) from which the equilibrated \( i \) are derived. The finite element method then generally leads to a "flexibility matrix" type approach.
D. Near Incompressibility in the Fully Plastic Range

As discussed in the Introduction, an important aspect of numerical schemes for finite strain elastic-plastic analysis is the problem of the accurate treatment of nearly incompressible deformation rates. In a finite element application of the potential energy rate formulation of the type given by Eq. (43) or (54), if the assumed deformation rates do not a priori obey the incompressibility constraint, it may be necessary to retain this constraint as an a posteriori constraint through a Lagrange multiplier, the hydrostatic pressure. To this end, consider the rate potential \( W(\dot{\varepsilon}) \), Eq. (18), for a classical Prandtl-Reuss-type rate constitutive law

\[
W(\dot{\varepsilon}) = \frac{1}{2} \dot{\sigma}^* : \dot{\varepsilon} - \tau^*(\dot{\varepsilon} : \dot{\varepsilon})
\]  

(76)

The corotational rate of Kirchhoff stress \( \dot{\sigma}^* \), for a classical Prandtl-Reuss-type approximation, can be written in terms of \( \dot{\varepsilon} \), as suggested by McMeeking and Rice [18], as

\[
\dot{\sigma}^*_{ij} = 2\mu \left[ \delta_{ij} \dot{\varepsilon}_{ij} - \frac{9\nu\mu}{(2\alpha + 6\mu)} \left( \frac{3\lambda + 2\mu}{\lambda + \mu} \right) \dot{\varepsilon}_{kk} \delta_{ij} \right] + \lambda \dot{\varepsilon}_{kk} \delta_{ij}
\]  

(77)

where \( \alpha = 1 \) is at yield and \( \tau^*_{ij} > 0 \), and \( \alpha = 0 \) otherwise; \( \tau^*_{ij} \) is the deviatoric Cauchy stress in \( C_N \); \( \tau^* \) is the slope of the uniaxial stress/plastic strain curve; and \( \lambda \) and \( \mu \) are Lame's constants. Putting Eq. (77) in the form

\[
\dot{\sigma}^*_{ij} = 2\mu E_{ijkl} \dot{\varepsilon}_{kl} + \lambda \dot{\varepsilon}_{kk} \delta_{ij}
\]  

(78)

and letting \( \dot{\sigma}^*_{ij} = \dot{\sigma}^*_{ij}' + \frac{1}{2} \dot{\sigma}^*_{kk} \delta_{ij} \) and \( \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}' + \frac{1}{2} \dot{\varepsilon}_{kk} \delta_{ij} \), we obtain

\[
\dot{\sigma}^*_{ij}' = 2\mu E_{ijkl} \dot{\varepsilon}_{kl}, \quad \dot{\sigma}^*_{kk} = (3\lambda + 2\mu) \dot{\varepsilon}_{kk}
\]  

(79a, b)

and hence

\[
W = \mu E_{ijkl} \dot{\varepsilon}_{kl} \dot{\varepsilon}_{lj} + \frac{\lambda \dot{\varepsilon}_{kk}^2}{2} - \tau^*(\dot{\varepsilon} : \dot{\varepsilon})
\]  

(80)

or, equivalently,

\[
W = \mu E_{ijkl} \dot{\varepsilon}_{kl} \dot{\varepsilon}_{lj} + \left( \frac{3\lambda + 2\mu}{6} \right) (\dot{\varepsilon}_{kk})^2 - \tau^*(\dot{\varepsilon} : \dot{\varepsilon})
\]  

(81)

To obtain numerically accurate solutions in situations of fully developed plasticity, it may be advantageous to retain the hydrostatic pressure as an independent variable and thus derive a mixed variational principle which represents a modification of the potential energy rate principle given through Eq. (43), with \( W \) expressed as in Eq. (80). To this end we introduce both \( \dot{\varepsilon}_{ij} \) and \( \dot{\varepsilon}_{kk} \) as additional independent variables into the functional of Eq. (43) through the introduction of Lagrange multipliers \( \beta_{ij} \) and \( \alpha \), respectively. From this general variational principle (with \( \dot{u}_i \), \( \dot{\varepsilon}_{ij} \), \( \dot{\varepsilon}_{kk} \), \( \beta_{ij} \), and \( \alpha \) all as variables) we demand all the necessary field equations in the case of near incompressibility. When the Lagrange multipliers are identified with the relevant stress rates, this general rate variational principle applies to the functional

\[
W_{\alpha} = \int_{V_N} \left\{ \mu E_{ijkl} \dot{\varepsilon}_{kl} \dot{\varepsilon}_{lj} + \frac{\lambda \dot{\varepsilon}_{kk}^2}{2} + \frac{\lambda}{(3\lambda + 2\mu)} \dot{\sigma}^*_{kk} \dot{\varepsilon}_{ij} \right\} dV + \text{boundary terms}
\]  

(82)

The above general principle is valid in both the cases of near and precisely incompressibility. In the above, the notation \( \dot{u}_{(i,j)} = \dot{u}_{i,j} + \dot{u}_{i,k} \) and \( \dot{u}_{i,k} = \dot{u}_{i,k} / \dot{\varepsilon}_k^* \) has been used. If from Eq. (82) one eliminates \( \dot{u}_{(i,j)} \) by defining \( \dot{\varepsilon}_{ij} = \dot{u}_{(i,j)} \), and \( \dot{\varepsilon}_{kk} \) through the contact transformation

\[
\lambda \frac{\dot{\varepsilon}_{kk}^2}{2} - \frac{\lambda \dot{\varepsilon}_{kk}^2}{2} = \frac{\lambda}{2} \frac{(\dot{\sigma}^*_{kk})^2}{(3\lambda + 2\mu)}
\]  

(83)

one obtains a mixed variational principle, involving \( \dot{u}_i \) and \( \dot{\sigma}^*_{kk} \) as variables, governed by the functional

\[
W_{\alpha} = \int_{V_N} \left\{ \mu E_{ijkl} \dot{u}_{k,j} \dot{u}_{i,j} + \frac{\lambda \dot{\varepsilon}_{kk}^2}{2} + \frac{\lambda}{(3\lambda + 2\mu)} \dot{\sigma}^*_{kk} \dot{u}_{i,j} \right\} dV - \int_{E_P} t \dot{\varepsilon}_{ij} ds
\]  

(84)

which remains valid for nearly or even precisely incompressible behavior at large plastic strains for all assumed displacement rates \( \dot{u}_i \) that do not obey the constraint of incompressibility a priori. Equation (84) and the associated
varietal principle are analogous to the case of linear isotropic elasticity given directly, without explicit derivation, by Herrmann [8].

Likewise, using the definition of $\dot{W}$ as in Eq. (81) in Eq. (43), and introducing $\dot{e}_{ij}$ and $\dot{e}_{kk}$ as additional independent variables into the functional in Eq. (43) through appropriate Lagrange multipliers, one can derive another alternate general variational principle, which remains valid in the limit of incompressibility, with the associated functional

$$
\pi_{\text{mp}}^{\text{*}}(\dot{u}_{i}; \dot{e}_{ij}; \dot{e}_{kk}; \dot{e}_{kk}^{*}; \dot{e}_{kk}^{**})
= \int_{V_N} \left\{ \mu \mathbb{E}_{ijkl} \dot{e}_{ij} \dot{e}_{kl} + \frac{3\lambda + 2\mu}{6} \dot{e}_{kk}^{*} + \frac{\dot{e}_{kk}}{3} + \frac{\dot{e}_{ij}(\dot{e}_{ij})}{3} \right\} dV + \text{boundary terms}
$$

(85)

where $\dot{e}_{ij}(\dot{e}_{ij}) = \dot{u}_{ij}(\dot{u}_{ij})$. If from Eq. (85) one eliminates $\dot{e}_{ij}$ as a variable through a priori satisfying the condition $\dot{e}_{ij} = \dot{u}_{ij}(\dot{u}_{ij})$, and if $\dot{e}_{kk}$ is eliminated through the contact transformation

$$
\frac{(3\lambda + 2\mu)}{6} \dot{e}_{kk}^{2} - \frac{\dot{e}_{kk}}{3} = \frac{-\dot{e}_{kk}^{*}}{6(3\lambda + 2\mu)}
$$

(86)

one obtains an alternate mixed variational principle, also involving $\dot{u}_{i}$ and $\dot{u}_{kk}$ as variables, governed by the functional

$$
\pi_{\text{mp}}^{\text{*}}(\dot{u}_{i}; \dot{e}_{kk}) = \int_{V_N} \left\{ \mu \mathbb{E}_{ijkl} \dot{u}_{ij}^{*} \dot{u}_{kl}^{*} + \frac{\dot{e}_{kk}^{*}}{3} \dot{u}_{kk}^{*} - \frac{(\dot{e}_{kk}^{*})^{2}}{3} - \frac{\dot{e}_{ij}^{*} \dot{u}_{ij}^{*} \dot{u}_{ij}^{*}}{3} \right\} dV - \int_{\Gamma_T} \dot{t}_{i} \dot{u}_{i} ds
$$

(87)

Equation (87) and the associated principle are analogous to the ones derived by Key [15] for linear elastic infinitesimal deformation problems through Fraeijs de Veubeke's [5] interpretation of the Hellinger-Reissner theorem in linear elasticity.

Nagtegaal et al. [24, Appendix II therein], in order to improve the accuracy of UL rate finite element formulations for problems of large plastic flow, suggest a mixed formulation based on the functional

$$
\pi_{\text{mp}}^{\text{*}}(\dot{u}_{i}; \dot{e}_{kk}) = \int_{V_N} \left\{ \frac{1}{2} \dot{u}_{ij}^{*} \dot{u}_{ij}^{*} \right\} dV - \int_{\Gamma_T} \dot{t}_{i} \dot{u}_{i} ds
$$

(88)

where $\dot{e}_{ij}$ is related to $\dot{e}_{ij}$ through an equation of the type of Eq. (79a). It is worth noting that the above formulation, Eq. (88), is analogous to the present formulation Eq. (87), except that $\dot{e}_{kk}$ appears as a variable in Eq. (87) and $\dot{e}_{kk}$ appears in Eq. (88). It is interesting to observe that the procedure based on Eq. (88) ceases to be valid in the limit of precise incompressibility. Moreover, in the discrete (finite-element) version of the functional corresponding to Eq. (88) (when appropriate discrete approximations for $\dot{u}_{i}$ and $\dot{e}_{kk}$ are introduced), Nagtegaal et al. [24] proceed to eliminate $\dot{e}_{kk}$ as a variable at the element level and introduce a modified definition for the strain energy density functional $\dot{W}$. The rigorous theoretical validity of the modified discrete functional, as a variational basis for obtaining discretized equilibrium equations, is not fully authenticated.

We note that the above discussed difficulties with the incompressibility constraint are somewhat easier to handle in the case of assumed stress finite element methods based on a complementary rate principle of the type given in Eqs. (73) and (74). (For a treatment of incompressibility using assumed stress finite element methods, see, for instance, the works of Tong [38] and Pian and Lee [34] in linear elastic infinitesimal deformation cases and that of Murakawa and Atluri [22] in finite elasticity problems.)

IV. RATE VARIATIONAL PRINCIPLES IN TOTAL LAGRANGIAN FORMULATION

In the numerical solution of certain problems such as, for instance, plates and shells, it may be preferable to use rate formulations wherein all the variables in each subsequent increment are referred to a fixed Lagrangian or Total Lagrangian (TL) frame in which the initial configuration $C_0$, with coordinates $x_0$, is used as the reference frame in each of the subsequent configurations. Let $\mathbf{s}'$ and $\mathbf{t}'$ be the rates of the second and first Piola-Kirchhoff stresses in going from $C_N$ to $C_N + \Delta N$, but referred to and measured per unit area in the initial configuration $C_0$. Let $\mathbf{V}'$ be the gradient operator in the coordinates in $C_0$, and let $\dot{u}$ be the rate of displacement from the current state. Then the Total Lagrangian strain-rate

$$
E' = \frac{1}{2} [\mathbf{V} \dot{u} + (\mathbf{V} \dot{u})^T + (\mathbf{V} \dot{u}) \cdot (\mathbf{V} \dot{u}^T) + (\mathbf{V} \dot{u}^T) \cdot (\mathbf{V} \dot{u})]
$$

(89)

where $\dot{u}$ is the displacement at $C_N$ as measured from $C_0$, and the UL strain-
rate are related through

\[ E' = (F^N)^T \cdot \dot{\varepsilon} \cdot F^N \]  
(90)

where \( F^N = (I + \nabla^0 u)^T \) and \( \dot{\varepsilon} \) is defined in Eq. (33). Likewise, \( \dot{\varepsilon}' \equiv (V^0 u)^T \), the TL rate of displacement gradient, is related to the UL rate \( \dot{\varepsilon} \) through

\[ \dot{\varepsilon}' = \dot{\varepsilon} \cdot F^N \]  
(91)

and

\[ s' = J^N(F^N)^{-1} \dot{\beta}(F^N)^{-T} \]  
(92)

\[ t' = J^N(F^N)^{-1} \dot{\gamma} = s' \cdot F^N + s^N \cdot e' \]  
(93a, b)

where \( J^N \) is the value of the determinant of the matrix \([y^N_{ij}]\). Finally, the TL rate of Jaumann stress, \( r' \), is related to \( t' \) and \( s' \) through

\[ r' = \frac{1}{2} \left\{ t' \cdot \mathbf{a} + \mathbf{a}^T \cdot t' + t' \cdot \mathbf{a}^T + \mathbf{a}^T \cdot t' \right\} \]  
(94)

\[ = \frac{1}{2} \left[ \dot{\gamma} \cdot (I + h^N) + (I + h^N) \cdot \dot{\gamma} + h^N \cdot \dot{\gamma} \right] \]  
(95)

where \( t^N \) and \( s^N \) are, respectively, the first and second Piola-Kirchhoff stress tensors in \( C_0 \) as referred to \( C_9 \); \( h^N \) is the engineering strain tensor in \( C_0 \) referred to \( C_9 \); and \( \mathbf{a}^N \) and \( \alpha' \), which are rotation tensors such that \((\alpha^N + \alpha')\) is an orthogonal tensor, satisfy

\[ (V^0 y^N)^T = \mathbf{a}^N \cdot (I + h^N), \quad (V^0 u)^T = \alpha' \cdot (I + h^N) + \alpha^N \cdot h' \]  
(96)

We now consider the question of the forms of rate potentials for \( s' \), \( t' \), and \( r' \). First we note that if a rate potential for \( s \) of the form of Eq. (17) exists, then, in view of Eqs. (90) and (92), a rate potential, say \( W' \), can also be shown to exist for \( s' \). Specifically, let the potential \( W \) for \( s \) (Eq. 17) be of the form

\[ W = \frac{1}{2} M_{ijkl} \dot{\varepsilon}_{ij} \dot{\beta}_{kl} \]  
(97)

where the tensor \( M_{ijkl} \) can be expressed in terms of \( L_{ijkl} \) and the relevant plasticity parameters through Eqs. (18) and (16). Then, in view of Eqs. (90) and (92),

\[ W' = \frac{1}{2} M_{ijkl} \dot{e}_{ij} \dot{\beta}_{kl} \]  
(98, b)

Likewise, in view of relation (93b),

\[ U' = W' + \frac{1}{2} s^N \cdot (e'T \cdot e') \]  
(100a, b)

where the relation \( E' = \frac{1}{2}(e'T \cdot F^N + F^N \cdot e') \) is implied. Finally

\[ Q' = W' + \frac{1}{2} s^N \cdot (h' \cdot h') \]  
(101a, b)

with \( E' = \frac{1}{2} [h' \cdot (I + h^N) + (I + h^N) \cdot h'] \).

We now consider the rate form of the field equations and boundary conditions. Considering the rates of Eqs. (6)-(12c), the latter are as follows.

1. In Terms of \( s' \); \( E' \); and \( u' \)

(LMB) \[ \nabla^0 \cdot (s^N \cdot e'T + s' \cdot (F^N)^T) + \rho^0 \dot{B}' = 0 \]  
(102)

(AMB) \[ s' = s'T \]  
(103)

(Compatibility) \[ E' = \frac{1}{2} [e' + e'T + e'T \cdot e'N + e'NT \cdot e'] \]  
(104)

with

\[ \dot{e}' = (V^0 u)^T \]  
(105)

(TBC) \[ n \cdot (s^N \cdot e'T + s' \cdot F^N)^T \equiv t' = \mathbf{t} \text{ on } S_{a0} \]  
(105)

where \( \mathbf{t} \) are prescribed tractions per unit area of the surface segment \( S_{a0} \) of the boundary of the solid in \( C_0 \), and \( n \) is a unit outward normal to \( S_{a0} \).

(DBC) \[ \dot{u} = \dot{u} \text{ at } S_{a0} \]  
(106)
2. In Terms of \( t'; e'; \) and \( \dot{u} \) (or \( \tau', \alpha', h', \) and \( \dot{\alpha} \))

\begin{align}
\text{(LMB)} & \quad \nabla^0 \cdot t' + \rho B' = 0 \\
\text{(AMB)} & \quad F^N \cdot t' + e' \cdot t^N = t^N \cdot e'T + t' \cdot F^N \\
\text{or, equivalently,} & \quad \text{(AMB)} \\
\text{(Compatibility)} & \quad e' = (\nabla^0 \dot{u})^T \\
\text{or, equivalently,} & \quad e' \equiv \alpha' (I + h^N + h') (t' \cdot \alpha^N + t^N \cdot \alpha^N) = \text{symmetric} \\
\text{(TBC)} & \quad \mathbf{n} \cdot t' = \mathbf{i}' \text{ at } S_{so} \\
\text{(DBC)} & \quad \dot{u} = \dot{u} \text{ at } S_{so}
\end{align}

\[ \pi_\text{HLW}(s'; E'; \text{and } \dot{u}) = \int_{V_0} \left\{ W'(E') - \rho B' \cdot \dot{u} + \frac{1}{2} s^N \cdot (e^T \cdot e') \\
- \mathbf{s}' \cdot \left[ E' - \frac{1}{2} (e' + e^T + e^T \cdot e^N + e^N \cdot e') \right] \right\} dV \\
\times \left[ \int_{S_{so}} \mathbf{t}' \cdot \dot{u} \, ds - \int_{S_{so}} \mathbf{t}' \cdot (\dot{u} - \ddot{u}) \, ds \right] \tag{114} \]

where \( e' = (\nabla^0 \dot{u})^T, W' \) as defined in Eq. (98a), and \( t' \) as defined in Eq. (105). If one eliminates \( E' \) and \( s' \) from Eq. (114) by a priori satisfying Eqs. (98b), (104), and (106), one obtains a potential energy rate principle with the associated functional

\[ \pi_\text{ HLW}(t'; e'; \dot{u}) = \int_{V_0} \left\{ U'(e') - \rho B' \cdot \dot{u} - t^T \cdot [e' - (\nabla^0 \dot{u})^T] \right\} dV \\
- \int_{S_{so}} \mathbf{t}' \cdot \dot{u} \, ds - \int_{S_{so}} \mathbf{t}' \cdot (\dot{u} - \ddot{u}) \, ds \tag{119} \]

where \( U' \) is defined through Eq. (100a) and \( t' \) is defined in Eq. (112). As in the UL rate case, because of the special structure of \( U' \) as given in Eq. (100a), it can be shown that \( t' \), as derived from \( U' \) through Eq. (100b), identically satisfies the AMB condition, Eq. (108).

If from Eq. (119) one eliminates \( e' \) and \( t' \) as variables by a priori satisfying Eqs. (100b), (110), and (113), one can derive a rate form of a potential energy functional

\[ \pi_\text{HLW}(\dot{u}) = \int_{V_0} \{ U'(\dot{u}) - \rho B' \cdot \dot{u} \} \, ds \tag{120} \]
the stationarity of which leads to Eqs. (107), (108), and (112) as its Euler equations and natural boundary conditions. This variational principle is identical to the one in Eq. (115) because of Eq. (100a). Further, by inverting Eq. (100b), one may, under certain conditions analogous to those discussed in the UL rate case, achieve the contact transformation

\[ U'(e') - T'^T: e' = - T'^* (t') \]

\[ \frac{\partial T'^*}{\partial t'} = e'^T \]

(121)

(122)

However, analogous to the situation discussed earlier in connection with the UL rate formulation, the AMB condition Eq. (108) cannot be verified to be embedded in the structure of \( T'^* (t') \) as obtained from Eq. (121). Thus the Hellinger-Reissner-type principle in terms of \( t' \) and \( a \) (derivable by using Eq. 121 in Eq. 119), or the complementary energy rate principle in terms of \( t' \) alone (derivable by satisfying conditions of Eqs. 121, 107, and 112 a priori in Eq. 119) cease to be rational principles since the AMB condition for \( t' \) is neither embedded in the structure of \( T'^* \) nor does it follow as an Euler equation from these principles.

3. In Terms of \( \alpha'; h'; u' \); and \( r'(t'; \alpha') \). Once again, to avoid the above difficulties in formulating consistent complementary energy and Hellinger-Reissner-type rate variational principles, we transform the general variational principle associated with Eq. (119) into one involving \( r'; \alpha'; h' \); and \( u' \) as variables. Noting from Eqs. (100a) and (101a) that

\[ U' = Q' - \frac{1}{2} s^N: (h' \cdot h') + \frac{1}{2} s^N: (e'^T \cdot e') \]

(123)

and since

\[ e' = \alpha' \cdot (I + h') + \alpha'^T \cdot h' \]

\[ \alpha'^N \cdot \alpha'^N = I, \quad \alpha'^T \cdot \alpha'^N = - \alpha'^N \cdot \alpha' \]

(124)

(125)

and \( t^N = s^N \cdot F'^N \), we reduce Eq. (123) to

\[ U' = Q' - t'^N: (\alpha' \cdot h') - \frac{1}{2} t'^N: [\alpha' \cdot \alpha'^N \cdot \alpha' \cdot (I + h')] \]

(126)

Upon using Eq. (126) and (124), Eq. (119) becomes

\[ \pi^2_{mb}(u'; h'; \alpha'; t') \]

\[ = \int_{V_0} \left\{ Q'(h') - \rho B' \cdot \ddot{u} + t'^N: [(\nabla^0u')^T - \alpha' \cdot (I + h') - \alpha'^N \cdot h'] 
\right. \]

\[ - \frac{1}{2} \tau'^N: [\alpha' \cdot \alpha'^N \cdot \alpha' \cdot (I + h')] \right\} dV 
\]

\[ - \int_{S_{en}} \tau' \cdot \ddot{u} ds - \int_{S_{en}} t' \cdot (\ddot{u} - \ddot{u}) ds \]

(127)

where \( Q'(h') \) is the potential for \( r' \) as defined in Eq. (101a) and \( r' \) is related to \( t' \) and \( \alpha' \) through Eq. (94). Since the variations \( \delta \alpha' \) are required to be consistent with Eq. (125b), it can easily be shown that the stationarity condition of the above functional yields: (1) the LMB condition, Eq. (107); (2) the AMB condition, Eq. (109); (3) the compatibility condition, Eq. (111); (4) the rate constitutive law, Eq. (101b); (5) the TBC, Eq. (112), and (6) the DBC, Eq. (113).

By inverting (if only numerically) Eq. (101b) and through the contact transformation

\[ Q' = \frac{1}{2} [t'^N \cdot \alpha' + \alpha'^N \cdot t'^T + \alpha'^T \cdot \alpha'^N] : h' = - R'^*(r') \]

(128)

\[ \frac{\partial R'^*}{\partial r'} = h' \]

(129)

one can eliminate \( h' \) as a variable from Eq. (127) and obtain a functional \( \pi^2_{mb}(u'; \alpha'; t') \) corresponding to a Hellinger-Reissner-type variational principle.

Finally, by requiring \( t' \) to satisfy only the linear LMB condition, Eq. (107), and the TBC, Eq. (112), one can eliminate \( \dot{u} \) as a variable from \( \pi^2_{mb} \) and obtain a TL rate complementary energy functional

\[ \pi^2(t'; \alpha') = \int_{V_0} \left\{ - R'^*(r') - \frac{1}{2} t'^N: [\alpha' \cdot (I + h')] \right\} dV + \int_{S_{en}} t' \cdot \ddot{u} ds \]

(130)

wherein it is implied that \( r' \) is related to \( t' \) and \( \alpha' \) through Eq. (94). Since the variations \( \delta t' \) are now subject to the constraints \( \nabla^0 \cdot \delta t' = 0 \) in \( V_0 \) and \( \ddot{u} \cdot \delta t' = 0 \) at \( S_{en} \), and since the variation \( \delta \alpha' \) are subject to the constraint that \( \alpha'^N \cdot \delta \alpha' \) be skew-symmetric, it can be shown that the vanishing of the first variation of the above functional leads to: (1) compatibility condition, \( (\nabla^0 u')^T = \)}
\( \mathbf{x} \cdot (I + h^N) + \mathbf{x}^N \cdot h \); (2) the AMB condition, Eq. (109); and (3) the DBC, Eq. (113).

Once again, inasmuch as the AMB condition follows unambiguously as the Euler equation and the admissible \( \mathbf{t}^* \) is required to satisfy only the uncoupled linear LMB condition Eq. (107) which can be satisfied easily by letting \( \mathbf{v}^0 \times \Psi + \mathbf{t}^p \), where \( \mathbf{t}^p \) is any particular solution such that \( \mathbf{v}^0 \cdot \mathbf{t}^p = - \rho^0 \mathbf{B} \), and the TBC, Eq. (12), the TL rate complementary energy principle of Eq. (130), is most consistent and useful for purposes of engineering application.

As in Eq. (75) for the UL rate case, in the application of Eq. (130) to a finite element assemblage the interelement traction reciprocity condition is handled through the introduction of appropriate Lagrange multipliers. Then

\[
\pi_{m0}(\mathbf{t}^*; \mathbf{x}; \hat{n}_m) = \sum_m \int_{V_m} \left\{-R^*(\mathbf{t}^*) - \mathbf{t}^* \cdot [\mathbf{x} \cdot (I + h^N)]
\right.
\]
\[
- \frac{1}{2} \mathbf{t}^{N_T} \cdot [\mathbf{x} \mathbf{a}^N \mathbf{T} \cdot \mathbf{x} \cdot (I + h^N)] \right\} dV 
\]
\[
+ \sum_m \int_{S_{m0}} \mathbf{t}^* \cdot \hat{n}_m ds + \sum_m \int_{\rho_{m0}} (n_1 \mathbf{t}^* \cdot \hat{n}_m dho \right)
\]

As in the UL rate case (see discussion following Eq. 75), the functional in Eq. (131) can be used to develop a finite element "stiffness matrix" method, and Eq. (130) can be used to develop a finite element "flexibility matrix" approach.

It is noted that Eq. (130), whose development has been based on independent considerations, is analogous to a principle derived by Fraeijs de Vibeke [6] which, however, governs the total deformations of a compressible elastic solid. It is also noted that a TL rate principle equivalent to that of Fraeijs de Vibeke [6] was developed and used in the context of an assumed stress-rate finite element method to solve certain finite strain problems for nonlinear elastic, compressible as well as incompressible solids, by Murakawa and Atluri [21, 22].

Finally, we remark that the TL rate complementary energy principle for elastic-plastic solids given in Eq. (130) differs slightly from the one for nonlinear elastic solids given earlier by Murakawa and Atluri [21] in which the third term in the volume integrand on the right-hand side of Eq. (130) is replaced by the term \( - \mathbf{t}^{N_T} \cdot [\mathbf{x} \cdot (I + h^N)] \). In that case, as an Euler condition, the exact AMB condition Eq. (109) is approximated by the requirement that \( (I + h^N) \mathbf{t}^* \cdot \mathbf{a}^N + h^N \mathbf{t}^* \cdot \mathbf{a}^N \) be symmetric (although the iterative correction procedures employed by Murakawa and Atluri [21] served to correct the above approximation in the rate AMB condition). In this sense the principle currently stated through Eq. (130) [which is equally applicable to nonlinear elastic solids when the potential \( R^*(\mathbf{t}^*) \) is appropriately defined] is an entirely consistent TL rate complementary energy principle for finite strain analysis of elastic as well as of elastic-plastic solids.

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Addendum: On Some New General and Complementary Energy Theorems for the Rate Problems in Finite Strain, Classical Elastoplasticity

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In connection with Eqs. (62)-(67), we have recently verified this assertion: let \( \mathbf{r} \) and \( \mathbf{\dot{r}} \) be linearly related by Eq. (62) with the restriction on \( *L_{ijk} \); that Eq. (37) be satisfied; then Eq. (67) follows. Nevertheless, whether a complementary rate principle exists in the \( \mathbf{r} \)-\( \mathbf{\dot{r}} \) formulation remains doubtful. Given an invertible linear relation between \( \mathbf{\sigma} \) and \( \mathbf{\epsilon} \), the corresponding relation between \( \mathbf{t} \) and \( \mathbf{\dot{t}} \) need not be invertible; it is singular whenever a certain eigenvalue-like condition on the current stress holds. We will present these results in more detail elsewhere.