ALTERNATE STRESS AND CONJUGATE STRAIN MEASURES, AND MIXED VARIATIONAL FORMULATIONS INVOLVING RIGID ROTATIONS, FOR COMPUTATIONAL ANALYSES OF FINITELY DEFORMED SOLIDS, WITH APPLICATION TO PLATES AND SHELLS—I

THEORY†

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Abstract—Attention is focused in this paper on: (i) definitions of alternate measures of "stress-resultants" and "stress-couples" in a finitely deformed shell (finite mid-plane stretches as well as finite rotations); (ii) mixed variational principles for shells, undergoing large mid-plane stretches and large rotations, in terms of a stress function vector and the rotation tensor. In doing so, both types of polar decomposition, namely rotation followed by stretch, as well as stretch followed by rotation, of the shell midsurface, are considered; (iii) two alternate bending strain measures which depend on rotation alone for a finitely deformed shell; (iv) objectivity of constitutive relations, in terms of these alternate strain "stress-resultants" and "stress-couple" measures, for finitely deformed shells. To motivate these topics, and for added clarity, a discussion of relevant alternate stress measures, work-conjugate strain measures, and mixed variational principles with rotations as variables, is presented first in the context of three-dimensional continuum mechanics.

Comments are also made on the use of the presently developed theories in conjunction with mixed-hybrid finite element methods. Discussion of numerical schemes and results is deferred to the Part II of the paper, however.

NOMENCLATURE

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<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
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<tbody>
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<td>B</td>
<td>undeformed body</td>
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<td>A</td>
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<td>another set of base vectors in b</td>
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<td>N</td>
<td>second fundamental form of S</td>
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<tr>
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<td>unit normal to s</td>
</tr>
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<td>J</td>
<td>mid-surface deformation gradient tensor</td>
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<tr>
<td>R</td>
<td>rigid rotation tensor</td>
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<td>A</td>
<td>base vectors on S</td>
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<td>second fundamental form of S</td>
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<td>A</td>
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<td>mid-surface deformation gradient</td>
</tr>
<tr>
<td>E_0</td>
<td>polar decomposition of E</td>
</tr>
<tr>
<td>U_0</td>
<td>mid-plane in a Kirchhoff-Love theory</td>
</tr>
<tr>
<td>A_0</td>
<td>first Piola-Kirchhoff stress tensor</td>
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†This paper is presented to my good friend, Prof. Kyuchiro Washizu on the occasion of his 60th birthday and the completion of a distinguished academic career at the University of Tokyo.

‡Regents' Professor of Mechanics.
The subject of alternate stress measures in a finitely deformed solid continuum, to be sure, has been discussed in what the author believes to be, a somewhat of a fragmentary form in various works such as those of Eringen[4], Truesdell and Noll[11], Hill[5], Biot[6], Lure[7], Fraejs de Veubeke[8], and the author and his colleagues[9,10], and others. However, a concise summary of these measures, the motivation for their introduction, their physical interpretation, and the strain-measures which are work-conjugate to these stress measures, do not appear to have been documented in a single source, with an unified notation. An attempt is made herein to meet this objective while, in doing so, certain other new stress-measures and their conjugate strain measures are introduced.

Employing the various stress and strain measures, certain mixed variational principles, primarily those involving finite-rotations of material elements as direct variables, and which can form the basis for efficient mixed-hybrid finite element methods[9,10] are discussed. More comments of an introductory nature are included at the beginning of the section dealing with this topic later in this paper.

Turning now to the primary objective of this paper, we note that linear and nonlinear shell theories have been the objects of fruitful scientific preoccupation of many a distinguished mechanical such as Sanders[11], Koiter[12], Budiansky[13], Reissner[14], Simmonds[15], Pietraszkiewicz[16], Wempner[17], and many more. The works of Sanders[11] and Budiansky[13] deal with nonlinear shell theories wherein the field equations and boundary conditions are written in terms of mid-surface displacement components. The resulting sets of equations are well-known to be quite complicated. One can hardly, in the author’s opinion, take issue with the statement of Simmonds[15] that, “the equations of motion of (even) an elastically isotropic shell undergoing small strain but large rotation are of immense complexity. To mount a frontal assault on THE SHELL PROBLEM—to solve the equations of motion for arbitrary geometry and arbitrary initial conditions—seems patently absurd”. It is in this sense that it is imperative to look at alternative ways of formulating “THE SHELL PROBLEM” so that a rational application of computational mechanics can at least partially meet the objective of solving “the shell problem”, rather than relying solely on ad hoc, but perhaps expedient and “inexpensive” techniques such as employing 3-D isoparametric finite elements in connection with “reduced-integration”. However, all considerations of computations of “the shell problem” are deferred to Part II of the present paper.

Important contributions in the direction of novel ways of writing shell equations, with a view towards their simplification, have been made by Reissner[18-21], Simmonds and Danielson[22,23], and Pietraszkiewicz[16]. Reissner showed that the equations for nonlinear axisymmetric shells of revolution can be written in a much simpler form in terms of rotation and stress function than in terms of displacements. Later, Simmonds and Danielson[23] attempted to obtain similar simplifications for arbitrary shells, using a finite rotation vector and a stress function vector. The work in [23] may now be recognized to be based on a polar-decomposition of shell mid-surface deformation gradient into a rigid rotation followed by pure stretch. In
this process, they [23] define a bending strain measure that is dependent solely on rigid rotations. Pietraszkiwicz, on the other hand, while not necessarily having the same objectives as in [23], presents an exhaustive study of the formulation of basic relations of the nonlinear shell theory in the Lagrangean description, the theory of finite rotations in shells, and other associated problems.

The present work, motivated in part by that of Simmonds, attempts the following, instead: (i) the several alternate measures, some believed to be introduced newly here, of stress-resultants and stress-couples in a finitely deformed shell are defined naturally from their counterparts in 3-D continuum mechanics, (ii) the equations of force and moment balance for a finitely deformed shell are written down concisely in terms of these alternate stress measures. These equations exhibit their essential similarity to their 3-D counterparts, (iii) mixed variational principles involving the rigid rotation tensor and stress function vector are developed for arbitrary finite deformations (arbitrary mid-plane stretch as well as arbitrary rigid rotation) of an arbitrary shaped shell. In doing so, both types of polar-decomposition (a) pure mid-plane stretch followed by rigid rotation, as well as (b) rigid-rotation followed by a pure mid-plane stretch are considered. In the case of (b) the present results are compared with those of [23] and the differences are critically examined; (iv) in the context of a semilinear isotropic elastic material, the principle of objectivity is critically examined; (iv) in the context of a semilinear isotropic elastic material, the principle of objectivity is critically examined; (v) evenhough the present theory is valid for arbitrary mid-plane stretches, and arbitrary rotations, no ad hoc definitions of "modified stress-resultants" and "modified bending strains" are employed, as appears to be the case in the celebrated works of Koiter [12], Sanders [13], and Budiansky [14].

The application of the present theories in conjugation with the mixed hybrid finite element methods, currently underway, will be reported in the forthcoming Part II of the paper.

1. PRELIMINARIES

In the interest of clarity and completeness, the pertinent ideas in the paper are developed from first principles.

Let \( P \) be a point with a position vector \( R \), in the undeformed body, \( 'B' \). Let \( p \) be the map of \( P \) in the deformed configuration, \( 'b' \), and let the position vector of \( p \) be \( r \). In general, \( R \) and \( r \) may be measured in different coordinate frames. Let

\[
dR = Q_d \, d\xi^d (1.1)
\]

where \( \xi^d \) are general, curvilinear, coordinates in \( B \). Then, let

\[
d\tau - g_\xi \, d\eta^\xi - g_\eta \, d\xi^\eta (1.2)
\]

where \( \eta^\xi \) are another set of curvilinear coordinates introduced in \( b \), while \( \xi^d \) and \( b' \) are the "convected" coordinates

For instance, \( \xi^d \) in \( 'B' \), and \( \eta^\xi \) in \( 'b' \) can both be cartesian, while \( \xi^d \) in \( b' \) will, in general, be convected into curvilinear coordinates. The deformation gradient tensor \( F \), can be written as:

\[
d\tau = F \, dr = dR \, F^T (1.3)
\]

\[
d\tau = \gamma^{-1} \, d\tau = d\tau \, \gamma^{-T} (1.4)
\]

where, \( (\cdot)^{-1} \) denotes an inverse, and \( (\cdot)^T \) denotes a transpose. The component representations of \( F \) and \( F^{-1} \) are

\[
F = \nabla \eta^\xi \, g_{\xi \xi} G^\xi = \nabla \xi^\eta \, g_{\eta \eta} G^\eta; \quad F^{-1} = \nabla_{\eta^\xi} \, g_{\xi \xi} G^\xi = \nabla_{\xi^\eta} \, g_{\eta \eta} G^\eta.
\]

Because of the one-to-one nature of deformation, for every \( dR \), there is a non-zero \( d\tau \). Thus \( F \) is non-singular, and can be decomposed into a pure-stretch and rigid rotation, through the polar-decomposition theorem, as:

\[
F = R \cdot U = V \cdot R (1.6)
\]

such that

\[
R \cdot R^T = I = R^T \cdot R (1.7)
\]

and

\[
U^T = F^T \cdot F; \quad V^2 = F \cdot F^T (1.8)
\]

where \( U \) and \( V \) are symmetric, positive definite, stretch tensors, \( R \) a rigid-rotation tensor (and hence orthogonal), and \( I \) is an identity tensor. The physical interpretation of \( F = R \cdot U \) is that \( dR \) is first stretched at \( P \) to \( dR^* \) through \( dR^* = UdR \), then \( dR^* \) at \( P \) is transported parallelly to \( p \), and rotated at \( p \) to \( d\tau \) at \( p \) through \( d\tau = RdR^* \). Likewise, the interpretation of \( F = V \cdot R \) is that \( d\tau \) at \( p \) is transported parallelly to \( p \), and rotated at \( p \) to \( d\tau^* \) at \( p \) through \( d\tau^* = Rd\tau \). Then \( d\tau^* \) at \( p \) is stretched to \( dR \) at \( p \) through \( d\tau = Vd\tau^* \).

Now, suppose that there is an oriented area \( (da) \) with a unit outward normal \( n \) [i.e. when \( (da) \) is at the outer boundary of the solid, the normal is outwards], at \( p \). Let the image at \( P \) in \( 'B' \), corresponding to \( (da) \), be \( NdA \). It can be shown [1] from purely geometrical considerations, that

\[
(NdA) = \frac{1}{J} (dana) \cdot F (1.9)
\]

\[
(dan) = J (dAN) \cdot F^{-1} (1.10)
\]

where \( J \) is the absolute determinant of \( F \). It can also be seen that \( J = d\tau / d\tau^* = \rho_0 \rho \), where \( \rho_0 \) is the mass-density at \( P \), while \( \rho \) is that at \( p \). It is evident that,

\[
J^2 = \det (F^T \cdot F) = \det U^2 = \det V^2 (1.11)
\]

hence, \( J = \det U = \det V \), while the determinant of \( R \) is 1. It is noted that the parallel translation and rigid rotation through \( R \) does not produce any length, area, and volume changes of material elements, and these are solely due to stretches, \( U \) or \( V \). Thus, if an oriented area \( (dAN) \) is considered at \( P \), the stretch \( U \) maps this into an oriented area \( d\tau^* \); the rotation \( R \) then has the effect of mapping \( dAN \) to \( d\tau \); thus only the unit normal \( n^* \) has changed to \( g \), but the area \( da \) is unchanged by \( R \).

Thus, we may write:

\[
F = R \cdot U; \quad d\tau^* = JU^{-1} \cdot (dAN) = J(dAN) \cdot U^{-1} (1.12)
\]
Likewise in the case of rotation $\mathcal{R}$ followed by stretch $\mathcal{V}$, we may consider the undeformed $(dAN)$ to be mapped to $(dAN^*)$ by $\mathcal{R}$, while $(dAN^*)$ is mapped to $(dAN)$ by $\mathcal{V}$. Thus,

$$F = \mathcal{V} \mathcal{R}: (dAN^*) = \mathcal{R}(dAN) = (dAN)_\mathcal{R} \mathcal{V}^{-1}.$$

With the above preliminaries, we now consider the definitions of alternate "stress" measures that are useful in characterizing the internal forces in a finitely deformed (with finite rotations as well as finite stretches) solid.

### 2. Alternate Stress Measures

Let $(dag)$ be an oriented differential area at $p$, in the deformed solid $'b'$. Let the internal force acting on this area be $df$. The "true" stress tensor, or often-called as the Cauchy stress tensor, is defined through the fundamental relation,

$$df = (dag)_\tau.$$

In component form, we write:

$$\eta = n^m g_m = n^m g^m = n^* \cdot g^* = n^* \cdot g^*$$

wherein the usual summation convention is employed. The $(\ast)$ in $n^*_\tau$ is used to emphasize that these are components in convected bases $g_\tau^*$ at $p$ as opposed to the bases $G_\eta$ at $P$. Likewise, the component representation of $\tau$ in the well-known dyadic notation, is:

$$\tau = \tau \cdot g_\tau^* = \tau \cdot g^*_\eta = \tau \cdot g^* \cdot g_\tau^* = \tau \cdot g^* \cdot g^*_\eta = \tau \cdot g^* \cdot g_\tau^* = \tau \cdot g^* \cdot g_\tau^*.$$

We now introduce several alternative tensors of "stress measures" through their following fundamental relations to the differential force vector $df$ acting on an oriented surface $(dag)$ at $p$ in $'b'$:

$$df = (dag)_\tau.$$

The physical interpretations of the various above "stress" tensors can be given from the relations:

$$(dAN)_\tau = df, R = R^T df = df^*$$

The tensor $\tau$ is often referred to as the first Piola-Kirchhoff or the Piola-Lagrange [1, 8] stress tensor. It is "derived" by moving $df$ acting on $(dag)$ in parallel transport to the image $(dAN)$ in the undeformed configuration $'B'$. By using the geometrical relation (1.9) it is seen that:

$$\tau = J(F^{-1}) \cdot \tau.$$

As is known, and to be seen later, in the absence of body couples, $\tau$ is symmetric. From (2.6) above, it is seen that $\tau$ is in general unsymmetric. The component representation of $\tau$ can be seen to be:

$$\tau = J(F^{-1} \cdot \tau F^{-T}).$$

Assuming that $\tau$ is symmetric, $\tau$ is symmetric. Its component representation is seen to be:

$$(s)_\eta = (s)_\tau \cdot g_\tau^*.$$

Thus the raising and lowering of components of $s$ in $G_\eta G_\eta$ basis is performed with the metric $G_\eta^T G_\eta$ and $G_\eta^T G_\eta$.

The tensor $\sigma$ of (2.4g) is often referred to as the Kirchhoff stress. It is simply a scalar ($J$) multiple of $\tau$. When $\tau$ is symmetric, so is $\sigma$. Its component representation is:

$$\sigma = J \tau^T = J \tau^T.$$

Thus, the raising and lowering of components of $\sigma$ in $G_\eta G_\eta$ basis is performed with the metric $G_\eta^T G_\eta$ and $G_\eta^T G_\eta$.
and
\[ \sigma_{M^*N^*} = \sigma^{KJ^*} g_{K^*M} g_{J^*N}. \tag{2.14} \]

Thus the raising/lowering of indices of components of \( \sigma \) in the convected basis system is performed with the metric \( g^{KL} \), \( g_{KL} \). It is seen from (2.9) and (2.13)

\[ \sigma^{MN} = s_1^{MN} \text{ but } \sigma_{M^*N^*} \neq (s_1)^{M^*N^*}. \tag{2.15} \]

Thus the contravariant components of \( q \) in the convected bases \( g_j \) are numerically equal to the contravariant components of \( x_l \) in the bases \( G_k \). This, however, does not, of course, imply that \( \sigma = s_1 \); since the covariant and mixed components of these tensors are not equal. From this viewpoint, it is then evident that in fact we may create 4 different tensors \( s_\alpha \) (\( \alpha = 1, 2, 3, 4 \)) such that:

\[ s_1 = \sigma^{MN} G_M G_N = (s_1)^{M^{K^*}} G_{M^{K^*}} G_{N^{L}}, \tag{2.16} \]

\[ s_2 = \sigma^{MN} G^M G^N = (s_2)^{M^{K^*}} G^{M^{K^*}} G^{N^{L}}, \tag{2.17} \]

\[ s_3 = \sigma_{M^*N^*} G^{M^*} G^{N^*} = (s_3)^{M^{K^*}} G_{M^{K^*}} G_{N^{L}}, \tag{2.18} \]

\[ s_4 = \sigma_{M^*N^*} G^M G^N = (s_4)^{M^{K^*}} G^{M^{K^*}} G^{N^{L}}. \tag{2.19} \]

Since,
\[ g_m = g_{L^*} \frac{\partial L^*}{\partial \xi^m} \text{ and } g^m = \frac{\partial m^*}{\partial \xi^m} g_{K^*} \tag{2.20} \]

it is seen from the relations of the type \( \sigma_{mn} g^m g^m = \sigma_{M^*N^*} G^M G^N \), that
\[ (s_2)_{KL} = \sigma_{K^*L^*} \frac{\partial m^*}{\partial \xi^m} \frac{\partial m^*}{\partial \xi^m} \tag{2.21} \]

and
\[ (s_3)^{MN} = (s_2)_{KL} G^{M^*} G^{N^*}. \tag{2.22} \]

Thus, from (2.21) and (1.5) it is seen that:
\[ s_2 = F^T \sigma. F = (F^T, \tau. F). \tag{2.23} \]

When \( \tau \) is symmetric, so is \( s_2 \), which\(^*\) has been referred to as the "convected stress tensor" by Truesdell and Noll[1]. However a more apt definition of \( s_2 \), as sketched above, simply is that it is an induced tensor whose covariant components in the \( G_k \) bases are equal to the contravariant components of \( q \) in the convected bases \( g_k \). Likewise, one can show that:
\[ s_3 = \sigma_{M^*N^*} G^{M^*} G^{N^*} = F^{-1} \sigma. F = (F^{-1}, \tau. F) \tag{2.24} \]

when \( \tau \) is symmetric, \( s_3 \) is not, in general. The author did not succeed in finding a "historical" reference to \( s_3 \). We can also see that:
\[ s_4 = \sigma_{M^*N^*} G^{M^*} G_{N} = \sigma_{M^*N^*} \frac{\partial m^*}{\partial \xi^m} \frac{\partial m^*}{\partial \xi^m} g_{K^*} \tag{2.25} \]

\(^*\)More correctly, Truesdell and Noll[1] refer to \((1/J)s_2\) as the convected stress tensor.

We now consider the meaning of \( \Gamma \). First consider the polar decomposition \( F = R. U \). The stretch \( U \) transforms \( \mathcal{N} dA \) to \((\mathcal{N} d\alpha) \) as in (1.12), and \((\mathcal{N} d\alpha) \) is rotated to \((\mathcal{N} d\alpha) \) as in (1.13). After stretching by \( U \), internal forces are generated in the solid. Let the force vector acting on \((\mathcal{N} \times d\alpha) \) be \( df^\alpha \). During rotation by \( R \), \( df^\alpha \) is transformed to \( df = R. df^\alpha = df^\alpha R^T \); thus \( df^\alpha \neq df \). Thus \( \Gamma \) is the "stress" tensor derived from the traction \( df \) acting on \((\mathcal{N} d\alpha) \) as seen from (2.5c). Thus it is a stress tensor in the stretched but not rotated configuration, and, like the true stress \( \tau \), is measured per unit deformed area. Thus
\[ (dA^\alpha) \Gamma = df^\alpha \neq df^\alpha R - (dA^\alpha) \tau R \]
\[ = (dA^\alpha) R^T, \tau R. \tag{2.26} \]

Thus,
\[ \Gamma = R^T, \tau R. \tag{2.27} \]

If \( \tau \) is symmetric, so is \( \Gamma \), which "bears" no name as far as the author knows. Suppose now we move \( df^\alpha \) acting on the "stretched" configuration \((\mathcal{N} d\alpha) \) in parallel transport, onto the image \((\mathcal{N} d\alpha) \) at \( P \). We can then derive a stress tensor \( \tau^\ast \) such that
\[ (dA^\alpha) \tau^\ast = df^\alpha \neq df^\alpha R - (dA^\alpha) \tau R \]
\[ = (dA^\alpha) R^T, \tau R \tag{2.28} \]

or
\[ \tau^\ast = (dA^\alpha) R^T, \tau R \tag{2.29} \]

When \( \tau \) is symmetric, \( \tau^\ast \) is in general unsymmetric. It is often referred to [2,24] as the Biot-Lure' stress tensor. It is convenient to introduce a symmetric "stress" tensor \( \tau \) such that
\[ \tau = (\tau^\ast + \tau^\ast T)/2. \tag{2.30} \]

It is worth mentioning, however, that while \( \tau^\ast \) is naturally related to the concept of a "traction" as in (2.28), no similar explanation is possible, in general, for \( \tau \). The symmetric tensor \( \tau \) is sometimes referred to [8,9] as the Jaumann stress tensor.

Finally, we consider the meaning of \( \tau^\ast \). First consider the polar decomposition \( \mathcal{J} = \mathcal{V} R. \). The rotation \( \mathcal{K} \) transforms \((\mathcal{N} d\alpha) \) to \((\mathcal{N} d\alpha) \), which in turn is mapped to \((\mathcal{N} d\alpha) \) through the stretch \( \mathcal{V} \). If the force \( (\mathcal{N} d\alpha) \) acting on \((\mathcal{N} d\alpha) \) in the final configuration is moved, in parallel transport, onto its image \((\mathcal{N} d\alpha) \) in the rotated but not stretched configuration, we can derive a stress tensor \( \tau^\ast \) such that:
\[ (N^\ast d\alpha) \tau^\ast = df = (dA^\alpha) \tau \]
\[ = (N^\ast d\alpha) J^{V \supset \tau} \tag{2.31} \]

Thus,
\[ \tau^\ast = J^{V \supset \tau} \tag{2.32} \]

Even if \( \tau \) is symmetric, \( \tau^\ast \) is unsymmetric, in general. To the author's best historical knowledge, no name is
attached to $T^*$. It is convenient to introduce a symmetric $T$ such that

$$2T = T^* + T^*.$$  \hspace{2cm} (2.33)

It is noted that while $T^*$ is related to the concept of a traction, $T$ is not in general.

For convenience of later reference, we summarize the various “stress” measures and their interrelationship in the following.

$$t = J(F^{-1} \cdot \gamma) = s_1, F^T = JU^{-1}, T^* = \epsilon^* R^T = R^T T^*.$$  \hspace{2cm} (2.34)

$$s_1 = J(F^{-1} \cdot \gamma, F^{-T}) = \epsilon^* R^T,$$  \hspace{2cm} (2.35)

$$s_2 = K^T \cdot \sigma F$$  \hspace{2cm} (2.36)

$$s_3 = F^{-1} \cdot \sigma E = s_4 T$$  \hspace{2cm} (2.37)

$$\Gamma = R^T \cdot \epsilon R = \frac{1}{2} U_{ij} R_{ij} = \frac{1}{2} U_{ij} S_{ij} U$$  \hspace{2cm} (2.38)

$$\epsilon^* + JF^{-1} \cdot \Gamma = \frac{1}{2} \left( s_1 + U_{ij} S_{ij} \right) U$$  \hspace{2cm} (2.39)

$$\Gamma^T = J(F^{-1} \cdot \gamma) = \epsilon^* R^T,$$  \hspace{2cm} (2.40)

$$T^* = J(R, F^{-1} \cdot \gamma) = \epsilon^* R^T = R^T \epsilon R^T$$  \hspace{2cm} (2.41)

$$T^* = \frac{1}{2} (R_{ij} + \epsilon R, R^T) = \frac{1}{2} R_{ij} (\epsilon^* + \epsilon R^T) R^T$$  \hspace{2cm} (2.42)

$$T^* = \frac{1}{2} R_{ij} (\epsilon^* + \epsilon R^T) R^T$$  \hspace{2cm} (2.43)

3. STRESS-WORKING-RATE, CONJUGATE “STRAIN” MEASURES AND ELASTIC POTENTIALS

Consider the deformed body ‘$b$’, wherein the true stress tensor at point ‘$p$’ is $\tau$. Consider a small volume $V^*$ with a bounding surface $s$ at $p$. When viewed as a free-body, this material volume is acted on by: (i) body forces $(f-a)$, where $g$ is the absolute acceleration of the material particle, and (ii) surface traction $(\sigma \cdot n)$ per unit area, exerted by the surrounding medium. Let the instantaneous velocity of the material particle at $p$ be $\dot{v}$.

$$\dot{v} = v^m g^m = v^i \cdot g^i$$  \hspace{2cm} (3.1)

We define a second-order velocity gradient tensor $\epsilon$ as:

$$\epsilon = \epsilon^* R^T,$$  \hspace{2cm} (3.2)

where $(\cdot)$ denotes a covariant derivative w.r.t. the coordinate identified by the follower index. It is noted that the covariant differentiation in (3.2) involves the metric $g_{mn}$ or $g^{MN}$. It can be shown easily[5] that the stress-working rate (or the rate of increase of internal energy) per unit volume in $b$, denoted by $\dot{W}$ is given by:

$$\dot{W} = \frac{1}{2} \epsilon^* + (\epsilon + e^T)$$  \hspace{2cm} (3.3)

where, by definition, $\epsilon = 1/2 (\dot{\epsilon} + \epsilon^T)$, is the “strain” rate referred to the configuration ‘$b$’.

If $\rho$ is the mass density in ‘$b$’, the virtual internal energy per unit mass, denoted by $\dot{\mathcal{W}}$ is:

$$\dot{\mathcal{W}} = \frac{1}{\rho} \dot{W}.$$  \hspace{2cm} (3.4)

If $\rho_b$ is the mass density in the undeformed body ‘$B$’, the rate of increase of internal energy per unit of undeformed volume, denoted by $\dot{W}_0$, is:

$$\dot{W}_0 = \frac{\rho_b}{\rho} \dot{\mathcal{W}} = J \dot{\mathcal{W}}.$$  \hspace{2cm} (3.5)

Hence,

$$\dot{W}_0 = JF \cdot \epsilon = \sigma \cdot \epsilon.$$  \hspace{2cm} (3.6)

Since $g$ is symmetric, from the simple properties of trace operation given in Appendix 1, it is seen that

$$\dot{W}_0 = \sigma \cdot \epsilon = \sigma \cdot \epsilon.$$  \hspace{2cm} (3.7)

Recall that

$$\dot{\epsilon} = \epsilon^* k_{g^i \epsilon^*} = \epsilon^* + \epsilon^T,$$  \hspace{2cm} (3.8)

If one expresses $p$ in the basis system $G_K$ at $P$ in $B$.

$$v = v_K G^K,$$  \hspace{2cm} (3.9)

one can see that

$$\dot{\epsilon} = v_K \dot{G}^K,$$  \hspace{2cm} (3.10)

where $(\cdot)$ denotes a covariant differentiation at $P$ with respect to the coordinate $\xi^*$ identified by the follower index $L$, and thus involves the metric at $P$. Recall that

$$F = \frac{\partial \epsilon^*}{\partial \eta^i} G^i = (G_{MK} + \eta_{MK}) G^M G^K$$  \hspace{2cm} (3.11)

and

$$F^{-1} = \frac{\partial \epsilon^*}{\partial \eta^i} G^i,$$  \hspace{2cm} (3.12)

hence,

$$\dot{F} = v_{MK} G^M G^K.$$  \hspace{2cm} (3.13)

Thus, it follows from (3.10, 12 and 13) that

$$\dot{\epsilon} = \dot{\xi} \cdot \dot{\eta}.$$  \hspace{2cm} (3.14)

We now define the right-Cauchy-Green deformation tensor $C$, the Green-Lagrange strain tensor $\gamma$, the left-Cauchy-Green deformation tensor $B$, and the Almansi strain tensor $\mu$ as follows:

$$C = F^T F = I + 2 \gamma = U^2$$  \hspace{2cm} (3.15)

$$B = F^T F = (2 \mu + I)^{-1} = V^2$$  \hspace{2cm} (3.16)

where $I$ is the identity (metric tensor).
It then follows that:

\[
\hat{\mathbf{C}} = 2\hat{\varphi} = \mathbf{U}^T \mathbf{U} + \hat{\mathbf{U}}^T \hat{\mathbf{U}}
\]

(3.17)

\[
\hat{\mathbf{E}} = \mathbf{F}^T (\mathbf{F}^T + \epsilon) \mathbf{F} = \mathbf{F}^T (\mathbf{F}^T + \epsilon) \mathbf{F}
\]

(3.18)

\[
\hat{\mathbf{B}} = \mathbf{V}^T \mathbf{V} + \hat{\mathbf{V}}^T \hat{\mathbf{V}} + \mathbf{E}^T \mathbf{E} + \hat{\mathbf{E}}^T \hat{\mathbf{E}}
\]

(3.19)

\[
\hat{\mathbf{C}} = \frac{1}{2} (\mathbf{F}^T - \hat{\mathbf{C}})^{-1} = \frac{1}{2} (\mathbf{F}^T + \epsilon \mathbf{F}^\epsilon)^{-1}
\]

(3.20)

Using (3.14, 17-20) in (3.7) and employing the properties of trace operation as in Appendix 1, we see that:

\[
\mathbf{W}_0 = \sigma \epsilon = \sigma \epsilon (\mathbf{F}^T (\mathbf{U}^T \mathbf{U} + \hat{\mathbf{U}}^T \hat{\mathbf{U}}), \mathbf{F}^{-1})
\]

(3.21)

Also,

\[
\mathbf{W}_0 = \frac{1}{2} (\mathbf{F}^T (\mathbf{U}^T \mathbf{U} + \hat{\mathbf{U}}^T \hat{\mathbf{U}}), \mathbf{F}^{-1})
\]

(3.22)

Since \( \hat{\mathbf{W}} \) is symmetric,

\[
\mathbf{W}_0 = \frac{1}{2} (\mathbf{F}^T + \epsilon \mathbf{F}^\epsilon) : \hat{\mathbf{U}} = \mathbf{r}^T \hat{\mathbf{U}}.
\]

(3.23)

Also,

\[
\mathbf{W}_0 = \frac{1}{2} (\mathbf{T}^T (\mathbf{R}^T \mathbf{R}^T) : \hat{\mathbf{U}}
\]

(3.24)

In general \( \hat{\mathbf{U}} \) and \( \mathbf{U}^{-1} \) need not be co-axial. Now we try to evaluate the term \( \hat{\mathbf{U}}^{-1} \hat{\mathbf{U}} \). To this end, let \( \lambda_1, \lambda_2, \lambda_3 \) be eigen-values of \( \mathbf{U} \) and \( \hat{\lambda}_1 \) be the representation of \( \lambda \) in the eigen-directions. (Thus, \( \lambda \) is a diagonal matrix.) Then, there exists an orthogonal transformation \( \hat{\mathbf{q}} \) such that

\[
\mathbf{U} = \hat{\mathbf{q}}^T \lambda \hat{\mathbf{q}}
\]

(3.25)

and hence,

\[
\mathbf{U}^{-1} = \hat{\mathbf{q}}^T \lambda^{-1} \hat{\mathbf{q}}
\]

(3.26)

and

\[
\mathbf{U}^{-1} = \hat{\mathbf{q}}^T \lambda^{-1} \hat{\mathbf{q}} + \hat{\mathbf{q}}^T \lambda^{-1} \hat{\mathbf{q}}
\]

(3.27)

\[
\mathbf{U}^{-1} = \hat{\mathbf{q}}^T \lambda^{-1} \hat{\mathbf{q}} + \hat{\mathbf{q}}^T \lambda^{-1} \hat{\mathbf{q}}
\]

(3.28)

\[
\hat{\mathbf{U}}^{-1} = \hat{\mathbf{q}}^T \lambda^{-1} \hat{\mathbf{q}} + \hat{\mathbf{q}}^T \lambda^{-1} \hat{\mathbf{q}}
\]

(3.29)

Now, we consider a diagonal matrix \( \mathbf{In} \lambda \) wherein the diagonal elements are \( \mathbf{ln} \lambda_1, \mathbf{ln} \lambda_2, \mathbf{ln} \lambda_3 \).Thus, we see that

\[
\frac{d}{dt} (\mathbf{ln} \lambda) = \lambda^{-1} \hat{\lambda} = \hat{\lambda}^{-1}
\]

(3.30)

We note that

\[
\frac{d}{dt} (\mathbf{ln} \mathbf{U}) = \mathbf{q}^T \mathbf{ln} \lambda, \mathbf{q} + \mathbf{q}^T \frac{d}{dt} (\mathbf{ln} \lambda) \mathbf{q} + \mathbf{q}^T \mathbf{ln} \lambda \dot{\mathbf{q}}
\]

(3.31)

Using (3.30–32) in (3.29) we see that

\[
\frac{d}{dt} (\mathbf{ln} \mathbf{U}) = \mathbf{q}^T \mathbf{ln} \lambda, \mathbf{q} + \mathbf{q}^T \frac{d}{dt} (\mathbf{ln} \lambda) \mathbf{q} + \mathbf{q}^T \mathbf{ln} \lambda \dot{\mathbf{q}}
\]

(3.32)

Thus, from (3.24), (3.26) and (3.33) it follows:

\[
\frac{d}{dt} (\mathbf{ln} \mathbf{U}) = \mathbf{q}^T \mathbf{ln} \lambda, \mathbf{q} + \mathbf{q}^T \frac{d}{dt} (\mathbf{ln} \lambda) \mathbf{q} + \mathbf{q}^T \mathbf{ln} \lambda \dot{\mathbf{q}}
\]

(3.33)

However, for materials that behave isotropically in the deformed state, it is known[1] that \( \tau \) and \( \mathbf{V} \) are co-axial (have the same eigen directions). Since \( \mathbf{V} = \mathbf{R}^T \mathbf{V}, \mathbf{R} \), and \( \mathbf{U} = \mathbf{R}^T \mathbf{V}, \mathbf{R} \), it then follows that for isotropic materials, \( \hat{\mathbf{V}} \) and \( \mathbf{U} \) are co-axial. Thus, \( \hat{\mathbf{V}}, \mathbf{U}, \mathbf{U}^{-1} \) as well as \( \mathbf{ln} \mathbf{U} \) are co-axial for isotropic materials. Thus it follows that for isotropic materials,

\[
\frac{d}{dt} (\mathbf{ln} \mathbf{U}) = \mathbf{q}^T \mathbf{ln} \lambda, \mathbf{q} + \mathbf{q}^T \frac{d}{dt} (\mathbf{ln} \lambda) \mathbf{q} + \mathbf{q}^T \mathbf{ln} \lambda \dot{\mathbf{q}}
\]

(3.34)

Thus, for isotropic materials, the stress-working rate is:

\[
\mathbf{W}_0 = \mathbf{J} \frac{d}{dt} (\mathbf{ln} \mathbf{U})
\]

(3.35)
For general materials, however, it is to be stressed that (3.35) applies.

Finally,

\[ W_0 = \frac{1}{2} \varepsilon : (F F^{-1} + F^{-T} F^T) \]

\[ = \frac{1}{2} \sigma : [(V_R + V_R \cdot R^{-1} \cdot V^{-1} + V^{-1} \cdot R (R^{-T} \cdot V + R^T \cdot V)] \]

\[ = \frac{1}{2} \sigma : (V^{-1} + V^{-1} \cdot V), \]

\[ + \frac{1}{2} \sigma : [V_R \cdot R^T \cdot V^{-1} + V^{-1} \cdot R \cdot R^T \cdot V] \]

\[ \frac{1}{2} \sigma : (V^{-1} + V^{-1} \cdot V), \]

\[ + \frac{1}{2} \sigma : (V^{-1} - V^{-1} \cdot \sigma \cdot V) : R \cdot R^T. \] (3.38)

The above expression is true for general materials. However, only for materials that are isotropic in \( b \), it is seen that \( \sigma, V, \) and \( V^{-1} \) are all coaxial. Thus, for isotropy,

\[ V \cdot \sigma \cdot V^{-1} = V^{-1} \cdot \sigma \cdot V. \] (3.39)

Hence, for isotropic materials, it follows that

\[ W_0 = \frac{1}{2} \varepsilon : (V^{-1} + V^{-1} \cdot V). \] (3.40)

Once again, even for isotropic materials, \( V \) and \( \sigma \) need not be coaxial. Thus letting

\[ V = \beta^T \cdot u \cdot \beta \] (3.41)

\[ \ln V = \beta^T \cdot \ln \mu \cdot \beta \] (3.42)

where \( \mu \) is the diagonal matrix representation of \( V \) in its eigen-directions, we see as before that,

\[ V^{-1} = \beta^T \cdot \mu^{-1} \cdot \beta \]

\[ \ln V = \beta^T \cdot \ln \mu \cdot \beta \]

Analogous to the development in eqns (3.34, 35), since \( \sigma, V, \) and \( V^{-1} \) are coaxial for isotropic materials, it then follows that

\[ W_0 = \varepsilon : \frac{d}{dt} \ln V. \] (3.44)

For non-isotropic materials eqn (3.38) applies. Also, from (3.35), it follows in the case of isotropy:

\[ W_0 = \frac{1}{2} \varepsilon : (V \cdot \dot{V}^{-1} + V^{-1} \cdot \dot{V}) = \frac{1}{2} (\sigma \cdot \dot{V}^{-1} + V^{-1} \cdot \sigma) : \dot{V} \]

\[ = \frac{1}{2} (T^{-1} + T^{-1}) : \dot{V} = T : \dot{V}. \] (3.45)

Eqns (3.44, 45) were also noted in [25], but their limits of applicability to materials that are isotropic in the deformed state were not emphasized.

In summary, for convenience of future reference, we note that: (i) for any general material the following holds:

\[ W_0 = \sigma : \varepsilon = I : \dot{V}^T = I : \dot{V} = \tau : \dot{U} \]

\[ = J \left( \frac{d}{dt} (\ln U) + \frac{1}{2} (U^{-1} \cdot U^{-T} - U^{-T} \cdot U^{-1}) \right) \cdot \dot{U} \]

\[ - J (\ln U - \ln (U')) \cdot \dot{U}' \]

\[ = \frac{1}{2} \sigma : (V \cdot \dot{V}^{-1} + V^{-1} \cdot \dot{V}), + \frac{1}{2} \left[ \sigma \cdot V^{-1} - V^{-1} \cdot \sigma \cdot V \right] \cdot R \cdot R' \]

\[ = T : \dot{V} + \frac{1}{2} \left[ V \cdot T^{-1} - T^{-1} \cdot V \right] \cdot R \cdot R'. \] (3.46)

(ii) only for materials that are isotropic in \( b \), the following holds:

\[ W_0 = J \left( \frac{d}{dt} (\ln U) = \sigma : \frac{d}{dt} (\ln V) = I : \dot{V} \right. \] (3.47)

In view of eqns (3.46, 47) it is recognized that: (i) for any elastic material,

\[ \frac{\partial W_0}{\partial \xi} = \xi \]

\[ \alpha = \frac{\partial W_0}{\partial \xi} \quad \tau = \frac{\partial W_0}{\partial \sigma} \]

\[ = \frac{\partial W_0}{\partial \sigma} \]

\[ \tau = \frac{\partial W_0}{\partial \xi} \] (3.48)

whereas, (ii) for isotropic elastic materials,

\[ J = \frac{\partial W_0}{\partial \ln U} \quad \sigma = \frac{\partial W_0}{\partial \ln V} \quad T = \frac{\partial W_0}{\partial \sigma} \] (3.49)

However, for any elastic material, it is to be noted that the principle of frame-indifference [1] requires that \( W_0 \) be a function only of \( F^T \cdot F \). Thus,

\[ W_0 = W_0(F^T \cdot F) = W_0(U) = W_0(R \cdot R'). \] (3.50)

Thus \( W_0 \) is a function of either \( U \) or \( R \cdot R' \).

### 3. Momentum Balance Conditions

The local momentum balance conditions for the deformed body "b" can be written as,

\[ (1 \text{LMR}): \frac{d}{dt} \int_s \tau \cdot \ddm = \int_s \tau \cdot \ddf + \int_s \tau \cdot \dddm \quad (4.1) \]

\[ (\text{AMB}): \frac{d}{dt} \int_s \tau \cdot \ddf = \int_s \tau \cdot \ddf + \int_s (\tau \cdot \ddf) \cdot \ddm \quad (4.2) \]

where: (i) LMB and AMB stand for linear and angular momentum balance, respectively (ii) \( m \) denotes mass, (iii) \( \tau = g \cdot \nu \cdot \alpha \) (iv) \( \ddf \) are prescribed body forces per unit mass. Assuming that mass is conserved, i.e. \( \ddm/dt = 0 \), we have:

\[ (\text{LMB}): \int_s \ddf + \int_s (\tau - g) \cdot \ddm = 0 \quad (4.3) \]

\[ (\text{AMB}): \int_s (\tau \cdot \ddf) + \int_s \ddf (\tau - g) \cdot \ddm = 0 \quad (4.4) \]
Let \( \nabla \) be the gradient operator at \( p \) in \( b \): i.e.

\[
\nabla = \frac{\partial}{\partial \eta} = g^r \frac{\partial}{\partial \xi^r}.
\]

(4.5)

Then, upon application of the divergence theorem, it follows that

\[
(\text{LMB}): \ \nabla \cdot \rho (\dot{\mathbf{f}} - \mathbf{a}) = 0
\]

(4.6)

and

\[
(\text{AMB}): \ \mathbf{\tau} = \mathbf{\tau}^* \quad \text{or} \quad \mathbf{\sigma} = \mathbf{a}^T.
\]

(4.7)

Upon substituting for \( df = d\mathbf{g}_T \) the various alternate representations, as given in (2.4), one can derive alternate forms of (LMB) and (AMB) in terms of the alternate stress measures defined earlier. To this end let us define \( \nabla_0 \) to be the gradient operator at \( P \) in \( B \), i.e.

\[
\nabla_0 = G^f \frac{\partial}{\partial \xi^r}.
\]

(4.8)

Thus, it can be shown easily that (LMB) and (AMB) take the following forms:

\[
\text{(AMB): } F_{U} = t^* \cdot F_{T}^T \quad \text{(4.9a,b)}
\]

\[
\text{(LMB): } \mathbf{r}_1 = \mathbf{r}_1^*; \quad \mathbf{r}_2 = \mathbf{r}_2^*; \quad \mathbf{r}_3 = \mathbf{r}_3^* \quad \text{(4.10a,b)}
\]

\[
\text{(AMB): } r^* = r^*; \quad \mathbf{U} = \mathbf{U}^* \quad \text{(4.11a,b)}
\]

\[
\text{(LMB): } \mathbf{r}_4 = \mathbf{r}_4^*; \quad \mathbf{r}_5 = \mathbf{r}_5^* \quad \text{(4.12a,b)}
\]

\[
\text{(AMB): } r = r^*; \quad \mathbf{U} = \mathbf{U}^* \quad \text{(4.13a,b)}
\]

5. Mixed Variational Formulations

In the linear theory of elasticity, the principle of minimum potential energy involves the field of compatible displacements alone as a variable and can be considered as a "primal" principle. Likewise, in the linear theory, the principle of minimum complementary energy involves the field of stress that satisfies LMB as well as AMB alone as a variable and can be considered as the "dual primal" principle. In the principle of potential energy, the condition of compatibility of displacements is introduced as a subsidiary condition through a Lagrange multiplier field (which turns out to be the stress field), it is well known that one obtains the so-called Hu-Washizu [26] three-field variational principle, which involves the stresses, strains, as well as displacements as variables. Similarly, if in the principle of complementary energy, the condition of LMB and AMB is introduced as a constraint through a Lagrange multiplier (which turn out to be displacements) one obtains the so-called Hellinger-Reissner two-field variational principle. Thus, in the linear theory, the additional field variables appearing in a multi-field variational principle also play the role of Lagrange multipliers. It then becomes a matter of philosophy whether to consider a mixed variational principle, even in a linear theory, as one that simple involves more than one field as a variable, or as one wherein certain conditions of constraint are introduced through additional Lagrange multiplier fields.

In the case of finite deformation of elastic solids, the principle of stationary potential energy still involves the field of compatible displacements alone as a variable, and can again be considered a "primal" principle. However, there does not appear to exist a principle of stationary complementary energy, involving the stress field alone as a variable, in the case of finite elasticity. For instance, if one uses the second Piola-Kirchhoff stress tensor \( \mathbf{s} \) as variable, it is well known that the resultant complementary energy principle involves both \( \mathbf{s} \) and \( \mathbf{u} \) as variables, even though \( \mathbf{u} \) does not play the role of a Lagrange multiplier to enforce any constraint condition. Moreover, as seen from (4.10), the LMB conditions for \( \mathbf{s} \) involves the deformation gradient \( F \). Several attempts have been made in recent literature, by Zubov [27], Koiter [24], de Veubeke [8], and the author [28, 29] in formulating complementary energy principles. As discussed in [9], the most consistent of such developments for finite elasticity is due de Veubeke [8], which was later extended to rate theories of elasticity and inelasticity by the author [28, 29]. The principle in [8] involves both the first Piola-Kirchhoff stress \( t \) as well as rigid rotation \( \mathbf{R} \) as variables. While \( t \) satisfies LMB priori, the presence of \( \mathbf{R} \) leaves AMB for \( t \) as an Euler Lagrange equation corresponding to a variation in \( \mathbf{R} \). Thus, in a way, the principles in [8, 28, 29] can be considered to be mixed principles. Of course, other mixed or multfield principles can be generated from the principle of stationary potential energy by relaxing the non-linear conditions of compatibility [9] for the various kinematic variables \( \mathbf{R}, F, U, V \) and \( u \). However, because of their novel application to thin bodies (plates and shells) to be treated later in this paper, we shall restrict our attention here to mixed principles involving \( \mathbf{R}, U \) or \( V, u \) and \( t, t^* \) or \( T^* \) as variables.

As first shown in [8], and later generalized in [9], a general mixed principle, for a general elastic material, and involving \( t, U, \mathbf{R} \) and \( u \) as variables can be stated as the stationary condition of the functional:

\[
F_t(U, V, R, t, \mathbf{R}) = \left( \int_{S_0} \left[ W_d(U) + t^* \cdot (U + \nabla_0 b)^T - R \cdot U \right] - \mathbf{b} \cdot \nabla_0 \mathbf{b} \right) \mathrm{d}V
\]

(5.1)

and

\[
F = \left( U + \nabla_0 b \right)^T \quad \text{(5.2)}
\]

where

\[
\delta F_t = \int_{S_0} \left( \frac{\partial W_d}{\partial U} \right) \delta U + 8 \delta t^* \cdot \left[ (U + \nabla_0 b)^T - R \cdot U \right] + t^* \cdot \left[ (U, \nabla_0 b) \right] + \left[ R \cdot U - R \cdot \delta U \right] - \rho_o b \cdot \delta u \right) \mathrm{d}V + \int_{S_0} \mathbf{R} \cdot \delta U \mathrm{d}s + \int_{S_0} \left[ N, \delta U \right] \cdot \delta u \mathrm{d}s. \quad \text{(5.3)}
\]
Noting that \( \delta R \) is subject to the constraint that \( \delta R, R^T + R, \delta R = 0 \), it is seen that the stationarity condition, \( \delta F = 0 \), leads to the Euler Lagrange Equations (ELE) and natural boundary conditions (NBC): (i) the constitutive law (CL): \( \partial W/\partial U = 1/2 (\delta R, R^T + R, \delta R) \) (ii) LMB, eqn (4.9a), (iii) AMB: eqn (4.9b) or equivalently, \( R^T, \delta U \) is symmetric, (iv) "compatibility" condition \( F = \delta U; \delta U \) (v) traction b.c (TBC) \( N_t = \delta U \) at \( S_{on} \), and (iv) the displacement b.c (DBC): \( \psi = \delta U \) at \( S_{on} \).

We now establish the contact transformation such that

\[
W_o(U) - 1/2 (t^* R + R^T t^*) U = W_o(U) - c U = - W_c(t)
\]

such that

\[
\frac{\partial W_c}{\partial t} = U.
\]

We restrict our attention now to a finite elasto-static problem. Suppose that the field \( t \) in (5.1) is constrained to satisfy the LMB, eqn (4.9a), as well as the TBC, \( N_t = \delta U \) at \( S_{on} \). Then \( U \) can be eliminated as a variable from (5.1). In addition, one can eliminate \( U \) as a variable from (5.1) by using the contact transformation (5.4). Thus, when \( U \) and \( U \) are eliminated, one can formulate a mixed variational principle involving \( t \) and \( R \) as variables, as follows:

\[
F_1(t, R, y) = \int_{V_o} \left\{ W_o(U) - c^* U - c U - \psi(U) \right\} dV + \int_{S_{on}} N_t \delta U dS.
\]

(5.6)

Noting that the variations \( \delta t \) are subject to the constraint \( \delta V_o \delta t = 0 \), and the variations \( \delta R \) are subject to the constraint \( \delta R, R^T + R, \delta R = 0 \), one can easily verify that the conditions \( \delta F_1 = 0 \) leads as its ELE and NBC: (i) the compatibility condition: \( R U = \left( I + \psi \delta y \right) T \); (ii) the AMB, i.e. \( R^T, \delta U = \text{sym} \), and (iii) the DBC, \( \psi = \delta U \) at \( S_{on} \).

Alternatively, \( F_1 \) of eqn (5.1) can be expressed in terms of \( c^*, U, R, \) and \( t \) as follows:

\[
F(t, R, y, c^*) = \int_{V_o} \left\{ W_o(U) + c^* T \left( (I + \psi \delta y) R - U \right) - \psi(U) \right\} dV - \int_{S_{on}} \psi \delta U dS - \int_{S_{on}} \psi(U - \delta U) dS.
\]

(5.7)

One can easily verify that for the above given \( F_1 \), the condition \( \delta F_1 = 0 \) leads to the same ELE and NBC as those that followed from (5.3).

Suppose one establishes the contact transformation

\[
W_o(U) - 1/2 (t^* R + R^T t^*) U = - W_c(t)
\]

thereby eliminating \( U \) as a variable from (5.7), and, in addition, eliminates \( \psi \) from (5.7) by satisfying the LMB, eqn (4.12) as well as the TBC, then one formulates a mixed principle with \( c^* \) and \( R \) as variables, with:

\[
F_2(t^*, R) = \int_{V_o} (- W_c(t) + c^* R) dV + \int_{S_{on}} N_t c^* R^T \delta U dS.
\]

(5.9)

The above mixed principles, with \( F_1 \) and \( F_2 \) as the respective functionals, as in eqns (5.1), (5.6), (5.8) and (5.9) are valid for general elastic materials.

We now explore the mixed variational principles based on the polar decomposition, \( F = V . R \). In this connection, we first note the important restriction: (i) the principle of frame-indifference[1] requires that \( W_o \) be a function of \( R^T, R \) only, i.e. \( W_o \) is a function of \( U \) and hence a function of \( V \) only through the variable \( R^T, V, R \).

With this restriction in mind, one can formulate a mixed variational principle, which can be stated as the condition of stationarity of the functional:

\[
F_1(\psi, V, R, y) = \int_{V_o} \left\{ W_o(R T + V, R) + c^* T \left( (I + \psi \delta y) R - V, R \right) - \psi(U) \right\} dV - \int_{S_{on}} \psi \delta U dS - \int_{S_{on}} \psi(U - \delta U) dS.
\]

(5.10)

Thus we write (5.11) as:

\[
\frac{\partial W_o}{\partial \delta V} - 2 V . R . c^*.
\]

(5.12)

The ELE and NBC that follow from \( \delta F_1 = 0 \), with \( \delta F_1 \) as in (5.13), are:

\[
\frac{\partial W_o}{\partial \delta V} - 1/2 \left( t^* R + R^T t^* \right) = \psi
\]

(5.14)

\[
(\psi, \psi) = 0
\]

(5.15)

\[
(I + \psi \delta y) R = V . R
\]

(5.16)

\[
(2 V . R . t^* + V . t^* R) . \delta R = 0
\]

(5.17)

or

\[
(\psi, \psi) = 0
\]

(5.18)

It is worth noting that even though the stress working density \( W_o \) is equal to \( T . V \) only in the case of isotropic
elastico, the above variational principle, \( \delta F_1 = 0 \) of eqn (5.13), is valid for general elastic materials. This is due to the fact that \( \delta W_0 = \delta W_0/\delta V \cdot \delta V + (\delta W_0/\delta R) \cdot \delta R \) in the above principle, and \( W_0 \) is consistently expressed as a function of \((R^T, V, R)\). The above variational principle, \( \delta F_1 = 0 \) of (5.12), will be shown later to have certain interesting applications to thin bodies such as shells.

Let us now establish the following contact transformation:

\[
W_0(R^T, V, R) - \frac{1}{2} (t^T R^T + R^T t) V = W_0(R^T, V, R) - F(R^T, V, R)
\]

where

\[
W_0 = W_0(I) - \varepsilon:t' = W_0(t)
\]

Such that

\[
\partial W_0/\partial t_t = V.
\]

Using eqn (5.19), one may eliminate \( V \) from \( F_1 \) of (5.10). Likewise if the LMB for \( t \), viz. (4.9a) and the TBC for \( \varphi \) are met, \( \varphi \) may also be eliminated from (5.10). Thus, one may formulate a mixed principle involving \( t \) and \( R \) as variables, with the associated functional:

\[
F_1(t, R) = \int_V \left( - W_0(R^T, V, R) + t^T \cdot \tau \right) dV + \int_S \tau \cdot N dS
\]

which, of course, is identical to that in eqn (5.6); and has the same ELE and NBC as those associated with \( \delta F_2 = 0 \) of (5.6).

Equation (5.21) can be alternatively expressed in terms of \( T \) and \( R \) as:

\[
F_1(T, R) = \int_V \left( - W_0(T^T, R, R) + T^* \cdot \tau \right) dV + \int_S \tau \cdot N dS
\]

The mixed variational principles, with the associated functionals \( F_1 \) and \( F_2 \), all involving the rigid rotation \( R \), in addition to alternate stress measures (in \( F_2 \)) and strain measures and displacements (in \( F_1 \)) have interesting applications to the mechanics of thin bodies such as plates and shells. This is primarily due to the reason that in such bodies, finite deformations usually imply large rotations but only small stretches. The treatment of rigid rotations as independent variables often simplifies the description of kinematics in such problems, and leads to simpler sets of field equations. This is the subject of the remainder of this report.

6. APPLICATIONS TO THIN BODIES: KINEMATICS OF A FINITELY DEFORMED SHELL

First of all, we define a body to be thin when one of its dimensions ("thickness") is much smaller compared to the other two characteristic dimensions, and thus the definition encompasses the cases of thin "plates" and "shells".

We define the reference surface of the undeformed shell to be \( S \), on which a generic point is denoted by \( P_0 \). We define the surface \( S \) by two curvilinear coordinates \( \xi_1, \xi_2 \) \((\alpha = 1,2)\). We define the unit normal to \( S \) at \( P_0 \) to be \( N \), and \( t' \) be the coordinate along \( I \) such that an arbitrary point \( P \) in the shell domain in its undeformed configuration, is defined by \( \xi \) such that

\[
R = R_0 + N \xi^3
\]

where \( R_0 \) is the position vector of \( P_0 \). The base vectors at \( P \) are:

\[
\dot{B} = - (\partial R/\partial \xi^3) A^* A^0 = B \cdot A^0 A^*; \quad \dot{B} = - (\partial R/\partial \xi^3) A^* A^0 = B \cdot A^0 A^*.
\]

\( B \) is the curvature tensor of \( S \) and is symmetric.

Because of their simplicity in characterizing the deformation in the special case of shells, plates, and beams, we shall use convected coordinates throughout the remainder of this report. Thus, after deformation let \( S \) be mapped into \( s \), and \( P_0 \) and \( P \) be mapped \( P_{0s} \) and \( P_s \) respectively. The point \( P_{0s} \) will still be defined by the convected coordinates \( \xi_1, \xi_2 \).

To define \( P_s \), we shall consider the well-known Kirchhoff-Love hypotheses, viz. (i) the material fibers originally normal to \( S \) are mapped into fibers normal to \( s \), and (ii) there is no thickness stretch, to be plausible. Let the position vectors of \( P_0 \) and \( P \) be \( R_0 \) and \( R \) respectively. Thus,

\[
R_0 = R_0 + u, \quad u = \delta R_0/\delta \xi^3 = R_0 + u + \xi^3
\]

The differential vector at \( P_0 \) is denoted by \( \delta P_0 \) and \( u \) is a unit normal to \( S \). The base vectors at \( P_0 \) and \( P \) are respectively:

\[
g_\alpha = \delta g_\alpha/\delta \xi^3 = A_{\alpha} + \delta y_{\alpha}/\delta \xi^3 = A_{\alpha} + y_{\alpha}; \quad g_\alpha = \delta g_\alpha/\delta \xi^3 = \xi^3 (\partial \beta/\partial \xi^3) g_\alpha
\]

where \( \beta \) is the symmetric curvature tensor of \( s \), and

\[
\dot{b} = a_{\alpha} q^0 q^\alpha = \delta g_\alpha/\delta \xi^3.
\]

A differential vector \( P_0 \) is denoted by \( \delta P_0 \), while that at \( P \) by \( \delta P \), such that

\[
dP_0 = A_{\alpha} d\xi^\alpha + N d\xi^3
\]

\[
dP = G_{\alpha} d\xi^\alpha + N d\xi^3
\]

while in the deformed configurations their maps are denoted by \( dP_0 \) and \( dP \) respectively, where:
\[ dp_0 = g_0 \, d\xi^0 + n \, d\xi^3 \]  
(6.12)

and

\[ dp = g_0 \, d\xi^0 + n \, d\xi^3. \]  
(6.13)

Thus, in the present convected coordinate system, the deformation gradients \( F_0 \) and \( \dot{F} \) defined by:

\[ \dot{dp} = F_0 \, dp_0 \]  
(6.14)

and

\[ dp = \dot{F} \, dP \]  
(6.15)

are given by:

\[ F_0 = g_0 A^a + nN. \]  
(6.16)

and

\[ F = g_0 G^a + nN. \]  
(6.17)

We will now consider the polar-decomposition of \( F_0 \) into stretch and rotation as follows:

\[ F_0 = R \cdot U_0 = \dot{V}_0 \cdot R. \]  
(6.18)

In the case of \( E_0 = R \cdot U_0 \), in the present convected coordinate system, we define, bearing in mind the presently invoked Kirchhoff-Love hypotheses,  

\[ U_0 = g_0 A^a + nN. \]  
(6.19)

and

\[ R = g_0 A^a + nN. \]  
(6.20)

whereby, \( a_0 = U_0 \cdot A^a \), \( a^a = U_0^{-1} A^a \), such that \( a^a \cdot a^a = \delta^a_a \); likewise, \( g_0 = R \cdot g_0 = A_0 \cdot R^T \), and \( g^a = R \cdot g^a \) such that \( g_0 g^a = \delta^a_a \). From (6.16), (6.19) and (6.20) it is seen that, under the Kirchhoff-Love hypotheses,  

\[ g = R \cdot N. \]  
(6.21)

Similarly, in the case of \( E_0 = V_0 \cdot R \), we define:

\[ R = A^a \cdot A^a + nN. \]  
(6.22)

and

\[ V_0 = g_0 A^a + nN. \]  
(6.23)

Such that, once again,

\[ A^a = R \cdot A^a = \delta^a_0 \cdot R^T; \quad A^a = R \cdot A^a; \quad A^a \cdot A^a = A^a \cdot A^a = \delta^a_a \]  
(6.24)

and

\[ g_0 = V_0 \cdot A^a; \quad g^a = V_0^{-1} \cdot A^a; \quad g_0 \cdot g^a = \delta^a_a \]  
(6.25)

once again,

\[ g = R \cdot N. \]  
(6.26)

Practical ways of representing \( R \) are given in Appendix 2 of this report. Using (6.26) in (6.9) it is seen that:

\[ \frac{\partial (R \cdot N)}{\partial \xi^3} = (R \cdot N)_{,3} \xi^3 \]  
(6.27)

Likewise, in terms of the second fundamental form \( b \) of \( s \), we note that:

\[ g_0 = (1 - \xi^3 b) \cdot g_0 = g_0 - \xi^3 b \cdot g_0 = g_0 - \xi^3 b \cdot R \cdot U_0 \cdot A_0 \]  
(6.28)

\[ = g_0 - \xi^3 b \cdot V_0 \cdot R \cdot A_0. \]  
(6.29)

Thus, the deformation gradient at any point in the shell is written as:

\[ \dot{F} = g_0 G^a + nN. \]  
(6.30)

\[ = (R \cdot U_0 - \xi^3 b \cdot R \cdot U_0) \cdot A_0 \cdot G^a + (R \cdot N) \cdot N \]  
(6.31)

\[ = (V_0 \cdot R - \xi^3 b \cdot V_0 \cdot R) \cdot A_0 \cdot G^a + (R \cdot N) \cdot N. \]  
(6.32)

At this point, it is worth noting some properties of \( b \), which is the curvature tensor of \( s \) and has the form

\[ b = b_\alpha (g^a) g^\alpha. \]  
(6.33)

or

\[ b_{\alpha \beta} = g_0 \cdot b_{\alpha \beta}. \]  
(6.34)

\[ = A_\alpha (R \cdot U_0)^T \cdot b \cdot (R \cdot U_0), A_\alpha. \]  
(6.35)

Let

\[ b^* = (R \cdot U_0)^T \cdot b \cdot (R \cdot U_0) = (V_0 \cdot R)^T \cdot b \cdot (V_0 \cdot R). \]  
(6.36)

Thus, even though \( b \) and \( b^* \) are evidently different tensors, entirely, they have the interesting property that: the covariant components of \( b \) in the basis system \( g_0 \) at \( p_0 \) are numerically equal to the covariant components of \( b^* \) in the basis system \( A_\alpha \), the contravariant components of the second Piola-Kirchhoff stress tensor \( \sigma \) and the Kirchhoff stress tensor \( \sigma \), respectively, as discussed before. Finally we consider the frame-indifferent form \( F^T \cdot \dot{F} \). It is seen that:

\[ F^T \cdot \dot{F} = (A_\alpha, U_0, A_\alpha - 2 \xi^3 b^* (R \cdot U_0)^T \cdot b \cdot (R \cdot U_0), A_\alpha \]  
(6.24)

+ \( \xi^3 A_\alpha (R \cdot U_0)^T \cdot b \cdot (R \cdot U_0) \cdot A_\alpha) G^a G^\alpha + NN \]

\[ = (R^T \cdot V_0 \cdot R - 2 \xi^3 b \cdot V_0 \cdot R + \xi^3 \cdot R \cdot V_0 \cdot b \cdot V_0 \cdot R) + NN \]

\[ = G^a A_\alpha (U_0 \cdot U_0) - 2 \xi^3 (R \cdot U_0)^T \cdot b \cdot (R \cdot U_0) \]  
(6.37)

\[ + (\xi^3 (R \cdot U_0)^T \cdot b \cdot (R \cdot U_0) \cdot A_\alpha) G^a + NN \]

\[ = G^a A_\alpha (R^T \cdot V_0 \cdot R - 2 \xi^3 (V_0 \cdot R)^T \cdot b \cdot (V_0 \cdot R) \]  
(6.38)

\[ + (\xi^3 (V_0 \cdot R)^T \cdot b \cdot (V_0 \cdot R) \cdot A_\alpha) G^a + NN. \]  
(6.39)
Thus, the strain energy density function for an elastic shell will be a function of $U_0$ and $b^*$. The Green-Lagrange strain tensor $\gamma$ in the shell is given by:

$$2\gamma = F^T F - I$$

(6.39)

where

$$I = G^{ab} G^{a*} G^{b*} + NN - G^{a*} (G^{b*} G_b) G^0 + NN.$$  

(6.40)

Thus,

$$I = G^{a*} A_a [(I_0 - \xi^* B) (I_0 - \xi^* B^*)] A_0 G^0 + NN.$$  

(6.41)

wherein (6.3) has been used. Thus, for instance, from (6.37) and (6.41) it is seen that:

$$\begin{align*}
2\gamma &= G^{a*} A_a [(I_0 - \xi^* B) (I_0 - \xi^* B^*)] A_0 G^0 + NN \\
&+ (\xi^* B^*)^T (I_0 - \xi^* B^*) - [(I_0 - \xi^* B^*)^T (I_0 - \xi^* B^*)] A_0 G^0 = 2\gamma_0.
\end{align*}$$

(6.42)

Thus, the Kirchhoff-Love hypothesis imply that the strain tensor $\gamma$ is in fact a 2-D $\gamma_{12}$ and the components related to the 3-D $\gamma_{13}$ are zero.

Now we shall consider the mechanical variables that characterize the stress-state in a deformed shell.

1. STRESS-MEASURES IN A FINITELY DEFORMED SHELL

In the deformed shell, let us consider an element with "lengths" $a_1$ and $a_2$ and "heights" $h_1$ and $h_2$ in the reference plane $s$, and "height" $h_3$ in the thickness direction. Let the Cauchy stress in the shell be $\tau$. The traction on a strip of area at height $h$ from the reference plane, spanned by $d\xi^0$ and $d\xi^1$, is given by:

$$dF^n = g^0 \sqrt{\det g} \sqrt{\det g^{-1}} g^{n*} \tau^{n*} g_d d\xi^1 d\xi^2.$$  

(7.1)

where $g$ is determinant of $g_{ij}$ (or equivalently $g_{ab}$ in this case).

The differential force per unit of $\xi^0$ is:

$$dN^n = \sqrt{(g)} \tau^{*n*} g_d d\xi^1 d\xi^2.$$  

(7.2)

We define a "stress-resultant" per unit of $\xi^0$ in the deformed state $\xi^0$ to be:

$$N^n = \int_\xi \sqrt{(g)} \tau^{*n*} g_d d\xi^1 d\xi^2.$$  

(7.3)

In the above the subscript $\tau$ for $N^n$ denotes that it is a true (Cauchy) stress-resultant. It can be shown that while $N^n$ is the "force", it is only $\tau^n$ defined by

$$\tau^n = \frac{N^n}{\sqrt{a}}$$

(7.4)

where $a$ is the determinant of the metric $a_{ab}$ of $s$, that behaves as a tensor component. Thus we may define a Cauchy stress-resultant tensor $\tau^n$ such that

$$\tau^n = (g_{ab} \tau^{a*} n^b)$$

(7.5)

so that, the force on an element of unit $d\xi^0$ in $s$ is:

$$\sqrt{(a_{ab})} d\xi^0 \left( \frac{\tau^{a*}}{\sqrt{a_{ab}}} \right) \sqrt{(g)} \sqrt{(g)} \tau^{*n*} g_d d\xi^1 d\xi^2 = N^n.$$  

(7.6)

As before, we define the tensor component $m^n$ as:

$$m^n = \sqrt{(a)} \tau^{n*} \n^a$$

(7.7)

and define a Cauchy stress-couple tensor $\gamma$ as:

$$\gamma^n = g_{ab} (\tau_{ab}^*) \n^b.$$  

(7.8)

We express the component representations as:

$$\gamma^n = a_1 \gamma^{a_1} + a_2 \gamma^{a_2} + a_3 \gamma^{a_3}.$$  

(7.9)

where $\gamma^{a_1}$, $\gamma^{a_2}$, and $\gamma^{a_3}$ are maps of $G_M$ after stretch and before rotation, while $G_L$ are maps of $G_L$ after rotation and before stretch.

Let us now consider the stress-resultant measures, and stress-couple measures in terms of the previously defined stress measures, $\gamma_1$, $\gamma_2$, $\gamma_3$, and $\gamma^*$ respectively. In terms of the presently used convected coordinates, the representation of these tensors are:

$$\begin{align*}
\tau^{a_1} &= \gamma^{a_1} G_L G_M \\
\tau^{a_2} &= \gamma^{a_2} G_L G_M \\
\tau^{a_3} &= \gamma^{a_3} G_L G_M \\
\tau^{*a_1} &= \gamma^{*a_1} G_L G_M \\
\tau^{*a_2} &= \gamma^{*a_2} G_L G_M \\
\tau^{*a_3} &= \gamma^{*a_3} G_L G_M.
\end{align*}$$

(7.10)

Let us now consider the stress-resultant measures, and stress-couple measures in terms of the previously defined stress measures, $\gamma_1$, $\gamma_2$, $\gamma_3$, and $\gamma^*$ respectively. In terms of the presently used convected coordinates, the representation of these tensors are:

$$\begin{align*}
\tau^{a_1} &= \gamma^{a_1} G_L G_M \\
\tau^{a_2} &= \gamma^{a_2} G_L G_M \\
\tau^{a_3} &= \gamma^{a_3} G_L G_M \\
\tau^{*a_1} &= \gamma^{*a_1} G_L G_M \\
\tau^{*a_2} &= \gamma^{*a_2} G_L G_M \\
\tau^{*a_3} &= \gamma^{*a_3} G_L G_M.
\end{align*}$$

(7.11)

But

$$\begin{align*}
l^{a_1} G_L G_M &= J F^{a_1} \gamma \gamma^{a_1} G_L G_M \\
&= \sqrt{(G)} \gamma^{a_1} G_L G_M.
\end{align*}$$

(7.12)

from which, it is seen that that

$$\begin{align*}
l^{a_1} N^n &= \int_\xi \sqrt{(g)} \tau^{n*} g_d d\xi^1 d\xi^2 = N^n.
\end{align*}$$

(7.13)

Once again it is only $\tau^n$ defined by
that behave as tensor components. Thus we may define a first Piola-Kirchhoff stress-resultant tensor as:

\[ \mathbf{r}_n = A \mathbf{s}_n \]

It then follows that:

\[ \mathbf{r}_n = \mathbf{R}_n \]

\[ \mathbf{r}_n = \mathbf{A}_0 \mathbf{r}_n \]

(7.20a, b)

Thus the first Piola-Kirchhoff stress-couple tensor can be defined as:

\[ \mathbf{r} = A \mathbf{s} \]

and \( \mathbf{r} \) can be expressed as:

\[ \mathbf{r} = \mathbf{V}(A) \mathbf{r}_n \]

(7.21)

It can be easily seen from (7.6) and (7.16) that

\[ \mathbf{r}_n = \mathbf{R}_n \]

\[ \mathbf{r}_n = \mathbf{A}_0 \mathbf{r}_n \]

We may use the alternate component representations:

\[ \mathbf{r}_n = \mathbf{R}_n \]

(7.35)

Recall that \( \mathbf{g}_0 = U_0 \mathbf{A}_0 \) and \( \mathbf{A}_0 = \mathbf{R}_0 \mathbf{A}_0 \) and \( \mathbf{g}_0 = \mathbf{R}_0 \mathbf{N} \).

Which component representation is used depends to a large extent on the convenience of stress-function representation, if any, in satisfying the momentum balance conditions a priori. Component representation of \( \mathbf{g}_0 \) in terms of \( \mathbf{A}_0 \) and \( \mathbf{g}_0 \) was used in [23], eventhough, as will be shown later, it may not be the most convenient form in a practical application.

8. MOMENTUM BALANCE CONDITIONS FOR A SHELL

It was seen that the “force” and “moment” acting on a unit of \( \xi^e \) of the reference surface \( s \) of the deformed shell were \( \mathbf{r}_n^e \) and \( \mathbf{r}_m^e \) respectively. We now derive the force and moment balance conditions for these “stress” measures as well as the alternate measures introduced in eqns (7.18)-(7.39). For simplicity, and without loss of generality, we restrict our attention to the case of “static” equilibrium only. Let \( \mathbf{r}^e \) and \( \mathbf{r}^e \) be the equivalent force and moment, respectively, per unit area of the reference surface \( s \) (of the deformed shell), due to external mechanical action on the shell. Let \( \mathbf{g}_0 \) and \( \mathbf{g}_0 \) be the external forces and moment defined per unit area of the undeformed reference surface, \( S \). It is seen:

\[ \mathbf{g}_0 \mathbf{A}_0 = A_0 \mathbf{g}_0 ; \mathbf{g}_0 \mathbf{A}_0 = A_0 \mathbf{g}_0 (8.1) \]

Let \( \mathbf{g}_0 \) and \( \mathbf{g}_0 \) be the gradient operators in the reference surface coordinates of \( S \) and \( s \) respectively. Thus:

\[ \mathbf{g}_0 = A_0 \mathbf{g}_0 \]

(8.2)

The linear and angular momentum balance conditions for \( \mathbf{r}_n^e \) and \( \mathbf{r}_m^e \) in \( s \) can be expressed in the well-known form[17] as:

\[ \frac{\partial}{\partial \xi} (\mathbf{V}(\mathbf{A}) \mathbf{r}^e) + \mathbf{V} \cdot (\mathbf{g}^e \mathbf{A}_0) + (\mathbf{g}^e \mathbf{A}_0 \mathbf{V}(\mathbf{A}) \mathbf{r}^e) = 0 \] (LMB).

The AMB is:

\[ \frac{\partial}{\partial \xi}(\mathbf{g}^e \mathbf{A}_0 \mathbf{V}(\mathbf{A}) \mathbf{r}^e) = 0 \] (AMB).

To summarize, we have the following relations.
Making use of eqns (7.18–7.39), as in the 3-D continuum case, the above can be written in alternate forms as follows:

LMB:
\[
\frac{\partial}{\partial \xi} (\sqrt{A} \sigma^\alpha) + \sqrt{A} \rho = 0 \quad \text{or} \quad \nabla_S \cdot (\sigma) + \rho = 0
\]
(8.5a,b)

\[\text{or} \]
\[
\frac{\partial}{\partial \xi} (\sqrt{A} \cdot \sigma^\alpha \cdot \mathbf{E}_0^T) + \sqrt{A} \rho = 0 \quad \text{or} \quad \nabla_S \cdot (\sigma \cdot \mathbf{E}_0^T) + \sqrt{A} \rho = 0
\]
(8.5a,b)

AMB:
\[
\frac{\partial}{\partial \xi} [g_x \sqrt{A} \sigma^\alpha] + [a_{\alpha x} \sqrt{A} \sigma^\alpha] + \sqrt{A} \rho = 0
\]
(8.9)

\[\text{or} \]
\[
\frac{\partial}{\partial \xi} [g_x \sqrt{A} (\sigma^\alpha \cdot \mathbf{E}_0^T)] + [a_{\alpha x} \sqrt{A} (\sigma^\alpha \cdot \mathbf{E}_0^T)] + \sqrt{A} \rho = 0
\]
(8.10)

\[\text{and} \]
\[
\frac{\partial}{\partial \xi} [g_x \sqrt{A} (\tau^\alpha \cdot \mathbf{R}^T)] + [a_{\alpha x} \sqrt{A} (\tau^\alpha \cdot \mathbf{R}^T)] + \sqrt{A} \rho = 0
\]
(8.11)

\[\text{and} \]
\[
\frac{\partial}{\partial \xi} [g_x \sqrt{A} \tau^\alpha] + [a_{\alpha x} \sqrt{A} \tau^\alpha] + \sqrt{A} \rho = 0
\]
(8.12)

The following interesting observations can be made: (i) the LMB conditions (8.5a) and (8.8a) for \( \sigma^\alpha \) and \( \tau^\alpha \) are identical in form, while in direct tensor notation, the equivalent conditions (8.5b) and (8.8b) are identical in form to their 3-D continuum counterparts, eqns (4.9a) and (4.13a) respectively, (ii) the AMB condition for the measures \( \sigma^\alpha \cdot \mathbf{E}_0^T \) and \( \tau^\alpha \cdot \mathbf{R}^T \) are identical in form (see 8.9 and 8.17).

9. MIXED VARIATIONAL PRINCIPLES FOR A FINITELY DEFORMED SHELL

Here, our primary interest is to consider variational principles based on stress-functions and rigid rotations being independent variables. To this end, we first consider the mixed variational principle with the associated functional, \( F_1 \) of (5.1). Upon substituting for \( E, U, R, \) and \( t \) the relevant plausible assumptions for the shell as in (6.32), (6.31), (6.41), and (7.11) in terms of the presently used convected coordinates, we obtain:

\[
F_1 = \int_{S_0} \left\{ W(dU_o, b^*) + \int_{S_0} \left[ (A_u + y_u) + (R \cdot N)_o \epsilon^\alpha \right] \right\} dS
\]

\[\text{or} \]
\[
- \int_{S_0} \left[ \epsilon \cdot y \cdot \sqrt{A} \cdot \epsilon^\alpha \right] dS = 0
\]
(9.1)

where \( y = v' G' \) is a unit normal to the edge-surface of the shell \( S_0 \), in its undeformed configuration, where deformations are prescribed, and \( t \) are prescribed tractions on the edge surface of the shell, \( S_{ed} \).

Upon carrying out integrations w.r.t. \( \xi^i \) and using the definitions for \( y^\alpha \) (and \( \sigma = A_u, y^\alpha \)) and \( t, \) (and \( T^\alpha \)), we obtain:

\[
F_1(U_o, b^*, R, t, \epsilon, \tau, \sigma, y) = \int_{S_0} \left\{ \sqrt{A} \cdot \tilde{W}(U_o, b^*) + \int_{S_0} \left[ (A_u + y_u) + (R \cdot N)_o \epsilon^\alpha \right] \right\} dS
\]

\[\text{or} \]
\[
- \int_{S_0} \left[ \epsilon \cdot y \cdot \sqrt{A} \cdot \epsilon^\alpha \right] dS = 0
\]
(9.2)

where \( y_0 = (v_0 A^*) \) is the unit outward normal to the boundary curve \( \partial S \) of \( S \), and \( C_m \) and \( C_n \) are portions of \( \partial S \) where tractions and deformation, respectively, are prescribed.

With the motivation of defining a "bending strain" measure that depends on the rotation tensor \( R \) alone, we express \( F_2 \) of (9.2) in terms of alternate "stress-couple" and "bending-strain" measures as follows. As a prelude, we first recall:

\[
\epsilon = \sigma^\alpha \cdot \mathbf{E}_0^T = \sigma^\alpha \cdot \mathbf{E}_0^T
\]
(9.3)

\[\text{and} \]
\[
\tau = \tau^\alpha \cdot \mathbf{R}^T = \tau^\alpha \cdot \mathbf{R}^T
\]
(9.4)

From which it can also be seen that:

\[
\tau = \tau^\alpha \cdot \mathbf{R}^T = \tau^\alpha \cdot \mathbf{R}^T = \tau^\alpha \cdot \mathbf{R}^T
\]
(9.5)

Now consider the term \( \tau^\alpha \cdot \mathbf{R}^T \cdot b \cdot U_o \) in \( F_1 \) of eqn (9.2). It is seen that:

\[
\tau^\alpha \cdot \mathbf{R}^T \cdot b \cdot U_o = \tau^\alpha \cdot \left( (R \cdot U_o)^* \right)^T \cdot (R \cdot U_o)^* \cdot b \cdot (R \cdot U_o)
\]
(9.6)

\[\text{and} \]
\[
\tau^\alpha \cdot \tau = \tau^\alpha \cdot \tau = \tau^\alpha \cdot \tau
\]
(9.7)

\[\text{and} \]
\[
\tau^\alpha \cdot \tau = \tau^\alpha \cdot \tau = \tau^\alpha \cdot \tau
\]
(9.8)
wherein, the new "bending-strain" measure, $\hat{\beta}^*$, is defined to be:

$$\hat{\beta}^* = b^* \cdot U_0^{-1}. \tag{9.10}$$

We now postulate the strain energy density $W_o$ to be a function of stretching strain $U_0$, and the bending strain $\hat{\beta}^*$. We discuss the justification and ramifications of this now. First, consider the frame-invariant form $F^T \cdot F$ discussed in (6.37). It was shown that:

$$F^T \cdot F = G^o A_o [(U_0 - \xi^0 b^*) \cdot (U_0 - \xi^0 b^*)^T] \cdot A_o G^o + NN. \tag{9.11}$$

In terms of the new strain measure $\hat{\beta}^*$, it is seen easily that:

$$F^T \cdot F = G^o A_o [(U_0 - \xi^0 \hat{b}^*) \cdot (U_0 - \xi^0 \hat{b}^*)^T] \cdot A_o G^o + NN. \tag{9.12}$$

The proof is as follows. From (9.12),

$$F^T \cdot F = G^o A_o [(U_0 - \xi^0 b^*) \cdot (U_0 + U_0 \hat{b}^*^T)] + (\xi^0)^2 b^*^T \cdot A_o G^o + NN.$$ 

Thus, we may take $(U_0 - \xi^0 b^*)$ as a measure of the 3-D strain state in the shell.

We will now show that $\hat{\beta}^*$ is an objective strain measure. First note that:

$$b = -(n)_{\omega} a^*.$$ 

In the presently considered case of the shell-mid surface element being first subjected to stretch $U_0$, and then to a rotation $R$, consider the case of a superimposed rigid rotation $R'$ of the entire shell. Then the new mid-surface deformation gradient is:

$$R_* = R' \cdot R \cdot U_0 = R' \cdot R \cdot U_0.$$ 

The mid-surface normal in the final state is:

$$n' = R \cdot n.$$ 

The curvature tensor in the final state is:

$$-b^* = b^* - (R')_{\omega} a^* = (R')_{\omega} a^*.$$ 

wherein the fact that $R'$ is a rigid motion of the entire shell has been used. Now we define the equivalent curvature strain referred to the undeformed configuration to be $b^*$, where:

$$b^* = (R')_{\omega} b^*,$$

Thus $\hat{\beta}^*$ is an objective measure. Since $U_0$ is clearly objective, it is seen that

$$\hat{\beta}^* = b^* \cdot U_0^{-1} = b^* \cdot U_0^{-1} = \hat{\beta}^*.$$ 

is objective, even though unsymmetric, in general.

We now introduce the concept of a "semi-linear isotropic elastic" material. In this constitutive theory the "strain tensor" of the infinitesimal elasticity theory is replaced by the "extension" tensor $(U - I)$ in the strain-energy density of a 3-D elastic continuum [24]. This is equivalent to replacing the "stress tensor" of the infinitesimal elasticity theory by the symmetrized Biot-Lure stress tensor, $\tau$, of (2.41), in the stress-strain relation for a 3-D continuum. Thus, digressing for a moment, we consider a 3-D continuum wherein the deformation gradient $F$ is decomposed as $F' = R \cdot U$, and $F^T \cdot F = U^2$. In this case, for a 3-D stress/strain state, the strain-energy density for a "semi-linear isotropic elastic" material is [24, 30]

$$W_o = \frac{1}{2} A [(U - I) : (U - I)] + \mu (U - I) : (U - I).$$ 

where $A$ and $\mu$ are Lamé constants, such that,

$$A \cdot W_o = \lambda [(U - I) : I] + 2 \mu (U - I).$$. 

For plane-stress conditions it is seen that

$$W_o = \frac{1}{2} \left( \frac{E}{1 - \nu} \right) [(U - I) : D] + \mu (U - I) : (U - I)$$

such that, for plane stress, with $k = \nu \left( \frac{1 - \nu}{1 - 2\nu} \right)$,

$$\tau = \frac{E}{(1 - 2\nu)} \nu [(U - I) : I] + (1 - \nu)(U - I).$$ 

Following Koiter [31], we consider the stress-state in a thin shell to be one of plane stress. Recall that for shell undergoing arbitrary deformation of Kirchhoff-Love type.

$$F^T \cdot F = G^o A_o [(U_0 - \xi^0 b^*) \cdot (U_0 - \xi^0 b^*)^T] \cdot A_o G^o + NN. \tag{9.24}$$

and that in the undeformed configuration,

$$U_0 = 0 \quad \text{and} \quad \hat{\beta}^* = b^* \cdot U_0^{-1} = B.$$ 

where $B$ is the curvature tensor of the undeformed shell. Assuming that the undeformed configuration is stress free, we may now write for a "plane-stress" shell the strain energy expression, analogous to the 3-D con-
Computational analyses of finitely deformed solids, with application to plates and shells—I

Continuing above, as:

$$W_0 = \frac{E_v}{2(1-v)} \left[ ((U_0 - l_0) - \xi^1(\hat{b}^* - B)) : I_0 \right]^2 + \mu \left[ ((U_0 - l_0) - \xi^1(\hat{b}^* - B)) : ((U_0 - l_0) - \xi^1(\hat{b}^* - B)) \right]$$

Expanding terms, we see that:

$$W_0 = \frac{E_v}{2(1-v)} \left[ ((U_0 - l_0) : I_0) - \xi^1(\hat{b}^* - B) : I_0 \right]^2 + \mu \left[ ((U_0 - l_0) : I_0) - \xi^1(\hat{b}^* - B) : ((U_0 - l_0) : \xi^1(\hat{b}^* - B)) \right]

\text{(9.25)}$$

We now consider the thickness of the shell to be $h$ and that the mid-surface of the shell is the reference surface. We now define $W_0$ to be the strain-energy density/unit area of the undeformed mid-surface. Thus we define:

$$s \int \hat{W}_0 \sqrt{(A)} \, d\xi^1 d\xi^2 = \int W_0 \sqrt{(G)} \, d\xi^1 d\xi^2$$

\text{(9.27)}

or

$$\hat{W}_0 = \int \hat{W}_0 \sqrt{(G)} \, d\xi^2 = \int_{\partial \Sigma} W_0 \sqrt{(G)} \, d\xi^2.$$

\text{(9.28)}

Following Koiter [31], we see that

$$\sqrt{\left(\frac{G}{A}\right)} = 1 - 2h^2 K x + K \xi^2$$

\text{(9.29)}

where $H$ and $K$ are the mean and Gaussian curvatures of the undeformed mid-surface. Following Koiter's [31] argument for a consistent first approximation to evaluate the integral (9.28), it is seen that:

$$\hat{W}_0 = \frac{E_v}{2(1-v)} \left[ h((U_0 - l_0) : I_0) + h^3 \left( (\hat{b}^* - B) : I_0 \right) \right] + \mu \left[ h((U_0 - l_0) : I_0) + h^3 \left( (\hat{b}^* - B) : (\hat{b}^* - B) \right) \right]$$

\text{(9.30)}

such that.

$$\frac{\partial \hat{W}_0}{\partial U_0} = \frac{E_h}{(1-v)} \left[ v((U_0 - l_0) : I_0) \right] + (1-v)(U_0 - l_0)$$

\text{(9.31)}

and

$$\frac{\partial \hat{W}_0}{\partial \hat{b}^*} = \frac{E_h^3}{12(1-v)^2} \left[ \xi^1((\hat{b}^* - B) : I_0) \right] (1-v) \xi^1((\hat{b}^* - B)).$$

Having this constitutive theory on hand, we now return (9.2) and express it as:

$$F_\xi(U_0, \hat{b}^*, R, \xi, \tau, \rho) = \int S(\sqrt{(A)} \hat{W}_0(U_0, \hat{b}^*) + \sqrt{(A)} \xi^1 R^T : (\xi + \rho) \xi^1 \hat{b}^* \sqrt{(A)}\, d\xi^1 d\xi^2$$

\text{(A^* - R \cdot U_0 + \sqrt{(A)} R^T : (\xi + \rho) R \hat{b}^* \sqrt{(A)}\, d\xi^1 d\xi^2 = \int_{\Sigma} \hat{W}_0 \sqrt{(A)} \, d\xi^1 d\xi^2}$$

\text{(9.32)}

We now consider the first variation of $F_\xi$, which is:

$$\delta F_\xi(\delta U_0, \delta \hat{b}^*, \delta R, \delta \eta, \delta \tau)$$

\text{(9.33)}

We first note that the variation $\delta R$ must be subject to the constraint of orthogonality, i.e. $R^T R = I$, or $\delta R^T \delta R = 0$, or that $\delta R^T \delta R$ is skew-symmetric tensor. Likewise, $\delta U_0$ is a symmetric tensor. To account for this, and for reasons of added clarity, we rearrange terms and rewrite $\delta F_\xi$ as:

$$\delta F_\xi = \int S \left\{ \sqrt{(A)} \left[ \frac{\partial \hat{W}_0}{\partial U_0} - \frac{1}{2} (\hat{b}^* R + R \hat{b}^*) \sqrt{(A)} \right] d\eta \right\}$$

\text{(9.34)}

Consider the following identities:

$$\int S \left( \sqrt{(A)} \xi^1 R^T (\delta R \hat{b}^*) d\xi^1 d\xi^2 = \int S \sqrt{(A)} \xi^1 d\xi^1 d\xi^2$$

\text{(9.35)}
Also,
\[ \left( \frac{\partial}{\partial r} \right)_{\theta} \delta R \cdot N = (\nabla (A) \cdot r^\alpha) \delta R \cdot R^T \cdot N = \left( \frac{\partial}{\partial r} \right)_{\theta} \delta R \cdot R^T. \] (9.36)

Similarly,
\[ \int_S \left( \nabla (A) \cdot g^\alpha \cdot \delta g \cdot d^\xi \cdot d^2 \xi \right)^2 = \int_S \left( (\nabla (A) \cdot g^\alpha) \delta g \right) d^2 \xi. \] (9.37)

Upon using (9.35-37) in (9.34), the condition that \( \delta F_i = 0 \) can be seen to lead to the following ELE and NBC:

\[ \delta W_0 = \frac{1}{2} \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right). \] (9.38)

\[ \delta W_0 = \frac{1}{2} \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right). \] (9.39)

\[ \delta W_0 = \frac{1}{2} \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right). \] (9.40)

\[ \delta W_0 = \frac{1}{2} \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right). \] (9.41a)

\[ \delta W_0 = \frac{1}{2} \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right). \] (9.41b)

\[ \delta W_0 = \frac{1}{2} \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right). \] (9.41c)

But, in the presently used polar decomposition,
\[ \left( \frac{\partial}{\partial r} \right)_{\theta} U_0 \] (9.41d)

and
\[ g^\alpha = R^T \cdot g^\alpha \] (9.41e)

Thus,
\[ \hat{b}^* = b_{\alpha \beta} A^\gamma g^\gamma. \] (9.41f)

Hence, the physical meaning of the curvature strain measure \( \hat{b}^* \) is evident: it is a tensor whose covariant components in the mixed-basis \( A^\gamma g^\gamma \) are numerically equal to the components \( b_{\alpha \beta} \) of the second-fundamental tensor \( b \) of the deformed mid-surface, in the basis \( g^\alpha \). Recall at the same time that \( \hat{b}^* \) was a tensor whose covariant components in \( A^\gamma g^\gamma \) bases were numerically equal to the components \( b_{\alpha \beta} \) of \( b \) in the basis \( g^\alpha \). Note also that \( \hat{b}^* \) is a function of \( R \) alone.

\[ \left( \frac{\partial}{\partial r} \right)_{\theta} \delta F_i = 0 \] (9.42)

\[ - \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) - \left( \nabla (A) \cdot g^\alpha \right) \cdot \delta g + \nabla (A) \cdot R^T \cdot \hat{b}^* \cdot R^T = \text{symmetric} \] (9.43)

which is a consequence of the skew-symmetry of \( \delta R \cdot R^T \). Assuming that \( \delta S = C_{in} + C_{on} \) without loss of generality, the following NBC are recovered.

\[ u = \tilde{g} \] (9.44)

\[ v = n_{\alpha \beta} \tilde{g}^\alpha \] (9.45a,b)

Clearly, (i) and (ii) above are the constitutive relations for \( r_i \) (and the conjugate \( U_0 \)) and \( \sigma_t \) (and the conjugate \( b^* \)) respectively; (iii) is the compatibility condition for mid-plane stretching; (iv) is the compatibility condition for curvature strain; (v) is the force balance or LMB; and (vii) and (viii) are the deformation and traction b.c.

We now examine the rather abstract form of angular momentum balance (AMB) as in (9.46) above, and see if it corresponds to the other known forms of AMB. It is seen that (9.43) implies that:

\[ - \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) + \nabla (A) \cdot R_{\alpha \beta} \tilde{g}^\alpha = 0. \] (9.46)

Consider the \( g^\alpha \) component of the above tensor identity:

\[ - \left( \frac{\partial}{\partial r} \right)_{\theta} \left( \delta R \cdot R^T \right) + \nabla (A) \cdot R_{\alpha \beta} \tilde{g}^\alpha = 0. \] (9.47)

To simplify (9.47), we recall the following identities/relations:

\[ (\delta R \cdot R_{\alpha \beta}) = g_{\alpha \beta}, \quad \delta R \cdot R_{\alpha \beta} = A_{\alpha \beta} \] (9.48a)

\[ (\delta R \cdot R_{\alpha \beta}) = g_{\alpha \beta}, \quad (\delta R \cdot R_{\alpha \beta}) = 0. \] (9.48b)

\[ \nabla (A) \cdot g^\alpha = \nabla (A) \cdot g^\alpha; \quad \nabla (A) \cdot R_{\alpha \beta} = \nabla (A) \cdot R_{\alpha \beta}. \] (9.48c)

Further, for convenience of comparison of the present result with the well-known\[17\] result for moment balance condition, we express \( \delta g^\alpha \) and \( \delta g^\alpha \) as:

\[ \delta g^\alpha = n_{\alpha \beta} \tilde{g}^\beta + n_{\alpha \beta} g^\beta; \quad \delta g^\alpha = n_{\alpha \beta} \tilde{g}^\beta + n_{\alpha \beta} g^\beta. \] (9.49a,b)

When (9.48, 49) are used in (9.47), we obtain the result:

\[ - \nabla (A) \cdot (n_{\alpha \beta} \tilde{g}^\beta + n_{\alpha \beta} g^\beta) + \nabla (A) \cdot (n_{\alpha \beta} \tilde{g}^\beta + n_{\alpha \beta} g^\beta) \] (9.50)

Thus, from (9.50) we obtain:

\[ - n_{\alpha \beta} + n_{\alpha \beta} \tilde{g}^\beta + n_{\alpha \beta} g^\beta = 0. \] (9.51)

\[ - n_{\alpha \beta} + n_{\alpha \beta} \tilde{g}^\beta + n_{\alpha \beta} g^\beta = 0. \] (9.52a)

or

\[ n_{\alpha \beta} + n_{\alpha \beta} \tilde{g}^\beta + n_{\alpha \beta} g^\beta = 0. \] (9.52b)

In (9.51), \( \Gamma_{\alpha \beta} \) is the Christoffel symbol of the deformed
mid-surface, and \((\cdot)\) implies a covariant derivative with respect to the deformed mid-surface and hence involves \(\Gamma^\alpha_{\beta\gamma}\). It can be seen that (9.51, 52) are identical to the moment balance conditions that are well-documented in literature [17].

Thus the present variational principle is consistent: (a) it leads to all the shell-equations as its ELE and NBC (b) the curvature strain measure \(b^*\) involves \(R\) alone (c) the ELE corresponding to a variation in \(R\) is the exact angular momentum balance condition, (iv) more interestingly, unlike in the well-known nonlinear theories of Koiter [12], Budiansky [13], and Symonds [23], and others due to Sanders [11], Pietraskiewicz [16], the present development does not involve any ad hoc modifications to the definitions of stress-resultants and/or bending strain tensors.

We now consider certain simplifications to the general mixed principles stated in (9.32). While several such simplifications are possible (along the lines of reducing the well-known Hu-Washizu principle [26] of linear elasticity to principles such as those of potential energy, complementary energy, and Helinger-Prange-Reissner), we shall deal with only two of the possible simplifications.

First we note that \(\tilde{W}_0\) of (9.30) can be written as:

\[
\tilde{W}_0 = \tilde{W}_{os} + \tilde{W}_{ob}
\]

where

\[
\tilde{W}_{os} = \frac{1}{2} \left( \mu (U_0 - L_0) \right)^2 + \frac{(1 - \nu)}{2} \left( \mu (U_0 - L_0) \left( U_0 - L_0 \right) \right)
\]

and

\[
\tilde{W}_{ob} = \frac{Eh^3}{12(1 - \nu)} \left[ \frac{1}{2} \left( \dot{b}^s - B \right)^2 + \frac{(1 - \nu)}{2} \left( \dot{b}^b - B \right)^2 \right]
\]

where the subscripts \(s\) and \(b\) stand for "stretching" and "bending", respectively, such that

\[
\frac{\partial \tilde{W}_{bs}}{\partial U_0} = \frac{1}{2} \left[ \mu R + R^T \mu R \right] = \eta.
\]

Suppose that we establish the Legendre contact transformation:

\[
\tilde{W}_{os}(U_0) - \eta; U_0 = - \tilde{W}_{os}(\eta).
\]

Using (9.57), \(U_0\) can be eliminated as a variable in (9.32). Likewise, suppose that the linear momentum balance condition, for \(\eta\), namely (9.42), as well as the traction b.c. that \(\tilde{N} = \nu_0 \eta\), are satisfied \textit{a priori}. Then, it can be seen that \(\mu\) can be also eliminated as a variable from (9.32). Further, suppose that the relation between \(\dot{b}^s\) and \(R\), namely (9.41b), is also satisfied \textit{a priori}, thus eliminating \(\dot{b}^s\) as a variable from (9.32). Thus, all in all, if \(U_0, \eta,\) and \(\dot{b}^s\) are eliminated as indicated, we can reduce (9.32) to

\[
F_2(\eta, R) = \int_S \left[ - \sqrt{(A)} \tilde{W}_{es}(\eta) + \sqrt{(A)} \tilde{W}_{ob}(R) \right]
\]

+ \sqrt{(A)} \eta^T : \left( \epsilon - \epsilon^s \right) d\tilde{\xi} \tilde{d}^2 \eta

- \int_{C_m} \tilde{\eta}(R - \lambda) \cdot \tilde{d} \sqrt{(A)} \tilde{d} \sqrt{(A)}

- \int_{C_m} \tilde{\eta}(R - \lambda) \cdot \tilde{d} \sqrt{(A)} \tilde{d} \sqrt{(A)}

To be precise, the following \textit{a priori} constraints hold in (9.58):

\[
\dot{\eta} = \frac{1}{2} \left[ (\eta, R + R^T \eta) \right]; \quad \tilde{W}_{es}(\eta) = \eta; U_0 = - \tilde{W}_{ob}(U_0)
\]

\[
(\sqrt{(A)} \eta^s)_s + (\sqrt{(A)} \eta) \tilde{G} = 0; \quad \tilde{\eta} = \tilde{N} \text{ at } C_m
\]

and

\[
\tilde{W}_{es}(\tilde{b}^s) = \tilde{W}_{es}(\tilde{b}^s(R, N, \ldots, R)).
\]

It can be verified easily that \(\delta F(\delta \eta, \delta R) = 0\) leads to the following ELE and NBC: (i) compatibility of mid-plane strains, viz. (9.40); (ii) the angular momentum balance condition, (9.43); (iv) the moment boundary conditions, (9.45b); and (v) the deformation b.c., (9.44).

The above principle may be viewed as a mixed-complementary energy principle. The constraint of LMB on \(\eta^a\) can be satisfied easily through the introduction of stress-functions as shown below, where upon the above function (9.58) becomes expressible in terms of a stress-function vector and the rotation tensor. As shown in [32, 33] we see that the equation

\[
(\sqrt{(A)} \eta^s)_s + (\sqrt{(A)} \eta) \tilde{G} = 0
\]

can be satisfied identically by setting:

\[
\eta^a = \left[ \epsilon^{a\alpha} \frac{\partial F}{\partial \epsilon^\alpha} + \beta^a \right]
\]

where \(F\) is a stress function vector, \(\epsilon^{a\alpha} = (1/\sqrt{A}) \epsilon^{a\alpha}\), \(\epsilon^{12} = - \epsilon^{12} = 1\); \(\epsilon^{11} = \epsilon^{22} = 0\); and \(\beta^a\) is a particular solution given by:

\[
\beta^a = - \frac{1}{2 \sqrt{A}} \int (\sqrt{(A)} \tilde{G} d\dot{\xi}^a.
\]

We can express \(F\) and \(\beta^a\) in component form along the basic vectors of the undeformed mid-surface as

\[
F = F^\alpha \tilde{A}_\alpha + F^j N^j
\]

and

\[
\beta^a = P^a \tilde{A}_\alpha + P^a N^j.
\]

Thus, taking

\[
\tilde{\eta} = \tilde{\eta}^a \tilde{A}_\alpha + \tilde{\eta}^j N^j
\]

\[
\tilde{\eta}^a = \epsilon^{a\alpha} \left( \frac{\partial F^\alpha}{\partial \epsilon^\alpha} + F^a \tilde{G}^a - F^b B_{ab}^a \right) + P^a
\]

and

\[
\tilde{\eta}^j = \epsilon^{j\alpha} \left( \frac{\partial F^\alpha}{\partial \epsilon^\alpha} + F^a \tilde{G}^j + F^j B_{ja}^a \right) + P^a.
\]
In (9.64), \( \Gamma^m_{ij} \) is the Christoffel symbol of the undeformed mid-surface.

The present principle based on \( F_2 \) of eqn (9.58), is very convenient for numerical application, via, say the finite element method. First note that (i) all the integrals in \( F_2 \) involve the undeformed mid-surface (ii) the stress function vector \( F \) and the first Piola-Kirchhoff-stress resultant, \( \eta^* \) are expressed in component form along the base vectors of the undeformed mid-surface: \( F = F^P A_p + F^O N; \eta^* = \eta^* A_p + \eta^* N \) (iii) the components of the rotation \( R \) can also be expressed along the basis of the undeformed mid-surface, as: \( R = R^P A_p A_p + R^O A_p N + R^P A_p A_p + R^P N \). Second, when \( \eta^* \) are derived from a stress function \( F \), which is independently assumed for each element, then the stress-resultants at the interelement boundary will not, in general, be reciprocated (i.e. Newton’s 3rd Law would be disobeyed). This constraint of interelement-stress-resultant reciprocity can be enforced through Lagrange multipliers introduced at the interelement boundaries. It can be shown[33, 9] that these Lagrange multipliers are, in fact, the interelement boundary displacements. Thirdly, it can be shown[9] that if \( \eta \) is continuous at the interelement boundary, then the moment resultants would be automatically reciprocated at the interelement boundary. Since \( R \) can be expressed in terms of a finite rotation vector \( \Omega \) as discussed in Appendix 2, this addition of \( \Omega \) continuous at the interelement boundary. This can be accomplished by using an interpolation for \( \Omega \) over each element, in terms of the values \( \Omega \) at nodes on the element boundary. Finite elements developed along the above lines are referred to as Mixed-Hybrid finite elements[34]. The development of such a finite element methodology for finite deformation analysis of shell, and its verification through carefully chosen numerical examples is the object of the forthcoming Part II of this paper.

At this point it is worth comparing the present principle based on \( F_2 \) of (9.58) with an analogous principle given by Simmonds and Danielson[23]. Briefly, the following comments can be made: (i) the decomposition \( F_0 = R \eta_0 \) is made in the present, while it can be supposed that the decomposition \( F_0 = \eta_0 R \) is made in [23], (ii) the present variational principle involves the first Piola-Kirchhoff stress resultants \( \eta_0 \) and the rotations \( R \) explicitly. Also \( \eta_0 \) in the present work is expressed in components along the base vectors at the undeformed mid-surface. Moreover, in the present work, \( \eta_0 \) and \( R \) occur in such a combination that the stress-resultant variable occurring in the variational principle may be identified as the symmetrically Biot-Lire stress resultant \( \eta_0 \). On the other hand, the stress-resultant appearing in [23] seems to the author to be that one which is identified as \( \eta_0 \) in the present work. Note however, that the vector \( r r^* \) is decomposed in [23] along the rotated base vectors, identified as \( A_P \) in the present work: (iii) the measures of bending strain are quite different in the present work as compared to those in [23], eventhough both the measures involve rotation \( R \) alone: (iv) an objective constitutive theory is developed from the first principles in the present work, while such treatment is not immediately clear in the work of [23]. (v) interesting enough, the present work does not involve any ad hoc modifications to the definition of stress-resultants or to that of the bending strain, while the origin of the modified stress variables in eqns (40) and (41) of [23] is less apparent. (vi) the fact that the exact AMB condition is an ELE corresponding to a variation in \( R \) is unambiguously demonstrated here, while a similar result is not so apparent in [23], and finally, (vii) the present work makes an unambiguous transition from 3-D continuum mechanics to shell theory in a direct tensor notation. In doing so, at least in the belief of the author, the essential aesthetics and beauty of structure of mechanics is preserved in the developed shell theory.

However, we rederive a theory analogous to that in [23] later in this report, and come back to a further technical discussion. Before doing this, however, we concentrate on a second alternate simplification of the general functional of eqn (9.32).

Suppose that, in addition to the contact transformation in (9.57), we also consider the contact transformation,

\[
\bar{W}_{66} (\vec{R}^*; \vec{\mu}, \vec{\eta}) = - \bar{W}_{66} (\vec{R}, \vec{\eta}) \tag{9.566}
\]

whereby, we can eliminate \( \vec{R}^* \) as a variable in (9.32). If, in addition, \( U_0 \) and \( y \) are eliminated through the same process as in the first simplification (i.e. \( F_0 \) discussed above, we obtain a three-fold principle involving \( \vec{R}, \vec{\eta} \) and \( \vec{\mu} \) as variables, stated as the stationary condition of the functional:

\[
F(\vec{R}, \vec{\mu}, \vec{\eta}) = \int_a^b \left( \bar{W}_{66}(\vec{R}, \vec{\mu}, \vec{\eta}) + \bar{W}_{66}(\vec{R}, \vec{\eta}) \right) \, dt + \int_{\Gamma_1} \bar{W}_{66}(\vec{R}, \vec{\mu}, \vec{\eta}) \, ds + \int_{\Gamma_2} \bar{W}_{66}(\vec{R}, \vec{\mu}) \, ds
\]

Precisely, the ELE and NBC of the statement \( dF_i = 0 \) are: (i) the compatibility condition for midplane stretch, rotation, and displacement (9.40), (ii) the compatibility condition for curvature strains (9.41), (iii) the moment balance condition (9.43), and (iv) the moment balance condition (9.43), and (iv) the moment and deformation boundary conditions (9.45b) and (9.44).

The new functional \( F_1 \) while involving \( \vec{\mu} \) as an additional variable as compared to \( F_2 \), does allow for a direct approximation of the moments \( \vec{\mu} \) in a numerical solution, while in \( F_1 \), \( \vec{\mu} \) depends on \( \vec{R} \) through the constitutive relation involving \( \vec{R}^* \) as a function of \( \vec{R} \). This may have certain advantages such as a possible added accuracy in the solution for \( \vec{\mu} \). This will be discussed further in Part II of the paper.

Now, we return to a discussion of a mixed principle involving stress functions and rotations for a finitely deformed shell, based on the polar-decomposition \( F_0 = V_0 R \) as is the case in [23].

First, consider the term \( r^* \cdot b \vec{R}, U_0 \) in (9.2). We rewrite this as:

\[
r^* \cdot b \vec{R}, U_0 = r^* \cdot b \vec{V}_0 R \tag{9.68}
\]

\[
= r^* \cdot b \vec{V}_0 R = r^* \cdot b \vec{V}_0
\]

\[
= r^* \cdot b \vec{V}_0 = r^* \cdot b \vec{V}_0
\]

\[
= r^* \cdot b \vec{V}_0
\]

\[
= r^* \cdot b \vec{V}_0
\]

\[
(9.69)
\]
wherein, the new bending strain measure is defined as:

$$\hat{b}^* = R_0 b^* R_0^T V_0^{-1}.$$  

(9.70)

Recall that,

$$b^* = \hat{b}^* U_0 = \hat{b}^* R_0^T V_0 R_0.$$  

(9.71)

Thus,

$$\hat{b}' = R_0 \hat{b}^* R_0^T; \quad V_0 = R_0 U_0 R_0^T.$$  

(9.72a, b)

Thus, it is interesting to note: (i) since \(\hat{b}^*\) is a bending strain measure that is solely determined by the rotation \(R\), so is the second bending strain measure \(b'^*\); (ii) since the frame-indifference requirements have been shown earlier to lead to the definition of an isotropic semilinear material in terms of \(U_0\) and \(\hat{b}^*\), these requirements will again be met if, in eqns (9.25) and (9.30), \(U_0\) and \(\hat{b}^*\) are replaced by \(R_0^T V_0 R_0\) and \(R_0^T b'^* R_0\) respectively; (iii) more importantly, note that \(\bar{W}_0\) involves not only \(V_0\) and \(\hat{b}^*\) but also the rotations \(R\) explicitly. To be precise, we write:

$$\bar{W}_0 = \frac{E h}{2(1-\nu^2)} [(R_0^T V_0 R - L_0)^2]$$

$$+ \frac{E \nu}{2(1-\nu)} \frac{h^2}{[(R_0^T b^* R - B)]^2} (9.73)$$

such that,

$$\frac{\partial \bar{W}_0}{\partial \hat{b}^*} = R_0 \frac{\partial \bar{W}_0}{\partial \hat{b}^*} R_0^T = \frac{E h}{1-\nu^2} R_0 \left[ \frac{\nu}{(R_0^T V_0 R - I) I_0} \right] I_0$$

$$+ (1-\nu)(R_0^T V_0 R - I_0) R_0^T (9.74)$$

$$\frac{\partial \bar{W}_0}{\partial b'^*} = \frac{E h^3}{12(1-\nu^2)} R_0 \left[ \frac{\nu}{(R_0^T \hat{b}^* R - I) I_0} \right] I_0$$

$$+ (1-\nu)(R_0^T \hat{b}^* R - I_0) R_0^T (9.75)$$

and

$$\frac{\partial \bar{W}_0}{\partial R} = 2 V_0 R \frac{\partial \bar{W}_0}{\partial U_0} + \hat{b}' R_0 \left( \frac{\partial \bar{W}_0}{\partial \hat{b}^*} \right)^T + \hat{b}' R_0 \left( \frac{\partial \bar{W}_0}{\partial b'^*} \right). (9.76)$$

The fact that a frame-indifferent constitutive theory involves a strain-energy density function \(\bar{W}_0\) that depends on the strain measures \(V_0\) and \(\hat{b}^*\) as well as \(R\) explicitly does not appear to have been accounted for by Simmonds and Danielson[23]. Further, the definition of \(\hat{b}^*\), which appears to be the definition of the bending strain measure in [23], is not as apparent as (9.22) as above. We now define \(F_1\) of (9.32) in terms of \(V_0\), \(R\), \(\hat{b}^*\) and \(b'^*\) as follows:

$$F_1 = \left[ \frac{\partial \bar{W}_0}{\partial \hat{b}^*} \right] + \frac{\partial \bar{W}_0}{\partial \hat{b}^*} R_0 = \left[ \frac{\partial \bar{W}_0}{\partial b'^*} \right] + \frac{\partial \bar{W}_0}{\partial b'^*} R_0 (9.76)$$

or

$$\hat{b}' = - R_0 A^*(R_0 N_m) = A^* b_m g^* = b_m A^* g^* (9.83)$$

(as opposed to the earlier relation \(\hat{b}^* = b_n A_n g^*\)). Thus \(\hat{b}'\) is a tensor whose covariant components in the mixed basis \(A^* g^*\) are numerically equal to the components \(b_m\) of the second fundamental tensor of the deformed mid surface, in the basis \(g^* a^*\).

$$\left( \frac{\partial \bar{W}_0}{\partial \hat{b}^*} \right) = \left[ \frac{\partial \bar{W}_0}{\partial b'^*} \right] (9.83)$$

where \(A^* b_m g^* = b_m A^* g^*\).
(vi) the boundary conditions

\[ u = \hat{u} \] and \((R, N) = (R, N)\) at \(C_m\)

(9.85)

and

\[ \nu_{o_m} e^{\gamma} = \nu N \] and \(\nu_{o_m} e^{\gamma} = \nu R\) at \(C_m\)

(9.86a,b)

and

\[
\begin{align*}
& (vii) \quad \gamma(A) V_o R, n : \delta R + \nu(A) \left[ \hat{\gamma} R, \left( \frac{\partial W_o}{\partial \delta} \right)^T \right] \\
& \quad + \left( \frac{\partial W_o}{\partial \delta} \right) \delta R - \left( \hat{\nu} + \gamma \right) N : \delta R \\
& \quad - (V_o + \nu R, n : \delta R - \hat{\gamma} R, \gamma T : \delta R) \sqrt{\gamma} = 0.
\end{align*}
\]

(9.87)

Noting that (a) \(\delta R, R^T\) is skew-symmetric, (b) \(\eta = (1/2)(n, R + R^T, n^T)\), (c) \(\gamma = \eta R = - (\hat{\nu} \partial / \partial \eta^T)\) we see that (9.87) can be written as:

\[
\begin{align*}
& \gamma(A) V_o R, n : \delta R - \hat{\gamma} R, \gamma T : \delta R \\
& \quad - (V_o + \nu R, n : \delta R - \hat{\gamma} R, \gamma T : \delta R) \sqrt{\gamma} = 0.
\end{align*}
\]

(9.88)

or that

\[
\begin{align*}
& \gamma(A) V_o R, n - \gamma(A) \hat{\gamma} R, \gamma T - (V(A) \eta)_{\alpha \beta} \eta = \text{symmetric}
\end{align*}
\]

(9.89)

where \(n = N, R T = R, N\) is the normal to the deformed mid-surface according to our original definition. Noting that \(\hat{\gamma} T = R, \hat{\gamma} T R, R T\), eqn (9.89) can be written as:

\[
\begin{align*}
& \gamma(A) V_o R, n - \gamma(A) \hat{\gamma} R, \gamma T + \nu R, \nu T = 0
\end{align*}
\]

(9.90)

If one notes that \(V_o R = R U_o\), it is seen that (9.90) is identical to the earlier relation (9.40). Thus eqn (9.90), as was (9.94), is in fact a disguised form of the exact moment balance conditions, as derived earlier, via (9.51) and (9.52b).

As before, we consider certain simplifications to the functional \(F;\) of (9.77). First we note that

\[
\hat{W} = \hat{W}_o \left[ R, V_o, R \right] + \hat{W}_o \left[ R^T, V_o, R \right].
\]

(9.91)

We consider the contact transformation,

\[
\hat{W}_o \left[ R^T, V_o, R \right] = \frac{1}{2} \left( \hat{\gamma} R, n \right) \left( R^T, n \right) - \nu R, V_o = \hat{W}_o \left[ R^T, V_o, R \right]
\]

(9.92)

Using (9.92), we may eliminate \(V_o\) as a variable from \(F_i\) of (9.77). We also eliminate \(u\) from \(F_i\) by satisfying: (i) the force balance condition (9.98), and (ii) the force boundary condition (9.86a). Finally, we eliminate \(\hat{\gamma}\) as a variable by satisfying the curvature strain compatibility condition (9.83). Thus we obtain a mixed variational functional involving \(R\) and \(\eta\) which appears only through \(\eta R = (1/2) \left( \left[ R, n + R, n T \right] R, n \right)\) as variables, as follows:

\[
\begin{align*}
& F_i (\eta, R) = \int_{C_m} \left[ - \gamma(A) \hat{W}_o \left( R^T, \gamma T, \hat{\gamma} R \right) \\
& \quad + \gamma(A) \hat{W}_o \left( R, \gamma T, \hat{\gamma} R \right) \\
& \quad + \left( \frac{\partial \hat{W}_o}{\partial \delta} \right) \delta \left( R \right) - \hat{\nu} R, \gamma T N : \sqrt{\gamma} \right] \, d \gamma \left( A \right) dc.
\end{align*}
\]

(9.93)

where

\[ \hat{\gamma} = - R, \hat{\gamma} T (R, N) \ldots \]

(9.94)

The ELE and NBC of \(\delta F_i = 0\) can again be easily verified to be: (i) compatibility of midplane strains, viz. \((\Delta_o + \beta_o) A^m = V_o R, \gamma T\), (vi) the moment balance condition, (9.90), (iii) the deformation boundary conditions (9.85), and (iv) the moment boundary conditions (9.86b).

The \textit{a priori} constraint of force balance, which as before is:

\[
\begin{align*}
& \gamma(A) \eta_{\alpha \beta} + \nu(A) \hat{\gamma} = 0
\end{align*}
\]

(9.95)

can be satisfied, as before, by setting,

\[
\beta_o = \epsilon^{\rho \gamma} \frac{\partial \gamma}{\partial \rho^\gamma} + \rho^\gamma
\]

(9.96)

and once again, as before, we may take:

\[
\beta_o = \epsilon^{\rho \gamma} \rho^\gamma = \epsilon^{\rho \gamma} \rho^\gamma + \rho^\gamma N
\]

(9.97)

which as before, involve only the geometry of the known undeformed mid-surface.

We now draw some comparisons between the present \(F_i\) and what should be an analogous result in [23]: (i) the frame-indifference of the constitutive relations are studied here from first principles. In fact, \(\beta_o\) is seen from eqns (9.76) and (9.87), that making \(\beta_o\) depend on \(\beta_o \partial W_o / \partial \eta^T\) is necessary to recover the exact moment balance condition. This issue is not explicitly addressed in [23], while the author has not verified that this is implicit in the results of [23], (ii) the linear momentum (LMB) or force balance condition in the present still involves \(\eta\). Thus, as in (9.97) and (9.98), the components of \(\beta_o\) and the stress function vector \(F\) still involve only the geometry of the known undeformed mid-surface. However, as noted earlier, the LMB in [23] is written directly in terms of \(\gamma\), and thus the components of \(\beta_o\) and \(\gamma\) are written in terms of the rotated base vectors \((A^m, \gamma\) in the present notation). For practical applications, the author believes that the present representation may be more convenient, (iii) the definitions of the new stress resultants \(\gamma\) and the stress couples \(\gamma\) appear quite naturally in the present work, without involving any \textit{ad hoc} approximations. While the definitions of stress-resultants and couples in
[23] should be analogous to $\gamma$ and $\gamma'$ of the present, their origin is not so apparent in [23], (iv) the direct tensor notation employed here, at least in the belief of the author, preserves the essential beauty and clarity of modern continuum mechanics.

We conclude this paper by noting that a second simplified functional $F^r$ which involves $\bar{E}$, $\bar{G}$, $\mathbf{I}$ through $\mathbf{R}$, $\gamma$ and $\gamma'$ can be derived in a fashion entirely analogous to that in (9.67).

Note added in proof. It is with a deep sense of sadness the author wishes to note the passing away of Prof. Washizu in November 1981.

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REFERENCES


APPENDIX I

Simple properties of trace operation

By definition, for any $A$ and $B$,

$$A: B = (A^m_{n}g^{mn}) : B^p_{p} = A^m_{n}B^p_{p}g^{mn} = A^m_{n}(B^p_{p}g^{mn}) = A^m_{n}g^{mn}: B^p_{p} = A: B.$$  

(A1.1)

Thus $A: B = A: B = A^m_{n}B^p_{p}$ for any $A$ and $B$. If $A$ is symmetric, while $B$ may not be,


(A1.2)

Let $A$ be a general second order tensor, say, $A = C: D$. Then,

$$A: B = (C: D) : B = (C: D)(D^T g_{mn}C^T) = D^T (C: D) = B^T (D^T C): B.$$  

(A1.3)

By straight forward algebra, it is seen that the above trace operation has the cyclic property:
APPENDIX 2

Representation of a rotation tensor for a shell

As noted in [23, 24] it is convenient to express the rigid body rotation by a single rotation of magnitude \( \omega \) about an axis parallel to some unit vector \( \mathbf{e} \) on the reference surface of the shell. Even though alternate representations of the finite rotation vector are possible [23, 24] here we assume for convenience that the finite rotation vector \( \mathbf{\Omega} \) has the form

\[
\mathbf{\Omega} = \sin \omega \mathbf{\hat{e}} \tag{A2.1}
\]

As noted in [24], the action of the finite rotation vector \( \mathbf{\Omega} \) on a vector \( \mathbf{A} \) can be expressed as the transformation of \( \mathbf{A} \) to \( \mathbf{A}' \) such that:

\[
\mathbf{A}' = \mathbf{A} + \mathbf{\Omega} \times \mathbf{A} + \frac{1}{2} \mathbf{[\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{A})]} \tag{A2.2}
\]

Thus, from (A2.3) and (A2.6) it is seen that \( \mathbf{R} \) is expressed as a function of \( \mathbf{\xi}' \) alone, and further \( \mathbf{R} \) involves the geometric variables of the known undeformed reference surface only. This is convenient in numerical computations.

Using (A2.3) and (A2.6), the components of \( \mathbf{R} \) can be expressed through straightforward algebra in terms of the basis \( \mathbf{\Delta}_{\mathbf{N}} \). The differentiation of \( \mathbf{R} \) with respect to \( \mathbf{\xi}' \) can be carried out in a straightforward manner. This tedious algebra is omitted here.