CHAPTER 5

Energetic Approaches and Path-Independent Integrals in Fracture Mechanics

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1. Introduction

The success of modern fracture mechanics is due, in a large measure, to the celebrated work of Irwin in showing that, for elastic materials, the crack-tip fields are governed by the so-called stress-intensity factor $K$. Likewise, in elastic–plastic materials, the well-known work of Hutchinson, Rice, and Rosengren shows that for stationary cracks in quasi-statically and monotonically loaded bodies of pure power-law hardening materials, the stress and strain fields in the vicinity of the crack-tip under yielding conditions varying from small-scale to full yielding are controlled by the Eshelby–Cherapanov–Rice $J$-integral.

This chapter is concerned with a general discussion of crack-tip parameters governing quasi-static as well as dynamic propagation of cracks in elastic as well as elastic–plastic materials. These crack-tip parameters are, in general, defined, say for two-dimensional problems, as integrals over a circular path $I$, with radius $\varepsilon$ being “very small”. The integrand, which involves the crack-tip stress, strain, and displacement fields, is, in general, such that it is of $(1/\varepsilon)$ type at radius $\varepsilon$ from the crack-tip, which renders the integral over $I$ to be of a finite magnitude. This crack-tip integral parameter is then sought to be represented equivalently as a far-field integral plus a “finite domain integral”, using the divergence theorem. This alternative representation is convenient for computational analyses of fracture problems. Under some special circumstances, the aforementioned “finite domain integral” vanishes identically—thus making it possible to express the crack-tip integral parameter solely as a far-field contour integral. These special circumstances are clearly spelled out in this chapter. In Section 2 of this chapter, we consider self-similar dynamic crack propagation in an elastic solid subject to non-uniform temperature fields, and when the material is considered to be non-homogeneous. In Section 3, we discuss crack-tip parameters for quasi-static as well as dynamic crack propagation in elastic–plastic solids. Finally, some remarks are also made concerning crack-tip parameters in creep crack-growth at elevated temperatures.

2. Elasto-dynamic crack propagation

2.1. Preliminaries

We consider the material to be non-linearly elastic and finitely deformed. We employ a fixed (global) Cartesian coordinate system such that $X_i$ and $Y_i$, refer,
respectively, to the coordinates of a given material particle before and after
deformation. We introduce another “local” Cartesian system \( x_i \) such that \( x_1 \) is
locally normal to the crack border and in the crack plane, \( x_2 \) is normal to the crack
plane, and \( x_3 \) is locally tangential to the crack border and in the crack plane. The
deformation gradient is represented by \( F_{ij} = Y_{i,j} = (\partial Y_i / \partial X_j) \) such that \( dY_i =
F_{ij} \, dX_j \). Henceforth in this section, we shall employ the “nominal” stress, denoted
here by \( t_{ij} \), as the measure of stress in the deformed body. Note that \( t_{ij} = (T_R)_{ij} \)
where \( T_R \) is the first Piola–Kirchhoff stress [1].

The boundary-value problem in elasto-dynamics is, in general, posed by the
equations [2]:

(1) (linear momentum balance):
\[
t_{ij,i} + f_j = \rho \ddot{u}_j
\]
(2.1)

(angular momentum balance):
\[
F_{ik} t_{kj} = F_{jk} t_{ki}
\]
(2.2)

(1) constitutive law):
\[
t_{ij} = \partial W / \partial F_{ij} = \partial W / \partial e_{ij}
\]
where
\[
e_{ij} = u_{i,j}; \quad u_j = Y_j - X_j; \quad F_{ij} = e_{ij} + \delta_{ij}
\]
(2.3)

(1) traction b.c):
\[
n_i t_{ij} = \bar{t}_j \quad \text{at } S_i
\]
(2.4)

displacement b.c):
\[
\dot{u}_j = \ddot{u}_j \quad \text{at } S_u
\]
(2.5)

(initial conditions):
\[
u_j = u_j^0(X_k), \quad \ddot{u}_j = \dddot{u}_j^0(X_k) \quad \text{at } t = 0.
\]
(2.6)

In (2.1)–(2.6), all components refer to the fixed (global) Cartesian system. In
(2.1), \( f_j \) are body forces per unit initial volume, \( \rho \) is the mass density of the
undeformed body, and \( \ddot{u}_j \) are accelerations; where \( (\cdot) \) denotes a material de­

Let \( \sigma_{ij} \) be the true (Cauchy) stress in the deformed body. Then it is well known
that:
\[ t_{ij} = J \frac{\partial X_i}{\partial Y_k} \sigma_{kj} \]  
(2.7)

where \( J \) is the determinant of \( F_{ij} \). If displacements and their gradients are infinitesimal, \( t_{ij} \approx \sigma_{ij} \); and Eq. (2.2) reduces to the condition of symmetry of \( \sigma_{ij} \). Eq. (2.3) is valid, in general, for an inhomogeneous as well as anisotropic body. The condition of material frame indifference imposes certain restrictions \([1]\) on the structure of \( W \); and hence, in general, it is a function only of \( C_{ij} = F_{ki} F_{kj} \). When the structure of \( W \) is thus properly defined, condition (2.2) becomes inherently embedded in the structure of \( W \) (see, for instance, Ref. \([2]\)). In (2.4) and (2.5), \( S_i \) and \( S_u \) are parts of the external surface of the undeformed body, where tractions and displacements are respectively specified.

### 2.2. Self-similar crack propagation

Consider the dynamic propagation of a crack in a \textit{self-similar} fashion such that the crack length increases by \((da)\) in time \((dr)\), with a non-constant velocity of propagation, \( \dot{a} = (da/dr) \). The energy release to the crack-tip per unit of crack extension, denoted by \( G \), is given from global energy considerations as:

\[ G \dot{a} = \frac{DW_{ex}}{Dt} + \frac{DH}{Dt} - \frac{DU}{Dt} - \frac{DT}{Dt} \]  
(2.8)

wherein, on the right-hand side of (2.8), the first term represents the rate of external work done, the second the rate of heat input to the body, the third the rate of change of internal energy in the body, and the fourth the rate of change of kinetic energy in the body. The heat-flux relation is given by:

\[ \frac{DH}{Dt} = -\int_{S} h \cdot n \, ds = -\int_{V} \nabla \cdot h \, dv \]  
(2.9)

where \( h \) is the vector of heat-flux, and \( n \) is the unit outward normal to the boundary \( S \) of the body, \( \nabla = e_{i} (\partial / \partial X_{i}) \). The components of \( n \) are \( n_{i} \) in the \( x_{i} \) system and \( N_{k} \) in the \( X_{k} \) system. Also, in any thermo-mechanical process \([1]\), we have:

\[ \frac{DU}{Dt} = \frac{DW}{Dt} + \frac{DH}{Dt} \]  
(2.10)

where \((DW/Dt)\) is the stress power, and \( W \) is the density of total stress work. Note that \( DW/Dt = t_{ij} \dot{F}_{ij} \) \([2]\). Using (2.10) in (2.8), we obtain:
where $V$ is the total volume of the cracked body, $S_f$ is the surface of $V$ where tractions are prescribed, $f_i$ are body forces, $W$ is the density of total stress work (or equivalently, the strain energy density, only in the case of a non-linear elastic material), and $T = (\frac{1}{2} \rho \ddot{u} \dot{u})$.

Referring to Fig. 1 for nomenclature, we consider, for instance in a two-dimensional problem, an arbitrarily small loop $\Gamma_v$ surrounding the crack-tip, such that the “volume” (or area in a “plane” problem with unit thickness) inside $\Gamma_v$ is $V_v$ (including the crack-tip). For instance, in two-dimensional problems $\Gamma_v$ may be
considered to be a circle of radius $r$, while in three-dimensional problems $I'$ may be considered to be a toroidal surface whose axis coincides with the crack-front and whose cross-section is a circle of radius $r$. If we consider the volume $(V-V_r)$ which does not include the crack-tip, we see that the following equation of conservation of energy holds:

$$0 = \int_{S_t} t \frac{Du_i}{Da} \, ds - \int_{I'} t \frac{Du_i}{Da} \, ds + \int_{V-V_r} f_i \frac{Du_i}{Da} \, dv - D \frac{D}{Da} \int_{V-V_r} (W + T) \, dv.$$  

(2.13)

If $n$ is the unit "outward" normal in the conventional sense, it is seen that the "external" boundary of $(V-V_r)$ is $(S_r-I')$ if the "sense" of $I'$ is as shown in Fig. 1. Using (2.13) in (2.12), it is seen that:

$$G = \lim_{\epsilon \to 0} \left\{ \int_{S_t} t \frac{Du_i}{Da} \, ds + \int_{V-V_r} f_i \frac{Du_i}{Da} \, dv - D \frac{D}{Da} \int_{V-V_r} (W + T) \, dv \right\}.  

(2.14)

Referring to Figs. 2(a) and (b), $P_2$ and $P_2$ represent the same material particle at times $t$ and $(t + dt)$, respectively, when the crack propagates by an amount $(da)$ in time $(dt)$. On the other hand, points $P_1$ and $P_2$ are located at the same distance and orientation (i.e. $r$, $\theta$ as in Figs. 2a, b) from the crack-tips at times $t$ and $t + dt$, respectively. The fundamental idea, in self-similar, elasto-dynamic crack propagation, is that the crack-tip fields are self-similar at times $t$ and $t + dt$, respectively, except that their intensities may differ. From this concept, we see that the changes in displacement, velocity, and stress at the same material particle, due to crack growth by $(da)$, are given by:

$$u_i(P_2) - u_i(P_1) = [u_i(P_2) - u_i(P_1)] - [u_i(P_2) - u_i(P_1)]$$  

(2.15a)

$$= (\partial u_i/\partial a - \partial u_i/\partial x_1) \, da$$  

(2.15b)

with similar relations for changes in $\dot{u}_i$ and $t_{ij}$. It is important to note that $x_i$ is along the direction of (self-similar) crack propagation as in Figs. 1 and 2. Note that terms such as $(\partial u_i/\partial a)$ arise due to a change in the strength of the singularities of the crack-tip fields corresponding to an increase in crack length of $(da)$, while terms such as $(\partial u_i/\partial x_1)$ occur due to the translation of the crack-tip fields by $(da)$. If the material particle is at the external boundary of the specimen, it is easy to see from relations of the type (2.15) that:

$$\frac{\partial u_i}{\partial a} = \frac{\partial u_i}{\partial x_1} \at \text{at } S_u;$$  

$$\frac{\partial t_{ij}}{\partial a} = \frac{\partial t_{ij}}{\partial x_1} \at \text{at } S_r.  

(2.16)
Now consider the third term on the right-hand side of (2.14):

\[ - \frac{D}{Da} \int \left[ (W + T)dV = \int \left[ \langle W + T \rangle(p_2) - \langle W + T \rangle(p_2) \right] / da dV . \]

(2.17)

It is well known [3] that in the case of the propagating crack, \( W \) and \( T \) may possess singularities of the order \( (1/r) \) near the crack-tip. Further, \( (\partial \omega / \partial a) \) and \( (\partial T / \partial a) \) merely represent change in the intensities of the singularities, while the order of their singularities is still \( (1/r) \). However, since \( (\partial W / \partial x_1) \) and \( (\partial T / \partial x_1) \) may lead to
“non-integrable” singularities (which invalidate the application of the divergence theorem to terms of the type \( \int_{V} (\partial W/\partial x_i) \) \( dv \), as done by Eshelby [4] and others), it is more proper to use the concept of “subtracting out singularities” as illustrated in Fig. 2(c) and detailed in [3, 5]. Thus, (2.17) may be written as:

\[
- \frac{D}{Da} \int_{V} (W + T) \, dv = - \int_{V} \frac{\partial}{\partial a} (W + T) \, dv + \int_{l_i} (W + T)n_1 \, ds. \tag{2.18}
\]

Using Eqs. (2.15b) and (2.18), we rewrite (2.14) as:

\[
G = \int_{l_i} \left( (W + T)n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, ds
- \left\{ \int_{V} \left( \frac{\partial}{\partial a} (W + T) - f_i \frac{\partial u_i}{\partial a} - \frac{\partial f_i}{\partial x_1} u_i \right) \, dv - \int_{l_i} t_i \frac{\partial u_i}{\partial a} \, ds \right\}
= \lim_{\varepsilon \to 0} \int_{l_i} \left( (W + T)n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, ds. \tag{2.19}
\]

Eq. (2.20) follows from (2.19) since the second term of (2.19) vanishes in the limit \( \varepsilon \to 0 \), due to the fact that \( \partial W/\partial a \) and \( \partial T/\partial a \) are still of order \( 1/r \) near the crack-tip, \( t_i \) is of \( \mathcal{O}(r^{-1/2}) \), while \( (\partial u_i/\partial a) \) is \( \mathcal{O}(r^{1/2}) \). The result for \( (\partial G) \), analogous to that in (2.20), has been derived earlier by Atkinson and Eshelby [6] and Eshelby [7], even though not conclusively for a crack propagating with an arbitrary history of motion. Using arguments similar to those used above in deriving (2.19) from (2.14), one may likewise derive from (2.12) that:

\[
G = \int_{S} \left( (W + T)n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, ds
- \left\{ \int_{V} \left( \frac{\partial}{\partial a} (W + T) - f_i \frac{\partial u_i}{\partial a} - \frac{\partial f_i}{\partial x_1} u_i \right) \, dv - \int_{S} t_i \frac{\partial u_i}{\partial a} \, ds \right\} \tag{2.21}
\]

where \( S \), the external boundary of \( V \), is such that \( S = S_r + S_u \), and use has been made of (2.16). Note that the second term in (2.21) does not vanish; its evaluation in the practical problem of crack propagation in an arbitrary finite body involves, however, two solutions for slightly different crack-length, \( a \) and \( (a + da) \). However, if one considers the volume \( V_T - V_r \) (where \( T \) is any path surrounding the crack-tip; see Fig. 1) and thus excludes the crack-tip, it is a simple matter to apply the divergence theorem and rewrite (2.20) as:

\[
G = \int_{r + S_T} \left( (W + T)n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, ds
- \int_{V_r - V} \left( \frac{\partial}{\partial x_1} (W + T) - t_{i,1} \frac{\partial^2 u_i}{\partial x_1^2} - t_{i,1} \frac{\partial^2 u_i}{\partial x_1^2} \right) \, dv. \tag{2.22}
\]
We now restrict attention, without much loss of generality, to infinitesimal deformations of cracked non-linear elastic bodies which are subject to non-uniform temperature fields. In such a case, the infinitesimal strain tensor \( \varepsilon_{ij} \) may be decomposed as:

\[
\varepsilon_{ij} = \varepsilon_{ij}^m + \varepsilon_{ij}^e
\]

(2.23)

where \( \varepsilon_{ij}^m \) are "mechanical" strains, and \( \varepsilon_{ij}^e \) are "thermal" strains \( (\varepsilon_{ij}^e = \alpha \theta \delta_{ij}) \); where \( \alpha \) is the coefficient of thermal expansion, \( \theta \) is the temperature rise \( (T - T_0) \); \( \theta = \theta(x_k) \); and \( T_0 \) is the ambient temperature). The total stress-working density is:

\[
W = \int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij}
\]

(2.24)

where

\[
\sigma_{ij} = f_{ij}(\varepsilon_{kl} - \varepsilon_{kl}^e), x_k
\]

(2.25)

and

\[
W = W(\varepsilon_{ij}, \theta, x_k).
\]

(2.26)

In (2.25) \( f_{ij} \) is a tensor-valued function. Note that in thermoelasticity stress depends on the mechanical strains as well as explicitly on \( x_k \), since the material may be non-homogeneous (either naturally or due to temperature dependence of the material properties in a non-uniform temperature field). Thus

\[
\frac{\partial W}{\partial x_1} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1} + \frac{\partial W}{\partial x_1} \bigg|_{\text{explicit}}
\]

\[
= \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1} + \int_0^{\varepsilon_{ij}} \left[ \frac{\partial f_{ij}}{\partial x_1} \bigg|_{\text{explicit}} + \frac{\partial f_{ij}}{\partial \varepsilon_{kl}^e} \frac{\partial \varepsilon_{kl}^m}{\partial \theta} \frac{\partial \theta}{\partial x_1} \right] d\varepsilon_{ij}
\]

(2.27)

wherein the definition of \( (\partial W/\partial x_1)_{\text{explicit}} \) is apparent. Use of (2.27) in (2.22) results in:

\[
G = J' = \int_{r^1, S, \Gamma} \left( (W + T)n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds
\]

\[
- \lim_{\varepsilon \to 0} \int_{V_{\Gamma}} \left( \rho \dot{u}_i \frac{\partial \dot{u}_i}{\partial x_1} + (f - \rho \ddot{u}_i \frac{\partial u_i}{\partial x_1} + \frac{\partial W}{\partial x_1} \bigg|_{\text{exp}} \right) dv \quad (2.28a)
\]

\[
= \int_S \left( (W + T)n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds
\]

\[
- \lim_{\varepsilon \to 0} \int_{V-V_{\Gamma}} \left[ \rho \ddot{u}_i \frac{\partial \ddot{u}_i}{\partial x_1} + (f_i - r \ddot{u}_i \frac{\partial u_i}{\partial x_1} + \frac{\partial W}{\partial x_1} \bigg|_{\text{exp}} \right] dv. \quad (2.28b)
\]
If the material is homogeneous in the \( x_1 \) direction, and \( (T - T_0) = 0 \), i.e. isothermal conditions prevail in the solid, then \( (\partial W/\partial x_1)_{\text{explicit}} = 0 \). On the other hand, consider for example an isotropic linear elastic material with non-uniform temperature distribution \( \theta(x_k) \) and non-homogeneous material properties. Here,

\[
\sigma_{ij} = E_{ijkl} \varepsilon_{kl}^m = (2\mu\delta_{ik}\delta_{jl} + \lambda\delta_{ij}\delta_{kl})(\varepsilon_{kl} - \alpha\theta\delta_{kl})
\]  

(2.29)

such that

\[
\frac{\partial W}{\partial x_1} \bigg|_{\text{explicit}} = \int_0^{\varepsilon_{ij}} \frac{\partial E_{ijkl}}{\partial x_1} \mu_{ijkl} \varepsilon_{ij} \, d\varepsilon_{ij} - \int_0^{\varepsilon_{ij}} E_{ijkl} \alpha \delta_{kl} \frac{\partial \theta}{\partial x_1} \, d\varepsilon_{ij}
\]

\[
= \frac{1}{2} \varepsilon_{ij} E_{ijkl} \varepsilon_{kl}^m - \varepsilon_{ij} E_{ijkl} \varepsilon_{kl}^m - (2\mu + 3\lambda)\alpha \varepsilon_{kk} \theta_{,1}.
\]  

(2.30)

where \( (\ )_{,1} = \partial(\ )/\partial x_1 \). Now, from (2.29) one obtains:

\[
\sigma_{ij,1} = E_{ijkl,1} \varepsilon_{kl}^m + E_{ijkl} \varepsilon_{kl,1}^m - E_{ijkl} \varepsilon_{kl,1}^m - E_{ijkl,1} \varepsilon_{kl}^m.
\]  

(2.31)

Use of (2.31) in (2.30) results in:

\[
\frac{\partial W}{\partial x_1} \bigg|_{\text{exp}} = \frac{1}{2} \sigma_{ij,1} \varepsilon_{kl}^m - \frac{1}{2} \sigma_{ij} \varepsilon_{ij,1}^m - \frac{1}{2} (2\mu + 3\lambda)\alpha \varepsilon_{kk,1} \theta_{,1}
\]

(2.32)

Note that \( W \) in (2.24) and (2.26) is given for the present linear elastic isotropic case as:\n
\[
W = \mu \varepsilon_{ij} \varepsilon_{ij} + \frac{1}{2} \lambda \varepsilon_{kk}^2 - (2\mu + 3\lambda)\alpha \varepsilon_{kk} \theta.
\]

If the temperature dependence of \( \mu \) and \( \lambda \) is ignored, the path-independent integral representation for energy-release rate, (2.28), becomes:

\[
J' = \int_{r+S_{,1}} \left( (W + T) n_1 - t_j \frac{\partial u_j}{\partial x_1} \right) \, ds
\]

\[
- \int_{\nu_{l}^{-} - \nu_{l}^{+}} \left[ (\rho \ddot{u} \dot{u}_{l,1} + (f_l - \rho \ddot{u}_l) u_{l,1} - (2\mu + 3\lambda)\alpha \varepsilon_{kk} \theta_{,1} \right] \, dv
\]  

(2.33)

where \( (\ )_{,1} = \partial(\ )/\partial x_1 \). Now, note the identity:

\[
\int_{\nu_{l}^{-}}^{(2\mu + 3\lambda)\theta \varepsilon_{kk} \alpha n_1 \, ds = \int_{r+S_{,1}}^{(2\mu + 3\lambda)\theta \varepsilon_{kk} \alpha n_1 \, ds}
\]

\[
- \int_{\nu_{l}^{-} - \nu_{l}^{+}} \left[ (2\mu + 3\lambda)\alpha \theta \varepsilon_{kk} \right]_{,1} \, dv.
\]  

(2.34)

\(^{1}\)This assumes that a temperature field is initially prescribed, and remains stationary during the mechanical loading of the solid.
Adding (2.34) and (2.33), we may define a parameter $\tilde{G}$, such that:

$$
\tilde{G} = J' + \int_{\Gamma_1} (2\mu + 3\lambda)\theta \varepsilon_{kk} \alpha n_1 \, ds = \int_{\Gamma} [(W^* + T)n_1 - t_i u_{i,1}] \, ds
$$

$$
= \int_{\Gamma + S_{ij}} [(W^* + T)n_1 - t_i u_{i,1}] \, ds
$$

$$
- \int_{\Gamma + S_{ij}} [\rho \ddot{u}_j u_{i,1} + (f_j - \rho \ddot{u}_j) u_{i,1} + (2\mu + 3\lambda)\alpha \theta \varepsilon_{kk,1}] \, dv
$$

(2.35)

where $W^* = \mu \varepsilon_{ij} \varepsilon_{ij} + (\lambda/2) \varepsilon_{kk}^2$. On the other hand, the Navier equations of isotropic elasticity in the presence of non-uniform temperature fields is given by:

$$
\mu u_{i,kk} + (\lambda + \mu) \varepsilon_{kk,i} - (2\mu + 3\lambda)\alpha \theta_j = 0.
$$

(2.36)

Thus

$$
\int_{\Gamma + S_{ij}} (\lambda + \mu) \theta \varepsilon_{kk,i} \, dv = \int_{\Gamma + S_{ij}} [(2\mu + 3\lambda)\theta \varepsilon_{kk,i} - \mu \theta u_{i,kk}] \, dv
$$

$$
= \mu \int_{\Gamma + S_{ij}} \theta_j u_{i,k} \, dv + \int_{\Gamma + S_{ij}} \left[ \frac{(2\mu + 3\lambda)}{2} \theta^2 n_i - \mu \theta u_{i,k} n_k \right] \, ds.
$$

(2.37)

If the temperature field in the body is assumed to obey the harmonic equation

$$
\theta_{ii} = 0,
$$

then

$$
\int_{\Gamma + S_{ij}} \theta_j u_{i,k} \, dv = \int_{\Gamma + S_{ij}} \theta_j u_{i,k} n_k \, ds.
$$

(2.39)

Thus, when the process is quasi-static ($T = 0, \dot{u}_j = 0$), it is possible to use (2.37) and (2.39) in (2.35) and define a modified parameter $\tilde{G}$ such that

$$
\tilde{G} = J_{QS} + \int_{\Gamma} \left\{ (2\mu + 3\lambda)\theta \varepsilon_{kk} \alpha n_1 - \frac{(2\mu + 3\lambda)}{(\lambda + \mu)} \alpha \right. 
$$

$$
\times \left[ \mu \theta_{j,k} n_k u_1 + \frac{\lambda}{2} (2\mu + 3\lambda) \alpha \theta^2 n_1 - \mu \theta u_{i,k} n_k \right] \right\} \, ds
$$

(2.40)

$$
= \int_{\Gamma + S_{ij}} \left\{ W^* n_1 - t_i u_{i,1} - \left( \frac{2\mu + 3\lambda}{\lambda + \mu} \right) \alpha \right. 
$$

$$
\times \left[ \mu \theta_{j,k} n_k u_1 + \frac{\lambda}{2} (2\mu + 3\lambda) \alpha \theta^2 n_1 - \mu \theta u_{i,k} n_k \right] \right\} \, ds.
$$

(2.41)
Thus, (2.41) represents a *path-independent* integral expression (without the presence of a domain integral) for \( \tilde{G} \). This result is due to Gurtin [8]. Note that \( J'_{\alpha \beta} \) is the "quasi-static" value of the energy-release rate \( J' \) (i.e. when the material inertia is ignored). Thus, while \( \tilde{G} \) represents a mathematically convenient integral in the case: (1) of linear isotropic homogeneous elasticity; (2) when \( \mu \) and \( \lambda \) do not depend on temperature; and (3) when the temperature satisfies \( \theta_{d} = 0 \); its physical significance is somewhat obscure. The point to be made here is that the situation when a meaningful crack-tip parameter can be represented equivalently by a far-field contour-integral alone (i.e. without the presence of a domain-integral) is rather rare in practice.

Sometimes in thermoelasticity it is convenient to define the stress potential:

\[
V = \int_{0}^{\gamma} \sigma_{ij} \, d\epsilon_{ij}^m
\]  
(2.42)

such that

\[
V = V(\epsilon_{ij}^m, x_k)
\]  
(2.43a)

and

\[
\sigma_{ij} = g_{ij}(\epsilon_{ij}^m, x_k).
\]  
(2.43b)

Note that \( V \) is simply a mathematical potential and is not equal to *stress-working* density, we have:

\[
\frac{\partial V}{\partial x_1} = \sigma_{ij} \frac{\partial \epsilon_{ij}^m}{\partial x_1} + \int_{0}^{\gamma} \frac{\partial g_{ij}}{\partial x_1} \, d\epsilon_{ij}^m
\]  
(2.44)

Thus, one may define a crack-tip parameter:

\[
G^* = \int_{t_c}^{(V + T)n_1 - t_i u_{i,1}} [(V + T)n_1 - t_i u_{i,1}] \, ds = \int_{t_c}^{t_i} [(V + T)n_1 - t_i u_{i,1}] - \int_{V_{i}-V_{t}} \left[ \rho \ddot{u}_{i,1} + (f - \rho \ddot{u}) u_{i,1} - \sigma_{kk} \alpha \theta_{k,1} + \frac{\partial V}{\partial x_1} \right]_{\exp} \, ds.
\]  
(2.45)

If the material is linear elastically isotropic, and \( \lambda \) and \( \mu \) depend on temperature, we have:

\[
\frac{\partial V}{\partial x_1} \bigg |_{\exp} = \int_{0}^{\gamma} \frac{\partial E_{ijkl}}{\partial x_1} \, \mu_{kl}^m \, d\mu_{ij}^m
\]  
(2.46a)

\[
= \frac{1}{2} E_{ijkl} \epsilon_{ij}^m \epsilon_{kl}^m.
\]  
(2.46b)
From the relations:

\[ \sigma_{ij} = E_{ijkl} \varepsilon_{kl}^{m}, \quad \sigma_{0,1} = E_{ijkl} \varepsilon_{kl}^{m} + E_{ijkl} \varepsilon_{kl,1} \]  

one may write (2.46b) as:

\[ \frac{\partial V}{\partial x_1} \bigg|_{\text{exp}} = \frac{1}{2} \sigma_{ij,1} \varepsilon_{kl}^{m} - \frac{1}{2} \sigma_{ij} \varepsilon_{kl,1} \]  

The case when (1) \( T = 0 \), (2) \( \ddot{u}_i = 0 \), and (3) the material constants are independent of temperature, i.e. \( (\partial V/\partial x_1)_{\text{exp}} = 0 \), has been reported by Ainsworth et al. [9]. Note, however, even this case (or in general) \( G^* \neq J'(=G) \). It is more natural to deal with the density of stress-work, \( W \), in the case of a non-linear material.

When the material is homogeneous in the \( x_i \) direction, and when isothermal conditions prevail, Eq. (2.28) reduces to

\[ G = J' = \int_{F^*} \left[ (W + T) n_i - t_i \frac{\partial u_i}{\partial x_1} \right] ds 
- \int_{V^*} \left[ \rho \ddot{u}_i \frac{\partial \ddot{u}_i}{\partial x_1} + (f_i - \rho \ddot{u}_i) \frac{\partial u_i}{\partial x_1} \right] dv \]  

where \( F^* \) is any arbitrary contour that encircles the crack-tip.

The sense of path-independence embodied in Eq. (2.49) implies that for any \textit{closed volume} \( V^* \), with a boundary \( F^* \), \textit{not enclosing} the crack-tip, as in Fig. 1, we have:

\[ \int_{F^*} \left[ (W + T) n_i - t_i \frac{\partial u_i}{\partial x_1} \right] ds \, \int_{V^*} \left[ \rho \ddot{u}_i \frac{\partial \ddot{u}_i}{\partial x_1} + (f_i - \rho \ddot{u}_i) \frac{\partial u_i}{\partial x_1} \right] dv = 0 \]  

which may be verified easily under the assumption of (2.1), material homogeneity along \( x_1 \), and when \( W \) is a single valued function of \( F_{ij} \).

Because of the use of \( J' \) as defined for any path \( F \) as in (2.49) involves a \textit{volume-integral}, the above notion of \textit{path-independence} has been pronounced by many to be useless. This viewpoint, however, is somewhat orthodox. True, the evaluation of (2.49) involves taking the limit of the volume integral to the crack-tip; and thus, on the surface, it appears to involve a "knowledge of the crack-tip fields", which the so-called \( J \)-integral of \textit{elasto-statics} [when \( \ddot{u} = \dddot{u} = 0 \) in (2.49)] does not involve. First of all, it is clear from (2.49) that its use does not require a knowledge of the crack-tip stress--strain fields, but only of displacement, velocity, and acceleration. Furthermore, a comparison of (2.49) (when evaluated over the external surface \( S \)) and (2.21) reveals that
and thus, the left-hand side of (2.51) remains finite in the limit \( \varepsilon \to 0 \). This is interesting if one notes that, in known analytical asymptotic solutions [10], \( \ddot{u}_i \sim O(r^{-1/2}) \) and \( \ddot{u}_i \sim O(r^{-3/2}) \); and hence, on first glance, the left-hand side of (2.51) appears to contain non-integrable singularities. It has also been verified directly [10] that for known analytical asymptotic solutions for infinite bodies, the volume integral in (2.49) does have a finite limit, due to the fact that the angular variation of the integrand is such that:

\[
\lim_{\varepsilon \to 0} \int \left[ \int_{\Gamma} (\rho \ddot{u}_i u_{i,k}) r \, dr \right] d\theta \to 0.
\]

Even though finding the solution of \( u_i, \ddot{u}_i, \dddot{u}_i \) near the crack-tip in a finite body is a difficult problem analytically, it is a relatively simple task in computational mechanics. This has been demonstrated conclusively [11, 12] in a variety of crack-propagation problems in finite bodies, even while using the simplest of crack-tip finite elements which do not model any of the singularities in strain, velocity, or acceleration.

If one considers the energy-release rate per unit time in self-similar elasto-dynamic crack propagation, one sees that this quantity is represented by:

\[
CG = C \int_{\Gamma_i} \left( (W + T)n_1 - \frac{\partial u_i}{\partial x_1} \right) dS
\]

\[
= \int_{\Gamma_i} \left( (W + T)C_k N_k - C_k l_i \frac{\partial u_i}{\partial X_k} \right) dS
\]

where \( C \) is the non-constant velocity of crack propagation along the \( x_1 \) direction, \( n_1 \) is the component of a unit normal to \( \Gamma_i \) along \( x_1 \), while \( C_k \) and \( N_k \) are components of the instantaneous velocity vector and the unit normal to \( \Gamma_i \), respectively, along the \( X_k \) directions (see Fig. 1). (Note that the velocity vector \( C \) with \( |C| = C \), along the \( x_1 \) direction in self-similar propagation, may be considered to have components \( C_k \) along the \( X_k \) directions.) It is now a simple task (1) to apply the divergence theorem, (2) to use the coordinate-invariant forms of the linear momentum balance laws of (2.1), under the assumption:

\[
\frac{\partial W}{\partial X_k} = \frac{\partial W}{\partial u_{i,j}} \frac{\partial u_{i,j}}{\partial X_k}
\]

\( (2.54) \)
i.e. $W$ does not depend explicitly on all the $X_k$ (or the material is homogeneous in all the $X_k$ directions), to derive from (2.53b):

$$
CG = C_k J'_k = \left[ \lim_{\epsilon \rightarrow 0} \int_{\Gamma + S'_{\epsilon T}} \left( (W + T)N_k - t_i \frac{\partial u_i}{\partial X_k} \right) ds \right.
\]

$$
- \int_{V_{\epsilon T} - V_\epsilon} \left( \rho \dot{u}_i \frac{\partial \dot{u}_i}{\partial X_k} + (f_i - \rho \ddot{u}_i) \frac{\partial \dot{u}_i}{\partial X_k} \right) dV \right] C_k . \tag{2.55}
$$

The sense of path-independence embodied in (2.55) is similar to that in (2.49), (2.50). In the above, $S_{\epsilon T}$ which is equal to $(S'_{\epsilon T} + S'_{\epsilon T})$ (+ and − referring, arbitrarily, to the crack faces) is the crack surface enclosed within $\Gamma$, while $S_c$ is the total crack surface. Thus, an evaluation of $J'_k$ not only involves a volume integral, but also an integral along the crack faces. The infinitesimal strain counterparts of the $J'_k$ integrals have been first stated in [10], based on a simple modification to the $J_k$ integrals for dynamic crack propagation given in [3].

It is important to note the meaning of (2.55) – it still governs the energy release per unit time, due to self-similar propagation (along the $x_1$-axis). $J'_k$ would simply characterize the total energy change due to a unit translation of the crack as a whole, rigidly, in the $X_k$ direction. Thus, $J'_k$ does not characterize the energy release due to a unit motion of the crack-tip in the $X_k$ direction (and thus kinking the original crack). In fact there are no simple integrals that characterize the energy release due to kinking of a crack, as is often erroneously implied in the literature [13, 14]. This is due to the fact that in deriving (2.18), which forms the basis of all the ensuing path-integrals thereof, use has been made of the self-similarity of solutions at time $t$ and $t + dt$, which is valid only in self-similar elastic crack propagation but not in the case, in general, of a kinked crack.

Assuming for the moment that the global and the crack-tip coordinates coincide, one may define:

$$
J'_2 = \lim_{\epsilon \rightarrow 0} \int_{\Gamma + S'_{\epsilon T}} \left( (W + T)N_2 - t_i \frac{\partial u_i}{\partial X_2} \right) ds
\]

$$
- \int_{V_{\epsilon T} - V_\epsilon} \left( \rho \dot{u}_i \frac{\partial \dot{u}_i}{\partial X_2} + (f_i - \rho \ddot{u}_i) \frac{\partial \dot{u}_i}{\partial X_2} \right) dV \tag{2.56}
$$

which would characterize the total energy change for a unit rigid translation of the crack as a whole (and not a unit growth of the crack-tip alone) in the $X_2$ direction. Assuming zero body force, traction-free crack faces, and elasto-static deformations, one may reduce (2.56) to:

$$
J_2 = \int_{\Gamma} \left( WN_2 - t_i \frac{\partial u_i}{\partial X_2} \right) ds + \lim_{\epsilon \rightarrow 0} \int_{S'_{\epsilon T}} (W^+ - W^-) dS \tag{2.57}
$$
wherein, for a flat crack face, \( N_2^+ = -N_2^- = -1 \). The definition of \( J_2 \) of Budiansky and Rice [15], on the other hand, does not involve the crack-face integral, which accounts for discontinuities of \( W \) along the crack face. Thus, as also noted by [16, 17], \( J_2 \) as given by [15] is not path-independent. Even though (2.57) appears to involve a knowledge of crack-tip \( W \) for its successful application as a path-independent integral, the use of (2.57) has been conclusively demonstrated [12, 18] in computational approaches using simple (non-singular) crack-tip finite elements.

From the above discussions, it should be clear that neither the integrals \( J'_k \) nor any other similarly "path-independent" integrals provide any information as to kinking of a crack or of the direction of propagation of the crack-tip in anything other than a collinear fashion, contrary to speculations often made in the literature [14, 19, 20].

Using the asymptotic solutions in self-similar crack propagation, even under arbitrary time history of motion of the crack-tip, namely \( \dot{u}_i \sim -C\partial u_i/\partial x_1 \), it is seen that the energy-release rate expression in (2.53a) reduces to that of Freund [21]. It is worth noting that (2.53a) as well as Freund's result are valid for an arbitrary shape of the loop \( \Gamma_c \) near the crack-tip. On the other hand, if consideration is restricted to fully steady-state (i.e. the field everywhere is invariant with respect to an observer moving with the crack-tip) self-similar propagation at a constant crack-tip velocity, it is seen that everywhere in \( V \), one has: \( \dot{u}_i = -C\partial u_i/\partial x_1 \); \( \dot{u}_0 = -C\partial^2 u_i/\partial x_1^2 \); \( \dot{u}_i = C^2 \partial^2 u_i/\partial x_1^2 \). [Note, however, even at constant velocity, unsteady conditions in general imply that: \( \dot{u}_i = (\partial u/\partial t) - C(\partial u_i/\partial x_1) \) and \( \dot{u}_i = (\partial^2 u_i/\partial x_1^2) + C^2(\partial^2 u_i/\partial x_1^2) - 2C(\partial^2 u_i/\partial x_1 \partial t) \]. Thus, when body forces \( f_i = 0 \), for steady-state, constant velocity propagation, the volume integral in (2.49) disappears; and the resulting expression, with \( 2T = \rho C^2(\partial u_i/\partial x_1)^2 \), becomes identical to that given by Sih [22], even though the far-field contour considered in [22] moves along with the crack-tip at the same velocity. It may be noted, however, that such steady-state constant-velocity propagation seldom occurs in practical problems of fast fracture in finite bodies; see [23] for further details.

Inasmuch as \( J'(=G) \) as defined in (2.20) has a well-defined physical meaning as the crack-tip energy release rate and can be conveniently computed from simple numerical procedures from far-field quantities through (2.49), it can be used as a parameter governing elasto-dynamic crack propagation and arrest. The relations between \( J'_k \) and the dynamic stress-intensity factors are given in [10]. \( J' \) is, in general, a function of the crack-tip velocity [10]. In a dynamic fracture problem, initiation of propagation occurs at \( J' = J'_0 \) and during crack propagation, \( J' = J'_0(C) \) where \( J'_0 \) and \( J'_0 \) are material properties. Examples of prediction of crack-propagation histories and crack-arrest using these criteria and comparison with experimental results may be found in Refs. [11, 12, 18, 23, 24].

As noted, the far-field path \( \Gamma \) in (2.49) is fixed in space. On the other hand, considering a far-field contour \( \Gamma \) to be a rigid path surrounding the crack-tip and in translation at the same velocity \( C \) along the \( x_1 \)-axis, a path-independent integral, denoted here by \( J_{1B} \), was given by Bui [25, 26] and Erlacher [27] for infinitesimal
deformation:
\[
J_B = \int \left[ W n_1 - n_k \sigma_{kj} u_{j,1} - T n_1 - \rho \ddot{u}_{j,1} C n_1 \right] \, ds + \frac{D}{Dt} \int \rho \ddot{u}_{j,1} \, dV
\]
(2.58)

where \( (\cdot)_{,1} = \frac{\partial (\cdot)}{\partial x_1} \). When the material derivative for a moving control volume containing singularities is properly treated, it may be shown (see Refs. [10, 23]) that (2.58) is equivalent to (2.49). However, experience has shown that (2.49) with a fixed path is easier to use directly in a computational scheme [11, 12, 18, 23, 24].

For linear elastic materials undergoing infinitesimal deformations, Irwin [28] and Erdogan [29] gave the expression for energy-release rate in dynamic crack propagation, as:

\[
CG(t_0) = -\frac{1}{2} \left( \frac{d}{dt} \right)_{t_0} \int I_{x_1} (x_1, t_0) u_1 [\{x_1 - (a(t) - a(t_0))\}, t_0] \, dx_1 .
\]
(2.59)

Thus, it is the work of tractions at \( t_0 \) in moving through the displacements at the corresponding points at time \( t_0 + dt \). The validity of (2.59) has been established for linear elasto-dynamics by Gurkin and Yatomi [30]. On the other hand, Achenbach [31] gives, for finite deformations as well as non-linear elastic behavior, the expression for \( G \), as:

\[
G = \left( \frac{1}{C} \right) \lim_{\delta \to 0} \int_{x_2 - \delta}^{x_2 + \delta} \sigma_{x_2}(x_1, 0, t) [\ddot{u}_i(x_1, 0^+, t) - \ddot{u}_i(x_1, 0^-, t)] \, dx_1
\]
(2.60)

where \( u_i(x_1, 0^+, t) \) denote the particle velocities at the crack surfaces, \( x_2 = 0^+ \). The tip of the crack is denoted by \( x_1 = a \), and \( \varepsilon \) a small number. Now, consider the path \( I_x \) in (2.20) to be a rectangle of height \( 2\delta \) (in the \( x_2 \) direction) and width \( 2\varepsilon \) (in the \( x_1 \) direction) and centered at the crack-tip. Thus, we may write from (2.20) that:

\[
G = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int (W + T) n_1 - t_i \frac{\partial u_i}{\partial x_1} \, ds
\]
(2.61)

since \( n_1 \) is zero on segments parallel to the \( x_1 \)-axis, and the integral of \((W + T)n_1\) vanishes along segments parallel to the \( x_2 \)-axis in the limit \( \delta \to 0 \). Also, near the crack-tip, \( \ddot{u}_i \approx -C\partial u_i/\partial x_1 \). Thus, it appears on first glance that (2.60) is the correct limit of (2.62). However, this has conclusively been disproved by Yatomi [32] who shows that the limit \( \delta \to 0 \) must be taken after the integral in (2.62) is evaluated; and in any event, (2.20) always leads to the correct result, even for finite-
deformation non-linear elastic problems, irrespective of the shape of $I_\gamma$. For the special path described above, the general validity of (2.61), (2.62) is established by Gurtin and Yatomi [30].

Strifors [19] and Carlsson [20], on the other hand, apply the "principle of virtual work" to an arbitrary part $V_F$ of the body containing a crack (as in Fig. 1), which they consider to have a "finite cohesive zone". Their [19, 20] definition of "an apparent crack-extension force", written below, for instance, in the $x_1$ direction, is arrived at by them [19, 20] by considering virtual displacements of the form $\delta u_i = -\partial u_i / \partial x_1$ over $V_F$ as well as over a cohesive zone of size $\epsilon$, as:

$$F = \int_{V_F} \left[ t_{ij} \left( \partial u_j / \partial x_1 \right) - (f_i - \rho \bar{u}_i) (\partial u_i / \partial x_1) \right] ds - \int_{r+S_\epsilon} t_i (\partial u_i / \partial x_1) ds$$

$$= \lim_{r \to 0} \int_{r}^{r+S_\epsilon} \left[ (\partial u_i / \partial x_1)^- - (\partial u_i / \partial x_1)^+ \right] dx_1.$$  \tag{2.63}

From the preceding arguments, it may be seen that the extreme right-hand side of Eq. (2.63) does not have the meaning of an energy-release rate even in the limited situations when (2.60) may be valid, because of the only one-sided limit of integration appearing in (2.63). Further, since $t_{ij}(\partial u_j / \partial x_1)$ and $(\bar{u}_i, \partial u_i / \partial x_1)$ may have singularities of order greater than $(r^{-1})$, the limit of the integral over $V_F$ must be considered separately. Thus, even though $F$ in (2.63) is path independent, its meaning is not clear. Kishimoto et al. [13] define a parameter such that:

$$\dot{j} = - \int_{I_\epsilon} t_i (\partial u_i / \partial x_1) dS$$  \tag{2.64}

where $I_\epsilon$ is a non-distorting "small" contour which moves at the same speed as the crack-tip. Even though $\dot{j}$ as in (2.64) is defined in [13] as one of the components needed in analyzing crack growth at an angle to the initial direction, this concept is questionable for reasons discussed earlier. Further, for arbitrary $I_\gamma$, $\dot{j}$ as in (2.64) is the rate of work done on the process zone of size $I_\epsilon$, by the surrounding medium and is not the energy release to the crack-tip. From (2.64), and the divergence theorem, they [13] derive the "far-field" expression:

$$\dot{j} = \int_{I_\epsilon+S_\epsilon} \left( W_{ni} - t_i \partial u_i / \partial x_1 \right) dS + \int_{V-V_\epsilon} \left( \rho \bar{u}_i - f_i \right) (\partial u_i / \partial x_1) dV - \int_{I_\epsilon} W_{ni} ds.$$  \tag{2.65}

Note the presence of a near-field integral on $I_\epsilon$, in the "far-field" expression (2.65). In [13] this integral over $I_\epsilon$ on the right-hand side of (2.65) is dropped (see Eqs. (24), (25) of [13]) by considering a special case of $I_\epsilon$ to be a rectangle of size $(2\epsilon \times 2\delta)$ centered at the crack-tip. However, it should be noted that the integral
of \( W_{n_1} \) over \( \Gamma_r \) does not vanish for arbitrary \( \Gamma_r \). Also if the integral over \( \Gamma_r \) is dropped from (2.65) and the resulting integral is considered in the limit when \( G \) is shrunk to \( \Gamma_r \), one obtains a near-field definition of \( \hat{J} \) from (2.65) that is different from the original definition, (2.64). Kishimoto et al. [14] in a later paper, redefine \( \hat{J} \) as

\[
\hat{J} = \int_{\Gamma_r} [W_{n_1} - t_i(\partial u_i/\partial x_1)] dS
\]

\[
= \int_{\Gamma_r} [W_{n_1} - t_i(\partial u_i/\partial x_1)] dS + \int_{\Gamma_r} (\rho \ddot{u}_i - f_i)(\partial u_i/\partial x_1) dV \quad (2.66)
\]

and consider [14] \( \hat{J} \) in (2.66), (2.67) as the "energy release rate per unit of crack translation in the \( x_1 \) direction". Comparing (2.66) with (2.20), it is seen that such is not the case for arbitrary shapes of \( \Gamma_r \), since (2.66) does not contain the rate of change of kinetic energy in the energy balance for dynamic crack growth.

It is easy to see that:

\[
\int_{\Gamma_r} \rho \ddot{u}_i(\partial u_i/\partial x_1) dV = \int_{\Gamma_r} [\partial(\rho \ddot{u}_i t_i) - \rho \dot{u}_i(\partial \dot{u}_i/\partial x_1)] dV
\]

\[
= \int_{\Gamma_r} (\rho \ddot{u}_i t_i n_1 ds - \int_{\Gamma_r} \rho \ddot{u}_i t_i n_1 ds
\]

\[
= -\int_{\Gamma_r} \rho \dot{u}_i(\partial \ddot{u}_i/\partial x_1) dV \quad (2.68)
\]

Using (2.68) and the rather extraordinary case when \( f_i \) are constants [i.e. \( f_i \neq f_i(x) \)], one may derive from (2.67) what Ouyang [33] defines as a parameter \( Y_3 \) for elasto-dynamic crack propagation (and an associated parameter \( Y_0 \), slightly different from \( Y_3 \), to account for plasticity), as:

\[
Y_3 = \hat{J} + \int_{\Gamma_r} \rho \ddot{u}_i u_i n_1 ds = \int_{\Gamma_r} [(W + \rho \ddot{u}_i t_i)n_1 - t_i(\partial u_i/\partial x_1)] ds
\]

\[
= \int_{\Gamma_r} [W_{n_1} - t_i(\partial u_i/\partial x_1) + (\rho \ddot{u}_i - f_i)u_i n_1] ds
\]

\[
- \lim_{t \to 0} \int_{\Gamma_r} \rho \dot{u}_i(\partial \ddot{u}_i/\partial x_1) dV \quad (2.69)
\]

Comparing (2.69) with (2.20) it is seen that \( Y_3 \) is not in general an energy release rate for elasto-dynamic crack propagation, and hence its use as a fracture parameter is questionable. Likewise, the parameter \( Y_2 \) of [33], which is the integral in time of \( Y_1 \) [similar to the time integral of \( G \) of (2.17) which would give the total fracture energy up to the current time \( t \)] is not a meaningful fracture parameter.
More recently, however, Aoki et al. [34] define a parameter which, for elasto-dynamic crack propagation, may be defined as:

\[
\tilde{J} = \int_{\Gamma_{sld}} \left[ (W + T)n_1 - t_i(\partial u_i/\partial x_1) \right] ds
\]

\[
= \int_{r + r_c} \left[ W n_1 - t_i(\partial u_i/\partial x_1) \right] ds + \int_{V - V_c} \left( \rho \tilde{u}_i - f_i \right)(\partial u_i/\partial x_1) dV + \int_{r_c} T n_1 ds.
\]

(2.70)

Note the presence of a "near-tip" integral (over \( \Gamma_c \)) in the supposedly far-field expression (2.71) for \( \tilde{J} \). They [34] go on to consider the limit of (2.71) for two different shapes of \( \Gamma_c \). In any event, the presence of the integral over \( \Gamma_c \) makes it inconvenient to use (2.71) in a meaningful computational sense (i.e., without an accurate near-tip modelling). It is a simple matter to use the divergence theorem and eliminate the integral over \( \Gamma_c \) from (2.71); in which case, the resulting far-field expression for the energy release rate is none other than \( J' \) of Eq. (2.49).

Finally we mention the following path-independent integrals of Nilsson [35] and Gurtin [36], respectively, for a stationary crack in a linear elasto-dynamic field:

\[
I(p) = \int_{c} \left[ ((\tilde{W} + \frac{1}{2} \rho \partial^2 \tilde{u}, \tilde{u}_i) n_1 - \tilde{t}_i(\partial \tilde{u}_i/\partial x_1) \right] ds
\]

(2.72)

and

\[
I = \frac{1}{2} \int_{r} \left[ (\sigma_{jk} * u_{j,k} + \rho u_j * \tilde{u}_i) n_1 - n_k \sigma_{jk} * \partial u_i/\partial x_1 \right] ds.
\]

(2.73)

In (2.72), \( I(p) \) denotes a Laplace transform of \( I(t) \) and \( ( ) \) denotes a Laplace transform of ( ). Likewise, in (2.73), \( [(f) * (g)] \) denotes a convolution integral in the time domain, of two functions \( f(t) \) and \( g(t) \). Thus, both (2.72) and (2.73) do not easily give the instantaneous value of the crack-tip parameter, which is useful in analyzing dynamic crack propagation and arrest in a finite body. Further, in the case of a stationary crack in a dynamic field, the energy release due to incipient crack growth at any instant of time is given from (2.20) and (2.21) as:

\[
G_{\text{stationary}} = \int_{r_c} \left[ W n_1 - t_i(\partial u_i/\partial x_1) \right] ds
\]

(2.74)

\[
= \int_{r_c} \left[ W n_1 - t_i(\partial u_i/\partial x_1) \right] ds - \lim_{r \to 0} \int_{V_r} \left( f_i - \rho \tilde{u}_i \right) \frac{\partial u_i}{\partial x_1} dV
\]

(2.75)

(2.74) follows from (2.19) since \( T \) is no longer singular at the stationary crack-tip; (2.75) follows from (2.74) due to the divergence theorem.

We now turn to a class of "path-independent integrals" derivable from the
application of Noether's theorem [37] in the form of "conservation laws" [3, 38-42].

2.3. "Conservation laws" and their relevance to fracture mechanics

The density of the "Lagrangian" for a (linear or non-linear) elasto-dynamic problem is defined as $L = (W - T - P)$ where $W$ is the strain energy density, $T$ the kinetic energy density, and $P$ the potential of external forces. In Lagrange's description of motion (with material coordinates $X_i$ as independent variables), $L$ may be considered, in general, to be a function of the variables $Y_{i,j} = \partial Y_i / \partial X_j$ (or equivalently of $u_{i,j}$), $\dot{u}_i$, $u_i$ as well as that of the independent variables $X_i$ (for a non-homogeneous system) and $t$ (for a non-holonomic system). Thus,

$$L^* = \int_{V} L(X_i, u_i, \dot{u}_i, u_{i,j}, t) \, dv \, dt. \tag{2.76}$$

Noether's theorem [37] concerning the invariance of $L^*$ with respect to certain transformations of the arguments of $L$ leads to corresponding conditions which may be labelled as conservation laws. Eshelby [4, 7, 43] was the first to intuitively recognize the importance of these in connection with "forces" on point defects and cracks. Gunther [38] was apparently the first to apply the formalism of the Noether's theorem to obtain general conservation laws in elastostatics. Knowles and Sternberg [39] provided, independently, a thorough treatment also in the case of finite elastostatics; and this work was later extended by Fletcher [40] to linear elasto-dynamics, although the claim in [40] that Eqs. (3.1)-(3.4) and (3.6) therein can easily be extended to finite elasticity should be viewed with some caution. More recently, Golebiewska-Herrmann [42, 43] presented studies of conservation laws in finite elasto-dynamics using both Lagrange as well as Eulerian descriptions of motion.

Here we briefly discuss the case of Lagrange's description of motion and consider the conservation laws that arise from (2.76) when it is required to be invariant under various transformations, when body forces $f_i$ are present:

1. Invariance under time translation ($t = t + \epsilon$):

$$- \frac{d}{dt} (W + T) + f_i \dot{u}_i + \frac{\partial}{\partial X_k} (t_{kj} \ddot{u}_j) = - \frac{\partial L}{\partial t} \bigg|_{\text{explicit}}. \tag{2.77a}$$

when $L$ does not depend on $t$ explicitly, this leads to a "conservation law" for a closed volume $V^*$ (with a surface $\Gamma^*$, see Fig. 1) that does not contain the crack-tip:

$$\frac{d}{dt} \int_{V^*} (W + T) \, dv - \int_{\Gamma^*} f_i \dot{u}_i \, dv - \int_{\Gamma^*} N_k t_{kj} \ddot{u}_j \, ds = 0. \tag{2.77b}$$
Eq. (2.77b) is analogous to the energy-balance relation (2.8) except for a subtle difference: (2.77b) does not imply any crack growth, whereas (2.8) is written specifically for crack growth.

(2) invariance under translation of $Y_i(Y_i = Y_i + \varepsilon_i)$:

$$t_{ij,i} + f_j = \rho \ddot{u}_j.$$  \hspace{1cm} (2.78a)

Henceforth in this section we use the notation, $(\_)_i = \partial(\_)/\partial X_i$. Eq. (2.78a) may be written in integral form as:

$$\int_{V^*} N_j t_{ij} ds + \int_{V^*} f_j dv - \int_{V^*} \rho \ddot{u}_j dv = 0.$$  \hspace{1cm} (2.78b)

Eqs. (2.78a, b) are, respectively, local and global equations of balance of linear momentum.

(3) invariance under rotations of $Y_i(Y_i = Y_i + e_{ijk} \omega_j Y_k)$:

Here, $e_{ijk}$ is the alternating tensor. The balance law is:

$$-\frac{d}{dt} (e_{ijk} \rho \ddot{u}_k Y_i) + \frac{\partial}{\partial X_p} (e_{ijk} t_{pjk} Y_i) + e_{ijk} \ddot{f}_k Y_j = 0.$$  \hspace{1cm} (2.79a)

When (2.78a) is used, it is seen that (2.79a) is but a disguised form of the angular momentum balance, (2.2). The corresponding “conservation law” is:

$$\int_{V^*} e_{ijk} Y_i (f_k - \rho \ddot{u}_k) dv + \int_{I^*} e_{ijk} Y_j N_p t_{pjk} ds = 0.$$  \hspace{1cm} (2.79b)

Note that $V^*$ is the volume in undeformed configuration.

(4) invariance under translation of $X_i(X_i = X_i + \varepsilon_i)$:

Note that translation of the coordinates in the undeformed body are considered (or equivalently the translation of the elastic field referred to the undeformed geometry). The balance law is:

$$\frac{d}{dt} \left( \rho \ddot{u}_i u_{i,i} \right) - f_k u_{k,i} + \frac{\partial}{\partial X_k} \left[ (W - T) \delta_{ik} - t_{ki} u_{i,i} \right] = \left( -\frac{\partial L}{\partial X_i} \right)_{\text{exp}}.$$  \hspace{1cm} (2.80a)

When we consider: (1) a volume $V^*$ that does not contain any singularities such that each of the terms in (2.80a) is integrable in $V^*$, and (2) $L$ does not explicitly depend on $X_i$, i.e. the material is homogeneous in all the $X_i$ directions, we obtain from (2.80a) the conservation law:

$$\int_{V^*} \left( \frac{d}{dt} \left( \rho \ddot{u}_i u_{i,i} \right) - f_k u_{k,i} \right) dv + \int_{I^*} \left[ (W - T) N_i - t_{ki} u_{i,i} \right] ds = 0.$$  \hspace{1cm} (2.80b)

The above conservation law, and its alternative representations, were discussed in
13. Note that Eshelby [4, 7, 43] names the terms in square brackets in (2.80a) the "energy–momentum tensor".

(5) Invariance under rotations of \( X_i = X_j + e_{ijk} w_j X_k \):
This is possible only when the (linear or non-linear elastic) material is isotropic. The balance law is:

\[
0 = e_{ijk} X_k \left[ (\rho \ddot{u}_m - f_m) Y_{m,i} + \rho \dot{u}_m \dot{u}_{m,i} \right] + \frac{\partial}{\partial X_p} \left( e_{ijk} X_k \left[ (W - T) \delta_{ip} - t_{pm} Y_{m,i} \right] \right). \tag{2.81a}
\]

When the global angular momentum conservation law (2.79b) is used, the conservation law corresponding to (2.81a) can be written as:

\[
0 = e_{ijk} \left\{ \int_{V^*} \left[ (\rho \ddot{u}_m - f_m) u_{m,i} X_k + \rho \dot{u}_m \dot{u}_{m,i} X_k - \rho \dot{u}_i u_k \right] \, dv 
+ \int_{\Gamma} \left[ (W - T) X_k N_i + N_m t_{m,i} u_k - N_{m} t_{pm} u_{m,i} X_k \right] \, ds \right\}, \tag{2.81b}
\]
where \( V^* \) is the volume that is void of any singularities.

(6) Invariance under scale changes of \( X_i \), \( i \in \{1, 2, 3\} \):
This is possible only when the material is linear. The corresponding conservation law in linear elasto-dynamics is [40]:

\[
\int_{V^*} \frac{d}{dt} \left\{ \rho \ddot{u}_i \left( u_i + u_{i,k} X_k + \dot{u}_k \right) + L \right\} \, dv 
+ \int_{\Gamma} \left\{ L N_i X_i - N_{j,k} \left( u_j + u_{i,k} X_k + \dot{u}_k \right) \right\} \, ds = 0. \tag{2.82}
\]

Now we consider the application of the conservation laws (2.80b) and (2.81b) to non-linear elasto-dynamic crack propagation. We consider a volume \( V_{e} \) which does not contain the crack-tip, where \( \Gamma \) is any path enclosing the crack-tip, \( V_{e} \) is a small volume with the boundary \( \Gamma \); also enclosing the crack-tip; thus \( \Gamma = S_{\Gamma e} - \Gamma_e \) is the boundary of \( V - V_{e} \). Note that the divergence theorem, in the presence of possible non-integrable singularities, may be applied only in \( V - V_{e} \) in the limit as \( \varepsilon \to 0 \). Based on these arguments, further elaborated upon in [3], we obtain from (2.80b) the path-independent integrals:

\[
J_k = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} \left[ (W - T) N_k - t_{j} u_{j,k} \right] \, ds \tag{2.83a}
\]

\[
= \lim_{\varepsilon \to 0} \int_{\Gamma + S_{\Gamma e}} \left[ (W - T) N_k - t_{j} u_{j,k} \right] \, ds + \int_{V_{e} - V_{\varepsilon}} \left( \frac{d}{dt} \left( \rho \ddot{u}_i u_{i,k} \right) - f_m u_{m,k} \right) \, dv. \tag{2.83b}
\]
Comparing (2.83a, b) with (2.20) and (2.49), it should be evident that $J_k$ of (2.83a) are not associated with the concept of an energy release rate, but rather (as the roots of their derivation would indicate) are associated with the rate of change of Lagrangian $L^*$ of the system, due to unit translation of the crack in the $X_k$ direction [3]. That the equivalent “energy–momentum tensor” in elasto-dynamics does not lead to an energy release rate was also noted by Eshelby [7]. Thus, the relevance of (2.83a) as a “fracture parameter” is vacuous. Likewise, letting global $X_1$ coincide with the crack-tip local $x_1$, the integral of (2.83b) in time, say,

$$Y_1 = \int_{t_0}^{t} J_1 \, dt = \lim_{\epsilon \to 0} \int_{t_0}^{t} \left( \int_{R+S_{x_1}} \left[ (W - T) n_1 - t_1 u_{x,1} \right] \, ds \right) \, dt \nonumber$$

$$- \int_{t_0}^{t} \left( \int_{V_{y_1}}^{V_{y_1}} \left( f_k u_{k,1} \right) \, dv \right) \, dt + \int_{V_{y_1}}^{V_{y_1}} \rho \ddot{u}_j u_{j,1} \, dv \bigg|_{t_0}^{t} \tag{2.84}$$

has little relevance to fracture – it is the total change of Lagrangian from $t_0$ to $t$. Eq. (2.84), in a slightly less general form, for the case of infinitesimal deformation, along with the assumption of a rather special set of constant body forces, i.e. $f_k \neq \tilde{f}_k(x)$ (which renders the volume integral of $f_k u_{k,1}$ to be a surface integral of $f_k u_{k,1} n_1$), appears in a paper by Ouyang [33].

If one realizes the identity:

$$\int_{R_1}^{2TN_k} dS = \int_{R+S_{x_1}}^{2TN_k} dS - \int_{V_{y_1}}^{V_{y_1}} 2\rho \ddot{u}_m u_{m,k} \, dv \tag{2.85}$$

and adds (2.85) to (2.83b), one recovers the integrals $J_k'$ of (2.20) and (2.49) which are associated with the energy release rate, as was done originally in [10].

Analogously to the way in which (2.83) is derived from (2.81b), we may derive the following path-independent integrals from (2.81b):

$$L = e_{ijk} \int_{R_1}^{[(W - T) X_k N_i + N_m t_m u_k - N_p t_p m u_{m,j} X_k]} ds \nonumber$$

$$= e_{ijk} \int_{R+S_{x_1}}^{[(W - T) X_k N_i + N_m t_m u_k - N_p t_p m u_{m,j} X_k]} ds \nonumber$$

$$+ \int_{V_{y_1}}^{V_{y_1}} \left[ (\rho \ddot{u}_m - f_m) u_{m,i} X_k + \rho \ddot{u}_m u_{m,i} X_k - \rho \ddot{u}_k u_{x,k} \right] dv \tag{2.86}$$

which would have the meaning of the rate of change of Lagrangian $L$ per unit rotation of the crack. In order to obtain an equivalent “energy release” interpretation, we may add the identity
\[
\int_{V_r - V_i} \frac{\partial}{\partial X_i} (2T e_{ij} X_k) \, dv - \int_{r+S_r} 2 e_{ij} T X_k N_i \, ds = \int_{r_i} 2 e_{ijk} TX_k N_i \, ds \tag{2.87}
\]

to (2.86) and obtain:
\[
L'_j = e_{ijk} \left\{ \left[ (W + T) x_k n_i + n_m t_{mi} u_k - n_p t_{pm} u_{m,j} x_k \right] \right\} ds
\]
\[
= e_{ijk} \left\{ \int_{r+S_r} \left[ (W + T) x_k n_i + n_m t_{mi} u_k - n_p t_{pm} u_{m,j} x_k \right] ds
\]
\[
+ \int_{V_r - V_i} \left[ (\rho \ddot{u}_i - f_m) u_{m,j} x_k - \rho \ddot{u}_m \ddot{u}_{m,j} x_k - \rho \ddot{u}_i u_k \right] dv \right\}. \tag{2.88}
\]

Finally, we note that the so-called \(M\)-integral for linear elasto-dynamics can be derived from (2.82).

2.4. Complementary representation of path-independent integrals in (non-linear) elasto-dynamic fracture

Here we define the complementary energy density (per unit initial volume) of the material, denoted here by \(W_c\), through the contact (Legendre) transformation,
\[
W_c(t_{ij}) = t_{ij} u_{j,i} - W(u_{j,i}) \tag{2.89}
\]
Evaluation of (2.89), however, involves finding the inverse of the stress–strain relation,
\[
t_{ij} = \partial W/\partial u_{j,i} \tag{2.90}
\]
That the inverse of (2.90) is not unique is now well established [1, 2]. However, by defining the so-called second Piola–Kirchhoff stress tensor,
\[
S_{ij} = t_{ik} (\partial X_i/\partial Y_k) \tag{2.91}
\]
we may define a “valid” complementary energy:
\[
W_c(S_{ij}) = S_{ij} C_{ij} - W(C_{ij}) \tag{2.92}
\]
where \(C_{ij} = (Y_{k,i} Y_{k,j})\). Evaluation of (2.92) involves finding the inverse of
\[
S_{ij} = (\partial W/\partial C_{ij}) \tag{2.93}
\]
which is known to be unique [1, 2], such that:

\[ C_{ij} = \frac{\partial W_e}{\partial S_{ij}}. \]  

We now define a Lagrangian in terms of the complementary energy, \( W_c \), as:

\[ L(S_{ij}, u_i, \dot{u}_j, u_{i,j}, X_i, t) = \int \int [S_{ij} C_{ij} - W_c(S_{ij}) - T - P] \, dv \, dt. \]  

By applying Noether's theorem, it is now possible to derive a variety of conservation laws, and complementary path-independent integrals, from (2.95). We omit further details, but refer for examples of these to [5, 44].

We conclude this section by noting that of the seemingly infinite varieties of "path-independent integrals" and attendant "conservation laws" possible in non-linear (or linear) elasto-dynamic crack propagation, only \( J' \) of (2.28) and (2.49), and the equivalent \( J^* \) of (2.58) have the properties: (1) they characterize an energy release rate due to crack propagation, (2) they are measurable, and (3) they are measures of dynamic crack-tip fields. In linear elasto-dynamic crack propagation, even though some of the other "path-independent independent integrals", such as \( J \) of (2.64) and (2.66), and \( J_k \) of (2.83a), do not have the same physical meaning as \( J' \) and \( J^* \), they may be related to the dynamic stress-intensity factors \( [k(t)] \). Such relations, which are, of course, different from those between \( J' \) and \( k(t) \), are given in [10].

2.5. Experimental measurability of \( J' \) for elastic materials

We consider here the question of the measurability, on laboratory test specimens, of the material property, namely, the rate of energy release per unit crack growth, \( G \) of (2.8), which has been defined also as a path-independent integral \( J' \) through Eqs. (2.20), (2.28) and (2.49), for elastic materials.

We consider two test specimens of identical geometry except for the lengths of cracks which are \( (a) \) and \( (a + da) \), respectively, in the two bodies. We consider the two bodies to be subject to identical load histories. Let this "loading" be through prescribed displacements \( \vec{u}_i \) on a portion \( S_a \) and through prescribed tractions \( \vec{t}_i \) on a portion \( S_b \) of the boundaries of each cracked body; and let the body forces \( f_i \) be zero. Prior and up to the onset of crack propagation, the equilibrium process in each body implies (see [44]):

\[ \int (W + T)^{(a)} \, dv = - \int \int \vec{u}_i^{(a)} \, d\vec{u}_i \, ds + \int \int \vec{t}_i^{(a)} \, d\vec{t}_i \, ds + \int \int \vec{t}_i^{(a)} \, d\vec{u}_i \, ds. \]  

(2.96)
In (2.96) the superscript \( (\alpha) \), which varies from 1 to 2, denotes the cracked body in question. The fact that the crack length in the second specimen is larger than that in the first by \( (\text{da}) \) is reflected in: (1) the strain and kinetic energy densities \( W \) and \( T \), respectively, being different in the two specimens, (2) the displacements \( u_i \) at the traction-prescribed boundary \( S_i \) being different for the two specimens, and (3) the tractions at the displacement-specified boundary \( S_u \) being different for the two specimens. Thus, from (2.96) one may derive:

\[
\frac{d}{da} \int_{\Omega} (W + T) \, dv = \int_{S_i} \frac{du_i}{da} \, \overline{d}t_i \, ds + \int_{S_u} \frac{dt_i}{da} \, \overline{d}u_i \, ds + \int_{S_u} \frac{dt_i}{da} \, \overline{d}u_i \, ds .
\]

(2.97)

Let \( (dA) \) represent the sum of: (1) the difference in area under the load \((\overline{t}_i)\) versus displacement \((u_i)\) diagram for surface \(S_i\), and (2) the difference in area under the load \((t_i)\) versus displacement \((\overline{u}_i)\) diagram for \(S_u\). Note that the sign convention employed here implies that:

\[
\frac{du_i}{da} \bigg|_{\overline{t}_i} > 0 \quad \text{at } S_i ; \quad \frac{dt_i}{da} \bigg|_{\overline{u}_i} < 0 \quad \text{at } S_u .
\]

Thus

\[
dA = \int_{S_i} \frac{du_i}{da} \, \overline{d}t_i \, ds - \int_{S_u} \frac{dt_i}{da} \, \overline{d}u_i \, ds .
\]

(2.98)

Using (2.97) in (2.98), one has:

\[
dA = \int_{S_i} \overline{t}_i \frac{du_i}{da} \, ds - \frac{d}{da} \int_{\Omega} [W + T] \, dv .
\]

(2.99)

As long as the cracks in both the bodies remain stationary, the stress–strain fields in both the bodies may be assumed to be of similar mathematical form, at all times. Thus, using procedures similar to those in connection with Eqs. (2.15)–(2.20), one may write:

\[
dA = \int_{R_{\text{external}}} \left( (W + T) n_i - \overline{t}_i \frac{\partial u_i}{\partial x_i} \right) dv + \int_{R_i} \frac{\partial u_i}{\partial a} \, ds - \int_{\Omega} \frac{\partial(W + T)}{\partial a} \, dv .
\]

(2.100)

Recall that Eq. (2.51) holds only for purely elastic materials, whose properties are independent of loading–unloading histories. For such materials, using (2.51) (in the absence of body forces), one obtains from (2.100) that:
\[
\begin{align*}
\text{d}A &= \int_{r_{\text{ca}}} \left( (W + T)n_i - \frac{\partial u_i}{\partial x_1} \right) \text{d}s - \int_{V-V_e} \left( \rho \frac{\partial \tilde{u}_i}{\partial x_1} - \rho \frac{\partial \tilde{u}_i}{\partial x_i} \right) \text{d}v \\
&= \int_{r_c} \left( (W + T)n_i - \frac{\partial u_i}{\partial x_1} \right) \text{d}s. 
\end{align*}
\] (2.101)

It is important to note that Eq. (2.101): (1) is not valid during dynamic crack propagation even for purely elastic materials, since (2.96) is not valid for propagating cracks; and (2) is not valid for elasto-plastic materials.

On the other hand, during dynamic crack propagation, \( J' \) can be determined experimentally, from its definition as an integral over \( r_c \), if the relevant near-tip data can be measured experimentally. Beinert and Kalthoff [45] have been successful in applying the method of caustics in measuring the dynamic stress-intensity factor near the tip of a propagating crack. From the thus-measured dynamic \( K \)-factor, \( J' \) as a function of crack velocity for a propagating crack can be determined using the crack velocity-dependent relation between \( J' \) and the \( K \)-factor given in [10]. On the other hand, if pertinent data to evaluate the integral on the external boundary \( S \) of the specimen as in (2.49) can be measured, one may use a hybrid experimental-numerical procedure to evaluate the integral on \( V - V_e \) and determine \( J' \) from (2.49), wherein \( I \) is considered to be the external boundary.

### 3. Inelastic (and dynamic) crack propagation

We first consider crack-growth initiation under quasi-static conditions, in elastic-plastic materials. The most widely used parameter so far, and the one that has made possible certain impressive advances in elasto-plastic fracture has been the \( J \)-integral [46]. In the context of incipient self-similar growth, under quasi-static conditions, of a crack in an elastic material, \( J \) [which is equal to \( J' \) when \( \tilde{u}_i \) and \( \tilde{\tilde{u}}_i \) are set to zero in (2.49)] has the meaning of energy release per unit of crack extension. As in the case of \( J' \) of (2.49), the path-independence of \( J \), evaluated now only as a contour integral, can be established when the strain energy density of the material is a single-valued function of strain and the material is appropriately homogeneous and the body forces are zero. In a deformation theory of plasticity, which is valid for radial monotonic loading but precludes unloading (and thus is essentially and mathematically equivalent to a non-linear theory of elasticity), \( J \) still characterizes the crack-tip fields. However, in this case \( J \) does not have the meaning of an energy release rate; it is simply the total potential-energy difference between identical and identically (monotonically) loaded cracked bodies which differ in crack lengths by a differential amount. It should be emphasized that even this interpretation of \( J \) under a deformation theory of plasticity is valid only up to the point of crack growth initiation [44], as discussed in Chapter 3. Moreover, in a flow theory of plasticity, under arbitrary load histories, the path-independence of \( J \),
evaluated as a contour integral, is no longer valid; and further, under these circumstances, $J$ does not have any physical meaning.

However, significant advances have been made, in the past decade, in the problem of crack growth initiation in monotonically loaded structures, using the concept of $J$-integral. The principal contributions that made these possible may perhaps be identified, as: (1) the work of Hutchinson [47] and Rice and Rosengren [48], who show that the stresses and strains near the crack-tip in a monotonically loaded body of a pure power-law hardening material, under yielding conditions varying from small-scale to fully plastic, are controlled by $J$; (2) the work of Begley and Landes [49] and Rice et al. [50] on the measurement of $J$ from small laboratory test specimens; and (3) simple procedures for estimation of $J$, by interpolating between fully plastic solutions and elastic solutions, based on the works of Bucci et al. [51], Shih and Hutchinson [52], and Rice et al. [50]. On the other hand, a large amount of crack growth in a ductile material is necessarily accompanied by a significant non-proportional plastic deformation which invalidates the deformation theory of plasticity. Thus, the validity of $J$, as a contour integral as defined by Eshelby [4] and Rice [46], is questionable under these circumstances. For limited amounts of crack growth, however, Hutchinson and Paris [53] argue that the far-field $J$, denoted as $J_f$ in Chapter 3, is still a controlling parameter. For such situations of $J_f$-controlled growth, Paris et al. [54] introduced the concepts of a “tearing modulus” and “$J$ resistance curve” to analyze the stability of such growth. Using the above concepts and the related concepts of CTOA, engineering approaches to elastic–plastic fracture analyses were elaborated upon by Kumar et al. [55] and Kanninen et al. [56].

The mechanics of crack growth initiation, and substantial amounts of stable growth, in elastic–plastic materials subject to arbitrary load histories is not yet understood. This state of affairs is due, in part, to the reason cited by Rice [57] in 1968 that “... no success has been met in attempts to formulate similar general results for incremental plasticity”.

Among the first attempts to find a suitable parameter, that is theoretically valid in elastic–plastic fracture mechanics, were those by Bilby [58] and Miyamoto and Kageyama [59] who defined an integral:

$$J_{ext} = \int_{\Gamma} (W^e N_1 - i, \beta^e_{ij}) \, ds$$  \hspace{1cm} (3.1)

where $W^e$ is the elastic strain energy density, and $\beta^e_{ij}$ is the “elastic distortion tensor” such that the increments of elastic displacements are given by: $du^e_i = \beta^e_{ij} \, dx_j$. The integral (3.1) is path-independent only for paths in the region of the body that remains elastic, but is path-dependent for contours passing through the plastic region. Some studies on $J_{ext}$ were presented by Miyamoto and Kageyama [59, 60].
Also, from time to time, ideas of "energy balance" and "energy release rates", similar to those in the previous section (Section 2), are presented in the literature for elastic-plastic materials. However, such ideas of "energy release rate" are well known [61,62] to be unworkable for elastic-plastic materials wherein stress saturates to a finite value at large values of strain. In such materials, under quasi-static conditions, it has been shown [61-63] that the energy release rate vanishes [i.e. the value \((\Delta U/\Delta a)\) tends to zero when \(\Delta a \rightarrow 0\), where \(\Delta U\) is the total change in global energy due to crack growth by amount \(\Delta a\)]. Of course, the total energy release for a finite growth step \(\Delta a\), denoted as \(G^* \Delta\), remains finite and depends on \(\Delta a\) [62,63]. It is this dependence on the size \(\Delta a\) that precludes a rational utilization of the "energy release" concept or the generalization of the original Griffith energy balance concept, in elastic-plastic fracture mechanics. Also, the derivation of integrals, that may characterize "energy release" even in finite growth steps, along the lines of those in Section 2, are no longer possible in elastoplasticity, since the solutions near the crack-tip at time \(t\) and at time \((t + \Delta t)\) (during which the crack grows by \(\Delta a\)) are, in general non-steady crack propagation cases, no longer self-similar - due to the elastic unloading that accompanies crack growth.

First consider a cracked elasto-plastic body that is subject to quasi-static, monotonic, and proportional (radial) loading such that a deformation theory of plasticity may be valid. Further, we consider (1) the material to be homogeneous, at least in the \(x_1\) direction; (2) the loading to be only through surface tractions, i.e. the body forces are zero; and restrict our attention to crack growth initiation only. For this case of a stationary crack, one may define a crack-tip parameter:

\[
J_1 = \int \left( W_{n_1} - t_{i} \frac{\partial u_i}{\partial x_1} \right) d\Gamma .
\]  

(3.2)

For a stationary crack, the integral in (3.2) remains finite, for all values of \(\varepsilon\) such that \(3\delta < \varepsilon < R^*\) (see Chapter 3). For the elasto-plastic body, \(W\) is the total stress-work at a material point (per unit volume) and is defined as:

\[
W = \int_{\varepsilon_0}^{\varepsilon} \sigma_{ij} \varepsilon_{ij} \, d\varepsilon.
\]  

(3.3)

Under arbitrary history of straining, \(W\) in (3.3) is not a single-valued function of \(\varepsilon_{ij}\). However, under conditions of validity of the deformation theory of plasticity as delineated above, \(W\) may be considered to be a single-valued function of \(\varepsilon_{ij}\). Thus, under the restrictive assumptions stated above,

\[
\int_{\varepsilon = \varepsilon_{ij}} \left[ \frac{\partial W}{\partial x_1} - \frac{\partial}{\partial x_1} \left( \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \right] \, d\varepsilon = 0 .
\]  

(3.4)
Thus, \( J_i \) in (3.2) is a path-independent integral in itself, without the presence of a domain integral; and hence \( J_i = J \), where \( J \) is the integral over any arbitrary contour \( \Gamma \) of the integrand which is identical to that in (3.2).

On the other hand, consider a stationary crack in a homogeneous (at least in the \( x_1 \) direction) elasto-plastic body that is subject to dynamic surface tractions, such that material inertia plays a dominant role. In this case, one may define a crack-tip parameter [which remains finite for any \( \epsilon \) similar to \( J_i \) of (3.2)], as:

\[
J' = \int_{\Gamma} \left( (W + T)n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, dv, \tag{3.5}
\]

Under a general transient dynamic loading, a deformation theory of plasticity does not, in general, hold, i.e. \( W \) is not a single-valued function of \( \gamma_{ij} \). Thus,

\[
\int_{\Gamma} \left[ \frac{\partial(W + T)}{\partial x_1} - \frac{\partial}{\partial x_1} \left( \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \right] \, dv \\
= \int_{\Gamma} \left[ \left( \frac{\partial W}{\partial x_1} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) + \rho \left( \ddot{u}_i \frac{\partial \dot{u}_i}{\partial x_1} - \ddot{u}_i \frac{\partial u_i}{\partial x_1} \right) + f_i \frac{\partial u_i}{\partial x_1} \right] \, dv. \tag{3.6}
\]

In (3.6) it is implied that \( \frac{\partial W}{\partial x_1} \) is evaluated directly by first calculating \( W \) as per (3.3) and then differentiating it with respect to \( x_1 \). Thus, Eq. (3.6) implies that \( J' \) in (3.5) is not a path-independent integral by itself, and the domain-integral of (3.6) is present in any path-independent integral definition for \( J' \).

The physical meaning of \( J_i \) of (3.2) under the restrictive assumptions mentioned earlier, has been discussed in Chapter 3 of this book, while \( J' \) of (3.5), in the context of an elastic-plastic body, has no physical interpretation other than that it is a parameter that quantifies the crack-tip fields.

Now consider the case of quasistatic stable crack growth in an elastic-plastic body. If a two-dimensional situation is considered, any integral over an arbitrarily small circular path \( \Gamma' \) near the crack-tip (with radius \( \epsilon \) being small and tending to zero), with the integrand being such that: (1) it depends on the stress, strain, and displacement state near the crack-tip, and (2) it has a \((1/r)\) variation near the crack-tip, would serve as a valid crack-tip parameter. Since the integrand has a \((1/\epsilon)\) variation at \( \Gamma' \), it is seen that the integral crack-tip parameter remains finite. This crack-tip integral parameter is then sought to be represented equivalently as a far-field integral plus a “finite domain integral”, using the divergence theorem. This alternative representation is convenient for computational analyses of fracture problems. Under certain idealized and special circumstances, however, the aforementioned “finite domain integral” vanishes identically – thus making it possible to express the crack-tip integral parameter solely as a far-field contour integral. To define a crack-tip integral parameter of the aforementioned type, a knowledge of the steady- as well as non-steady-state asymptotic solutions near a
growing crack in a hardening material is necessary. While some progress has been made in recent years, such a complete asymptotic solution yet remains elusive for the practically important problem of mode-I crack growth in plane strain/plane stress. For an ideally plastic material, the asymptotic crack-tip fields in a mode-I problem have been recently studied by Slepyan [64], Gao [65], and Rice et al. [66]. Later, Rice refined these solutions [67]. While the solutions in [67] are valid during non-steady as well as steady-state growth, the other solutions [64–66] are valid only for steady-state growth. Finally, Gao [68] has developed an asymptotic solution for steady-state growth in a power-law hardening material; however, it was later noted by Gao [69] that there is a deficiency in these solutions, namely that the plastic part of the strain rate does not vanish as θ (the angular coordinate centered at the crack-tip) approaches the boundary between the plastic loading and elastic unloading sectors. Thus, the issue of asymptotic fields near the crack-tip in mode-I growth in a strain-hardening material is in need of further study.

If one considers the problem of initiation of growth from a stationary crack in an elastic–plastic solid under arbitrary load history, it is clear that \( J \), as defined in (3.2) remains finite for any value of \( ε \), including \( ε → 0 \). Moreover, under the restrictive assumptions which validate a deformation theory of plasticity, for a stationary crack, \( J \) is path-independent as discussed earlier. As the crack begins to grow, there is no reason to expect that the crack-tip integral as defined in Eq. (3.2), which had been finite until growth initiation, would not continue to remain finite. On the other hand, with elastic unloading accompanying large amounts of growth and the consequent invalidation of a deformation theory of plasticity, one would expect that crack-tip integral as defined in (3.2) would not remain path-independent by itself, without the presence of a domain integral. With these in mind, one may define a crack-tip parameter for quasistatically growing cracks in elastic–plastic solids under arbitrary histories, as:

\[
T^* = \int_{\Gamma_0} \left( Wn - t_i \frac{δu_i}{δx} \right) dΓ
\]  

(3.7)

where the density of stress-work, \( W \), is as defined in Eq. (3.3). The path-independent integral representation of \( T^* \), which now involves a domain-integral term, may be written, using the divergence theorem, as:

\[
T^* = \int_{Γ + S_Γ} \left( Wn - t_i \frac{δu_i}{δx} \right) dΓ - \int_{V_f - V_i} \left( \frac{∂W}{∂x} - δ_{ij} \frac{∂ε_{ij}}{∂x} - f_i \frac{δu_i}{δx} \right) dv
\]

(3.8)

For reasons fully discussed by Brust et al. [71, 72], the path \( Γ \) in the definition of \( T^* \) of (3.7) is as follows: In mode-I crack growth, at any crack length, \( Γ \) includes a semicircle centered at the current crack-tip and of radius \( ε \), and traverses along the wake of the advancing crack-tip and parallel to the crack axis at a distance \( ±ε \). Thus, \( Γ \), in the definition of \( T^* \) of (3.7) is no longer a circle or radius \( ε \) as in the definition of \( J \) of (3.2) for a stationary crack.
wherein \( \frac{\partial W}{\partial x_1} \) is evaluated by first computing \( W \) at each material point as per the definition given in (3.3) and then computing the partial of \( W \) with respect to \( x_1 \). The volume integral in Eq. (3.8) does not, in general, vanish for problems of crack growth in finite bodies. If one defines a far-field \( J_t \) through the equation:

\[
J_t = \int_{V_{\text{external}}} \left( Wn_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, d\Gamma,
\]

(3.9)

it is clear that the crack-tip parameter \( T^* \) differs from the far-field \( J_t \) through the term:

\[
T^* = J_t - \int_{V_{\text{far-field}}} \left( \frac{\partial W}{\partial x_1} - \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1} + \tilde{f}_i \frac{\partial u_i}{\partial x_1} \right) \, dv
\]

(3.10)

where \( V \) is the total volume. While the volume integrals in (3.8) and (3.10) do not vanish in the general cases of crack growth, there may be special circumstances when they do. Consider, for instance, the hypothetical case of a totally steady-state crack growth, i.e., a situation in which the stress/strain fields not only asymptotically close to the crack-tip, but also everywhere in the solid, remain invariant with respect to an observer moving with the crack-tip. To this end, consider a two-dimensional problem for instance; and let \((\xi_1, x_2)\) be a coordinate system centered at the moving crack-tip, such that:

\[
\xi_1 = x_1 - c(\lambda)
\]

(3.11)

where \( \lambda \) is a monotonically increasing time-like parameter, and \( c(\lambda) \) is the coordinate of the crack-tip in the space-fixed coordinate system \( x_1 \). It is easy to see from (3.11) that:

\[
\frac{\partial (\cdot)}{\partial x_1} \bigg|_{x_1} = \frac{\partial (\cdot)}{\partial \xi_1} \bigg|_{\xi_1}
\]

(3.12a)

and

\[
\frac{\partial (\cdot)}{\partial \lambda} \bigg|_{x_1} = \frac{\partial (\cdot)}{\partial \xi_1} \bigg|_{\xi_1} - \frac{\partial (\cdot)}{\partial \xi_1} \frac{\partial c}{\partial \lambda}.
\]

(3.12b)

In the totally steady-state problem, one has, everywhere in the solid,

\[
\frac{\partial (\cdot)}{\partial \lambda} \bigg|_{\xi_1} = 0
\]

(3.13)

and, hence,

\[
\frac{\partial (\cdot)}{\partial \xi_1} = \frac{\partial (\cdot)}{\partial x_1} = - \frac{\partial (\cdot)}{\partial \lambda} \bigg|_{x_1} \frac{1}{(\partial c/\partial \lambda)}.
\]

(3.14)
Thus
\[ \frac{\partial W}{\partial x_1} = -\frac{1}{(\partial c/\partial \lambda)} \frac{\partial W}{\partial \lambda} \bigg|_{x_1} = -\frac{1}{(\partial c/\partial \lambda)} \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial \lambda} \bigg|_{x_1} \] (3.15)

and
\[ \frac{\partial \varepsilon_{ij}}{\partial x_1} = -\frac{1}{(\partial c/\partial \lambda)} \frac{\partial \varepsilon_{ij}}{\partial \lambda} \bigg|_{x_1}. \] (3.16)

From (3.15) and (3.16), it follows that, when \( f_i = 0 \), the volume integrals in (3.8) and (3.10) vanish identically in the totally steady-state case. However, in practice, when stable crack growth occurs in a finite body, such as a “compact tension” type laboratory test specimen, the stress–strain field asymptotically close to the crack-tip may attain a steady state (thus resulting in a constant value of \( T^* \) during sustained crack growth), while the far-field stress–strain fields do not remain invariant with respect to an observer moving with the crack-tip (thus resulting in an ever-increasing value of far-field \( J_I \)). This situation is illustrated in Fig. 3, which shows results of numerical simulation of an experiment on a compact tension test specimen [70, 71]. The difference between the far-field \( J_I \) and the near-tip \( T^* \), i.e. the volume integral in (3.10), can be accounted for, mainly, by events immediately in the plastic zone near the crack-tip and elastic unloading which accompanies crack growth. Extensive studies of the use of \( T^* \) to analyze stable crack growth under monotonically rising load, as well as the predictive capability of \( T^* \) in situations of crack growth after a cycle of loading, unloading to zero load, followed by

![Diagram](https://example.com/diagram.png)

**Fig. 3** Typical results for \( J_I \) and \( T^* \) during stable crack growth in a compact tension specimen. (Based on numerical simulation of experimental data. See Ref. [71] for further details)
reloading, have been presented by Brust et al. [70–72] recently. The results of a
careful, combined numerical/experimental study showed [72] that the $T^*$ parameter
accurately predicted the experimentally observed behavior (crack growth to
begin only after 50% loading in the reloading phase), while the other parameters
[$J_f$ (i.e. the far-field $J$ as is widely used) and CTOA (crack-tip opening angle)] were
seriously anti-conservative (i.e. they predict crack growth to recommence only at
the 100% load level in the reloading phase).

Likewise, for the case of dynamic crack propagation in a dynamically loaded
inelastic body (such as a rate-dependent or visco-plastic body), one may define a

... crack-tip parameter $T^*$, and its equivalent path-independent integral (including a
domain-integral term) representation, as:

$$ T^* = \int_{r_i} \left( (W + T) n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) dI $$

(3.17a)

$$ = \int_{r_i} \left( (W + T) n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) dI $$

$$ - \int_{v_i} \left[ \left( \frac{\partial W}{\partial x_1} - \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1} \right) + \rho \left( \ddot{u}_i \frac{\partial u_i}{\partial x_1} - \ddot{u}_i \frac{\partial u_i}{\partial x_1} + f_i \frac{\partial u_i}{\partial x_1} \right) \right] dv $$

(3.17b)

where $W$ is the total stress work at a material point, as defined in (3.3), and
($\partial W/\partial x_1$) is evaluated directly from the computed values of $W$ at two infinitesimal-
ly close material particles. Once again, the domain integral in (3.17b) does not
vanish except in rather very special circumstances. Consider the case of mode-I
inelastic dynamic crack propagation along the $x_1$-axis. As in (3.11), we introduce a
coordinate system centered at the propagating crack-tip, such that

$$ \xi_1 = x_1 - c(t) $$

(3.18)

where $t$ is the Newtonian time.

We now consider the material derivative of $u_i$ (i.e. material velocity $\dot{u}_i$), material
derivative of $\dot{u}_i$ (i.e. material acceleration $\ddot{u}_i$), and the material derivative of stress
work $W$ (i.e. stress power $\dot{W}$), as follows:

$$ \dot{u}_i = \frac{\partial u_i}{\partial t} \bigg|_{x_1} = \frac{\partial u_i}{\partial t} \bigg|_{\xi_1} \frac{\partial \xi_1}{\partial t} $$

(3.19a)

$^3$Here the definition of the path $I_i$ is similar to that given in connection with the quasi-static case, as in
(3.7).
\[ \ddot{u}_i = \frac{\partial \dot{u}_i}{\partial t} \bigg|_{x_i} = \frac{\partial \dot{u}_i}{\partial t} \bigg|_{\xi_1} - \frac{\partial c}{\partial t} \frac{\partial \dot{u}_i}{\partial \xi_1} \]

\[ = \frac{\partial^2 u_i}{\partial t^2} \bigg|_{\xi_1} - 2 \frac{\partial c}{\partial t} \frac{\partial^2 u_i}{\partial t \partial \xi_1} + \frac{\partial^2 u_i}{\partial \xi_1^2} \left( \frac{\partial c}{\partial t} \right)^2 - \frac{\partial u_i}{\partial \xi_1} \frac{\partial^2 c}{\partial t^2} \]  

(3.19b)

\[ W = \frac{\partial W}{\partial t} \bigg|_{x_i} = \frac{\partial W}{\partial t} \bigg|_{\xi_1} - \frac{\partial W}{\partial \xi_1} \frac{\partial c}{\partial t} \]  

(3.19c)

\[ \dot{\varepsilon}_{ij} = \frac{\partial \varepsilon_{ij}}{\partial t} \bigg|_{x_i} = \frac{\partial \varepsilon_{ij}}{\partial t} \bigg|_{\xi_1} - \frac{\partial \varepsilon_{ij}}{\partial \xi_1} \frac{\partial c}{\partial t} \]  

(3.19d)

If one restricts attention to constant-velocity crack propagation: (1) with zero body forces, \( f_i = 0 \); and (2) with fully steady-state conditions, i.e. the stress, strain, displacement, velocity, and acceleration fields everywhere in the solid (asymptotically close to the crack-tip as well as in the far-field) remain invariant with respect to an observer moving with the crack-tip, we have from (3.19):

\[ \ddot{u}_i = -\frac{\partial c}{\partial t} \frac{\partial u_i}{\partial \xi_1} = -\frac{\partial c}{\partial t} \frac{\partial u_i}{\partial x_1} \]  

(3.20a)

\[ \ddot{u}_i = -\frac{\partial c}{\partial t} \frac{\partial \dot{u}_i}{\partial x_1} \]  

(3.20b)

\[ \frac{\partial \varepsilon_{ij}}{\partial x_1} = -\frac{1}{\frac{\partial \varepsilon}{\partial \xi_1}} \dot{\varepsilon}_{ij} \]  

(3.20c)

\[ \frac{\partial W}{\partial x_1} = \frac{\partial W}{\partial \xi_1} = -\frac{1}{\frac{\partial c}{\partial \xi_1}} \frac{\partial W}{\partial t} \bigg|_{x_i} = -\frac{1}{\frac{\partial c}{\partial \xi_1}} W = -\frac{1}{\frac{\partial c}{\partial \xi_1}} \sigma_{ij} \dot{\varepsilon}_{ij} \]  

(3.20d)

Thus, under fully steady-state conditions, Eqs. (3.20a–d) imply that the volume integral in (3.17b) is zero, and the contour-integral \( T^* \) as defined in (3.17a) is path-independent in itself. However, for the case of accelerating/decelerating crack motions in finite bodies, the attainment of fully steady-state conditions everywhere in the solid is an unlikely event, as in the case of quasi-static crack growth described earlier (and illustrated in Fig. 3).

It is thus seen that, in general, the path-independent integral representations for the crack-tip parameter, \( T^* \), in situations of quasi-static and dynamic crack propagation in inelastic solids, involve a domain integral term as in Eqs. (3.8) and (3.17b), respectively.

Since most inelastic crack propagation analyses have to rely on computational approaches, and since almost all computational approaches for inelastic analyses involve rate (or incremental) formulations, it is convenient to define rate (or incremental) crack-tip parameters, ab-initio \([3, 44, 73, 74]\). This approach will also

\( ^4 \) For a comprehensive discussion of the relation of these rate (or incremental) parameters to the concept of "conservation laws" in rate theories of inelastic solids (such as the general flow theory of elastoplasticity), see Refs. \([3, 44, 73, 74]\).
facilitate a computation of the derivative in the $x_i$ direction of the total stress-work in a more simple manner. To this end, we consider the more general dynamic case and define an incremental crack-tip parameter [with the path $\Gamma_c$ as defined in connection with (3.7)], as:

$$
\Delta T^* = \int_{\Gamma_c} \left( (\Delta W + \Delta T) n_i - t_i \frac{\partial \Delta u_i}{\partial x_i} - \Delta t_i \frac{\partial u_i}{\partial x_i} - \Delta t_i \frac{\partial u_i}{\partial x_i} \right) d\Gamma.
$$

(3.21)

Detailed discussions of the relevance of $\Delta T^*$ in the context of the flow theory of plasticity may be found in [44]. The last term in the integrand on the right-hand side of (3.21) is of second order but is retained for better accuracy in an incremental analysis. The incremental stress-work, $\Delta W$, is defined as:

$$
\Delta W = \sigma_{ij} \Delta \varepsilon_{ij} + \frac{1}{2} \Delta \sigma_{ij} \Delta \varepsilon_{ij}.
$$

(3.22)

where $\Delta \varepsilon_{ij}$ is the incremental strain, and $\Delta \sigma_{ij}$ the incremental stress. Eq. (3.21) is valid for any material model. Considering, for example, the case of elastic-plasticity, one may decompose the incremental strain as

$$
\Delta \varepsilon_{ij} = \Delta \varepsilon_{ij}^e + \Delta \varepsilon_{ij}^p.
$$

(3.23)

The incremental stress-strain relation may be written, in general, as:

$$
\Delta \sigma_{ij} = E_{ijkl} \Delta \varepsilon_{kl}.
$$

(3.24)

where $E_{ijkl}$ is the tangent constitutive matrix.

For classical elasto-plasticity, for instance,

$$
\Delta \sigma_{ij} = [E_{ijkl}^e - (\Gamma/g) N_{ij} N_{kl}] \Delta \varepsilon_{kl}
$$

(3.25)

where $\Gamma = 0$ in an elastic process (no change in plastic strain) and $\Gamma = 1$ in a plastic process (change in plastic strain); $g$ is a parameter related to isotropic and/or kinematic hardening of the material; and $N_{ij}$ is a unit normal to the yield surface.

Even though the material may be elastically homogeneous, Eqs. (3.24) and (3.25) make it evident that $E_{ijkl}^e$ is an explicit function of the location of the material particle, i.e. it depends on whether the material particle in question is undergoing an elastic process or a plastic process. It is thus seen that:

$$
\frac{\partial \Delta W}{\partial x_i} = \sigma_{ij} \frac{\partial \Delta \varepsilon_{ij}}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_i} \Delta \varepsilon_{ij} + \Delta \sigma_{ij} E_{ijkl}^e \frac{\partial \Delta \varepsilon_{kl}}{\partial x_i} + \frac{1}{2} \Delta \varepsilon_{ij} \frac{\partial E_{ijkl}^e}{\partial x_i} \Delta \varepsilon_{kl}.
$$

(3.26)

From (3.24) it is seen that:
\[
\frac{\partial \Delta \sigma_{ij}}{\partial x_1} = E_{ijkl}^{'} \frac{\partial \Delta \varepsilon_{kl}}{\partial x_1} + \frac{\partial E_{ijkl}^{'}}{\partial x_1} \Delta \varepsilon_{kl} \quad (3.27a)
\]

From (3.27) it follows that:
\[
\frac{1}{2} \Delta \varepsilon_{ij} \frac{\partial E_{ijkl}^{'}}{\partial x_1} \Delta \varepsilon_{kl} = \frac{1}{2} \Delta \varepsilon_{ij} \frac{\partial \Delta \sigma_{ij}}{\partial x_1} - \frac{1}{2} \Delta \varepsilon_{ij} E_{ijkl}^{' \prime} \frac{\partial \Delta \varepsilon_{kl}}{\partial x_1} = \frac{\partial \Delta \sigma_{ij}}{\partial x_1} - \frac{1}{2} \Delta \sigma_{ij} \frac{\partial \Delta \varepsilon_{ij}}{\partial x_1} \quad (3.27b)
\]

Using (3.27) in (3.26), one finds that:
\[
\frac{\partial \Delta W}{\partial x_1} = \left( \sigma_{ij} + \frac{1}{2} \Delta \sigma_{ij} \right) \frac{\partial \Delta \varepsilon_{ij}}{\partial x_1} + \left( \frac{\partial \sigma_{ij}}{\partial x_1} + \frac{1}{2} \frac{\partial \Delta \sigma_{ij}}{\partial x_1} \right) \Delta \varepsilon_{ij} \quad (3.28)
\]

Use of (3.28) along with the divergence theorem, results in the following path-independent integral representation for \( \Delta T^* \) of (3.21) (assuming that non-inertial body forces are zero):
\[
\Delta T^* = \int_{\Gamma^*} \left[ (\Delta W + \Delta T)n \right]_1 - (t_i + \Delta t_i) \frac{\partial \Delta u_i}{\partial x_1} - \Delta t_i \frac{\partial u_i}{\partial x_1} \right) \, d\Gamma \\
+ \int_{V^*} \left[ \sigma_{ij} \left( \frac{\partial \varepsilon_{ij}}{\partial x_1} + \frac{1}{2} \frac{\partial \Delta \varepsilon_{ij}}{\partial x_1} \right) - \Delta \varepsilon_{ij} \left( \frac{\partial \sigma_{ij}}{\partial x_1} + \frac{1}{2} \frac{\partial \Delta \sigma_{ij}}{\partial x_1} \right) \\
+ \rho(\dot{u}_i + \Delta \dot{u}_i) \frac{\partial \Delta u_i}{\partial x_1} - \rho(\dot{u}_i + \Delta \dot{u}_i) \frac{\partial \Delta \dot{u}_i}{\partial x_1} + \rho \Delta \dot{u}_i \frac{\partial u_i}{\partial x_1} - \rho \Delta \dot{u}_i \frac{\partial \dot{u}_i}{\partial x_1} \right] \, dV, \\
\quad (3.29)
\]

The above incremental parameter is such that: (1) it involves only the incremental stress-work which can be defined for any material constitutive model; (2) the far-field definition is inherently path-independent, but involves a domain integral; (3) its definition can easily be modified for non-isothermal conditions; (4) it is a path-independent type crack-tip parameter valid for large amounts of crack growth and general non-steady-state conditions; (5) it is a valid crack-tip parameter for arbitrary histories of loading and unloading; and (6) when specialized to the case of fully steady-state crack propagation, the domain integral in the far-field representation of \( \Delta T^* \) vanishes identically.

Consider, for instance, the case when material inertia, kinetic energy, and body forces are negligible as in situations of creep crack growth in structures operating at elevated temperatures. Ignoring second-order terms in rates, a rate parameter governing the crack-tip conditions in such situations may be written as:
\[
\dot{T} = \int_{t} \left( W_{n} - t \frac{\partial \dot{u}_{i}}{\partial x_{1}} - i \frac{\partial u_{i}}{\partial x_{1}} \right) d\Gamma \\
= \int_{t} \left( W_{n} - t \frac{\partial \dot{u}_{i}}{\partial x_{1}} - i \frac{\partial u_{i}}{\partial x_{1}} \right) d\Gamma + \int_{v_{1} = v_{2}} \left( \dot{\sigma}_{ij} \frac{\partial \dot{u}_{i}}{\partial x_{1}} = \dot{e}_{ij} \frac{\partial \sigma_{ij}}{\partial x_{1}} \right) dv .
\]

(3.30)

In the above,

\[
\dot{W} = \sigma_{ij} \dot{e}_{ij} 
\]

(3.31)

is the “stress power” and hence is defined for any material irrespective of the postulated constitutive relation. Specifically, the rate parameter \( \dot{T} \) of (3.30) remains valid for crack growth in situations wherein elastic, plastic, and creep strains may be simultaneously present. When the stress state in the body saturates and pure steady-state Norton’s power-law type creep occurs, i.e. \( \dot{e} \sim \sigma^{n} \), the situation is often referred to as “steady-state creep”. In this sense, the parameter \( \dot{T} \) of (3.30) remains valid in “arbitrary non-steady creep” conditions and, of course, in the limit as the so-called “steady-state creep” conditions attain. In the limit as “steady-state creep” conditions develop, \( \dot{\sigma}_{ij} \sim 0, i \sim 0; \) and \( \dot{T} \) of (3.30) becomes:

\[
\dot{T}_{SSC} = \int_{t} \left( W_{n} - t \frac{\partial \dot{u}_{i}}{\partial x_{1}} \right) d\Gamma \\
= \int_{t} \left( W_{n} - t \frac{\partial \dot{u}_{i}}{\partial x_{1}} \right) d\Gamma - \int_{v_{1} = v_{2}} \dot{e}_{ij} \frac{\partial \sigma_{ij}}{\partial x_{1}} dv 
\]

(3.32)

wherein the subscripts (SSC) on \( \dot{T} \) indicate “steady-state creep”.

Earlier, Landes and Begley [75] and Goldman and Hutchinson [76] considered the special case of “steady-state creep” and proposed a crack-tip parameter \( C^{*} \). As mentioned earlier, in the so-called “steady-state creep”, only creep strains are present; and, further, the creep strain-rate is proportional to the \( n \)th power of stress (Norton’s power law) which saturates to \( \sigma \) at any material particle, i.e. \( \dot{e} \sim \sigma^{n} \). This is entirely analogous to the case of deformation theory of plasticity, wherein \( \dot{e} \sim \sigma^{n} \). Based on this analogy, Landes and Begley [75] and Goldman and Hutchinson [76] define \( C^{*} \), analogous to the deformation theory \( J \), as:

\[
C^{*} = \int_{t} \left( W^{*} n_{1} - t \frac{\partial \dot{u}_{i}}{\partial x_{1}} \right) d\Gamma \\
= \int_{t} \left( W^{*} n_{1} - t \frac{\partial \dot{u}_{i}}{\partial x_{1}} \right) d\Gamma
\]

(3.33)

(3.34)
where
\[ W^* = \int_0^{\epsilon_0} \sigma_{mn} \, d\dot{\epsilon}_{mn}. \tag{3.35} \]

Note that \( W^* \) in (3.35) is not the stress-power; it is simply a pseudopotential for \( \sigma_{ij} \) in terms of \( \dot{\epsilon}_{ij} \), and thus \( W^* \) has no physical meaning. Note that \( W^* \) is a single-valued function of \( \dot{\epsilon}_{ij} \), just as \( W \) in the definition of deformation theory of plasticity \( J \) is a single-valued function of \( \epsilon_{ij} \). Thus, in "steady-state creep"
\[ \sigma_{ij} = \frac{\partial W^*}{\partial \dot{\epsilon}_{ij}}; \quad \frac{\partial W^*}{\partial x_1} = \sigma_{ij} \frac{\partial \dot{\epsilon}_{ij}}{\partial x_1}. \tag{3.36a, b} \]

It is because of (3.36b) that the domain integral, which arises from the application of the divergence theorem to the integral over \( \Gamma \) in (3.33), vanishes identically; and hence \( C^* \) is path-independent as a contour integral alone. On the other hand, the "stress-power" \( W \), which is defined for "non-steady creep", in general, such that
\[ W = \sigma_{ij} \dot{\epsilon}_{ij}; \quad \frac{\partial W}{\partial x_1} = \sigma_{ij} \frac{\partial \dot{\epsilon}_{ij}}{\partial x_1} + \sigma_{ij} \frac{\partial \dot{\epsilon}_{ij}}{\partial x_1}. \tag{3.37} \]

Note that even under the so-called steady-state creep conditions,
\[ (\dot{T}^*_{SSC} - C^*) = \int_{\Gamma} [\dot{W}_{SSC} - W^*] m_{ij} \, d\Gamma \neq 0 \tag{3.38} \]
where the subscripts (SSC) on \( \dot{W} \) denote "steady-state creep". For instance, for Norton's power-law type "steady-state creep" (\( \dot{\epsilon} \sim \sigma^{n} \)), we have:
\[ \dot{\epsilon}_{ij} = \frac{3}{2} \gamma (\sigma_{eq})^{n-1} \sigma_{ij}', \tag{3.39} \]

where \( \gamma \) is the fluidity parameter, \( \sigma_{eq} \) the equivalent stress, and \( \sigma_{ij}' \) the stress deviator. \( (\sigma_{eq}^2 = \frac{3}{2} \sigma_{ij}' \sigma_{ij}') \); \( n \) the hardening parameter. For the material model of (3.39), we have:
\[ \dot{W}_{SSC} = \sigma_{ij} \dot{\epsilon}_{ij} = \sigma_{eq} \dot{\epsilon}_{eq} = \gamma \sigma_{eq}^{n+1} \]
\[ = \left(\frac{1}{\gamma}\right)^{1/n} (\dot{\epsilon}_{eq})^{(n+1)/n} \tag{3.40} \]

where the subscripts (SSC) denote steady-state creep. On the other hand,
\[ W^* = \int_0^{\epsilon_0} \sigma_{mn} \, d\dot{\epsilon}_{mn} = \left(\frac{n}{n+1}\right) \left(\frac{1}{\gamma}\right)^{1/n} (\dot{\epsilon}_{eq})^{(n+1)/n}. \tag{3.41} \]
Thus,

\[ (W_{SSC} - W^*) = \frac{1}{n+1} \left( \frac{1}{\gamma} \right) (\dot{\varepsilon}_{eq})^{(n+1)/n} = \frac{\gamma}{n+1} (\sigma_{eq})^{n+1}. \] (3.42)

Also earlier, Stonesifer and Atluri [77, 78] considered a crack-tip parameter for general "non-steady creep" conditions, \( \hat{T}_c \), defined as:

\[
\hat{T}_c = \int_{r} \left( \dot{W}_N - t_i \frac{\partial u}{\partial x_1} \right) d\Gamma \]
\[
= \int_{r+s} \left( \dot{W}_N - t_i \frac{\partial u}{\partial x_1} \right) d\Gamma - \int_{\nu_k - \nu_k} \left( \dot{\varepsilon}_{ii} \frac{\partial \sigma}{\partial x_1} \right) d\nu. \] (3.43)

Comparing (3.30) and (3.43), it is seen that for general non-steady conditions,

\[
(\hat{T}^* - \hat{T}_c) = - \int_{r} \left( t_i \frac{\partial u}{\partial x_1} \right) d\Gamma; \] (3.44)

whereas, under the so-called "steady-state creep" conditions,

\[
(\hat{T}^*_{SSC} - \hat{T}_c_{SSC}) = 0 \] (3.45)

wherein the subscripts (SSC) indicate "steady-state creep".

Recently Brust et al. [70–72, 79] have presented a number of studies, using \( T^* \) (or \( \Delta T^* \) or \( \hat{T}^* \)), pertaining to stable crack growth under monotonic as well as cyclic loading in elastic-plastic bodies and creep crack growth at elevated temperatures. Under monotonic loading, \( T^* \) increases monotonically and is equal to \( J_\Gamma \) for small amounts of growth; while for moderate to large amounts of growth, \( J_\Gamma \) continues to increase substantially while \( T^* \) levels off to a constant value. Thus, \( T^* \) has the features of a combined \( (J_\Gamma-\text{CTOA}) \) criterion. A detailed account of these results is given in [70, 71]. Ref. [72] presents a study of the use of \( T^* \) in situations of cyclic loading. In current practice, the \( J_\Gamma \)-integral (far-field value) is commonly used to determine the resistance of plastically deformed structures to continued crack growth. This approach has been shown [71, 72] to be valid only for very small amounts of growth under monotonic loading; yet in the engineering community the \( J_\Gamma \)-resistance curve approach is believed to be conservative for all general loadings. On the other hand, the \( T^* \) parameter is formulated to be valid under a wide range of loading/unloading conditions. These two criteria (\( T^* \) and \( J_\Gamma \)) along with CTOA (crack-tip opening angle) were examined for their validity and predictive capability in situations of crack growth after a cycle of loading, unloading to zero load, followed by a reloading [72]. The results of a careful, combined numerical/experimental study showed [72] that the \( T^* \) parameter accurately predicted the
behavior (crack growth to begin only after 50% loading in the reloading phase),
while the other parameters (J and CTOA) were seriously anti-conservative. In
another study [79], the relevance of the parameters $T^*\,$, $C^*$, and $T_c$ in characteriz-
ing creep crack growth was examined. Experimental data on creep crack growth in
a 316 stainless steel single-edge-notched specimen was numerically simulated, and
the variations of various parameters during crack growth were ascertained. These
results were found to be in favor of the $T^*$ parameter in characterizing creep crack
growth under non-steady creep (not pure power-law creep) as well as in situations
wherein time-independent plastic strains are significant, in addition to creep
strains.

However, the present state of knowledge is too premature to conclude that
fracture-characterizing parameters are rationally established for inelastic and
dynamic fracture.

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