Direct and inverse multi-scale analyses of arbitrarily functionally graded layered hollow cylinders (discs), with different shaped reinforcements, under harmonic loads

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ABSTRACT

A multi-scale model is proposed to investigate arbitrarily functionally graded hollow cylinders (discs), with fibers, particles, or disc-shaped reinforcements, subjected to harmonic loading conditions. The stress analyses are performed by dividing the cylinders (discs) into several layers each with homogeneous properties, which are functionally graded through the thickness of the structures, with varying microstructural details. Good agreement can be obtained by comparing the present stress distributions against other analytical solutions used as boundary conditions or obtained for homogeneous and continuously graded structures. Furthermore, the Mori-Tanaka model is used to generate effective properties of each layer reinforced with fibers, particles or disc-shaped inclusions. The stress distributions in the cylinders along the radial direction are effectively investigated with the influence of either the shape or the volume fraction of reinforcements. Finally, the particle swarm optimization technique is combined with the present framework to provide inverse calculations for microstructural details, in the effort of finding proper inclusion volume fractions or minimizing the shear stress along the radial direction, which are necessary for the design of functionally graded structures. The present analysis for arbitrarily FG cylinders under arbitrary loading conditions provides benchmark solutions for other future analytical and numerical methods.

1. Introduction

Functionally graded materials (FGMs) have gained extensive attention since the concept was first proposed by a group of material scientists in Japan in 1984 [1–2]. FGMs have various applications in aerospace industry, civil and marine structures, as well as bio-technologies. The properties of the FGMs are always architected through non-uniform distributions of reinforcements in heterogeneous materials. The reinforcements can be categorized into different shapes, properties and sizes, depending on the purposes of the applications. Thus, the macroscopic effective properties of FGMs depend directly on the microstructural details. FGMs have numerous existing forms in the world, ranging from man-made materials reinforced with particles using manufacturing systems applied with centrifugal forces [3] to natural-occurring bio-materials (bamboos) reinforced with fiber vascular bundles [4].

The mechanical and thermal properties of FGMs have been investigated for inhomogeneous cylinders, spheres and plates. However, the development of general solutions of boundary value problems for inhomogeneous materials and structures is always a challenging task. Thus, almost all existing methods involving inhomogeneous media are based on simplifying assumptions and functions to describe functional gradations. Several analytical expressions are derived for functionally graded cylinders [5–8] and spheres [6] under symmetric loading conditions. Batra and his co-authors have also published a sequence of their work regarding the elastic solutions of FG cylinders and spheres and material tailoring designs [9–11]. Finite element (FE) analyses are also conducted to study cylinders or spheres in more complicated situations [12–15]. In addition, some authors also focused on the investigations of FGMs as heat-shielding materials [16–20].

It should be noted that solving boundary value problems of inhomogeneous materials has two strategies – discretizing the space with local trial functions or using global analytical trial functions. One could use a trial function expansion for the elastic solutions since the material properties vary in the elastic domain and no simple solutions can represent the stresses or displacements [5–6,8]. The alternative is to divide the cylinders, spheres, plates into several layers with perfect contact conditions; and each layer is treated as a homogeneous body,
leading to much simpler elastic solutions [7]. In addition, more general loading conditions can be considered with the easier equations.

Additionally, most of the work presented focused on the stress analyses by directly assuming macroscopic effective properties and ignoring the details of microscopic reinforcements. However, it should be noted that FGMs are usually manufactured in several different forms. One could use filament (fiber) wound laminate tubes (or shells), where the fiber direction varies to tailor the material properties of each lamina. This type of long fiber-reinforced composite tubes (shells) have wide applications in structures like pressure vessels and piping. Based on the general elastic theory of cylindrical anisotropy by Lekhnitskii [21], numerous studies have been presented to perform stress and displacements under several scenarios. A general elastic analytical solution is obtained for coaxial hollow circular cylinders made of orthotropic materials [22]. Ferreira et al. [23–24] analyzed the laminated shells by employing the meshless collocation method based on radial basis functions (RBFs) with higher-order shear deformation theory. Sarvestani et al. [25] studied the effects of lay-up sequences and orientations of laminae on the stress distributions along the tube thickness. Many other publications focused on the hygro-thermal-mechanical behavior of the laminate structures [26–28], just to name a few.

There are also other techniques of fabricating the FG layered cylinders or spheres, such as mixing several materials by powder metallurgy methods [6]. One example of the manufacturing process is applying the centrifugal force where particle or fiber distributions are formed [29]. Unlike the previous laminate structures, this type of FG structures tailor the material properties of the layers by interpreting the phase volume fractions or shapes of microstructural details. Although FGMs are mainly achieved through tailoring the microstructural reinforcements, there is still not much work to provide a unified multi-scale analytical or numerical framework to conduct stress analysis while also considering the microstructural details. Kukui and Yamanaka [29] provided an expression for the gradation of particle distribution and then applied it to a functionally graded (FG) thick-walled tube, but not much microstructural simulation was provided. Salzar [30] proposed an optimization algorithm to reduce the effective stress by changing the fiber volume fraction in FG metal matrix composite tubes. Reddy and Cheng [31] provided three-dimensional solutions for FGM plates using the transfer matrix method, while the overall properties are calculated by Mori-Tanaka (M-T) method. Nie et al. [10] provided a technique to tailor FG materials and used rule-of-mixtures (ROM) and the Mori-Tanaka method for finding the effective properties. Xin et al. [32] provided an approximate analytical solution by considering M-T model for FG cylinder subjected to internal pressure.

Although some work has been conducted for the material designs of FG structures [9–10], there is still not a systematic procedure for the inverse calculations of both structural and microstructural details based on practical engineering requirements. Combined with the fact of rapid development of computational technology, it is necessary to introduce a simple and stable optimization technique to fulfill this purpose. In addition, based on the recent review work by Dai et al. [14], too many hypotheses of different degrees are made to simplify the models of composite structures, which, in the meantime, affect the applicability of the results as well. Thus, a technique for arbitrarily FG cylindrical structures with various types of non-axisymmetric loading cases is required.

In order to present a sophisticated multi-scale framework in this contribution, composite N-layered cylinders and discs are investigated with each layer being considered as homogeneous, and the properties are graded through the thickness. The effective properties of each layer reinforced with fibers, particles, or discs of different aspect ratios are calculated using the Mori-Tanaka (M-T) model, firstly proposed by Mori and Tanaka [33], then illustrated by Benveniste [34], and popularized by Weng and his co-authors [35–36] by giving the explicit expressions. The effective properties generated by M-T model are then used for the stress and displacement analyses for multilayered composite cylinders subjected to harmonic loading conditions. Finally, the particle swarm optimization technique [37] is introduced to solve the inverse problem of providing a design procedure for thick cylinders with the consideration of microstructural details.

The present contribution is organized as follows: Section 2 introduces the theoretical framework by conducting the relevant derivations for hollow composite cylinders (discs) under harmonic internal and external loading conditions. Section 3 provides validation for the equations in Section 2 and illustrates that the analysis of N-layered composites can be effectively used to generate the analytical solutions for arbitrary functionally graded structures. Section 4 briefly explains the Mori-Tanaka model which is employed to generate the effective properties of fiber, particle, or disc reinforced layers. Section 5 presents the particle swarm optimization method for both the inverse calculation of volume fraction to generate the necessary properties in each layer as well as the efforts of minimizing shear stress in the FG structures. Section 6 presents some conclusions.

2. Theoretical framework

To provide a simple procedure to analyze functionally graded thick cylinders (wherein the gradation along the radial direction is of arbitrary type) under harmonic pressure loadings, we first present analytical solutions for an N-layered cylinder wherein each layer has different homogeneous elastic properties. Thus, first, N-layered composite cylinders (discs), wherein each layer has homogeneous properties, are investigated under harmonic internal and external pressures with different magnitudes. The inner and outer radii of the structures are \( r_1 = a \) and \( r_{N+1} = b \), respectively. Each layer is considered to be reinforced with secondary fibers, spheres, or discs with different aspect ratios \( \alpha \), while \( \alpha > 1 \) for fibers, \( \alpha = 1 \) for spheres, and \( \alpha < 1 \) for discs. The cross-section of a circular cylinder is shown in Fig. 1. In the stress analysis, each layer is treated as being homogeneous, with the effective material properties having been calculated using the Mori-Tanaka homogenization model, which is introduced in Section 4. In this section, the basic governing equations are first provided for the generic \( i \)-th layer of the cylinders, and the interlayer traction reciprocity and displacement continuity conditions, as well as external boundary conditions are then employed to establish the relationships among the layers. Thus, the quantities in this section are considered within the N-layered structural level.

First of all, the harmonic boundary loading conditions at the inner and outer peripheries of the layered cylinders can be expressed as

![Fig. 1. Cross section of N-layered composite cylinders (discs) with different shaped reinforcements and under internal and external loadings in terms of harmonic expansions, wherein each layer has homogeneous elastic properties.](image-url)
\[ p_{in} = P_{0m} + \sum_{m=1}^{M} [P_{m}^i \sin m\theta + P_{m}^r \cos m\theta] \]
\[ p_{out} = P_{0m} + \sum_{m=1}^{M} [P_{m}^i \sin m\theta + P_{m}^r \cos m\theta] \]

in which \( P_{0m}^i, P_{0m}^r, P_{m}^i, P_{m}^r \) are the amplitudes of the harmonic terms, and “m” stands for the m-th order. The problem can be solved by obtaining Navier’s equations of elasticity for the i-th, (i−1)-th, (i+1)-th layers, which can be derived by combining equilibrium equations, stress-strain relationship and strain-displacement relationship:

\[ C_i^{(1)} \left( \frac{\phi_{ap}^{(1)}}{\beta_{ap}} + \frac{1}{r} \frac{\partial \phi_{ap}^{(1)}}{\partial r} \right) + C_i^{(2)} \left( \frac{\phi_{ap}^{(2)}}{\beta_{ap}} + \frac{1}{r} \frac{\partial \phi_{ap}^{(2)}}{\partial r} \right) + \frac{1}{r} \frac{\partial \phi_{ap}^{(2)}}{\partial r} = 0 \]
\[ C_{ij}^{(1)} \left( \frac{\phi_{ap}^{(1)}}{\beta_{ap}} + \frac{1}{r} \frac{\partial \phi_{ap}^{(1)}}{\partial r} \right) + C_{ij}^{(2)} \left( \frac{\phi_{ap}^{(2)}}{\beta_{ap}} + \frac{1}{r} \frac{\partial \phi_{ap}^{(2)}}{\partial r} \right) + \frac{1}{r} \frac{\partial \phi_{ap}^{(2)}}{\partial r} = 0 \]

where \( E^{(i)} \) and \( \nu^{(i)} \) are the homogenized Young’s modulus and Poisson’s ratio for the i-th layer, respectively. Eq. (2) can be solved by assuming Fourier series expansions:

\[ u_i^{(1)}(r, \theta) = \sum_{m=1}^{\infty} \left[ f_m(r) \cos m\theta + g_m(r) \sin m\theta \right] \]
\[ u_i^{(1)}(r, \theta) = \sum_{m=1}^{\infty} \left[ f_m(r) \sin m\theta + g_m(r) \cos m\theta \right] \]

Substituting Eq. (3) into Eq. (2) leads to the expressions for displacement field with different harmonic terms:

\[ u_i^{(1)}(r, \theta) = F_{01}^{(i)} \alpha_1 \xi + F_{02}^{(i)} \alpha_2 \xi^{-1} + \sum_{j=1}^{3} \alpha_j F_{j}^{(i)} (\cos \theta) + F_{j}^{(i)} (\sin \theta) + \sum_{m=2}^{\infty} \sum_{j=1}^{4} \alpha_j g_m F_{j}^{(i)} (\sin \theta) \]

where \( \xi = r/a \) is introduced as a non-dimensionalized parameter. The introduction of \( \xi \) is very important for structures with large dimensions and helps mitigating the ill-conditioned equations that affect the accuracy of solutions.

The corresponding eigenvalues \( p_{mj} \) in Eq. (4) are given as

\[ p_{m1} = m + 1, p_{m2} = m - 1, p_{m3} = -(m + 1), p_{m4} = -(m - 1) \]

and eigenvectors are expressed in the form of

\[ \beta_{jm}^{(i)} = \frac{2C_{22}^{(i)} (1 - p_{mj}^2) + m^2 (C_{11}^{(i)} - C_{22}^{(i)})}{m(C_{11}^{(i)} + C_{22}^{(i)}) - 3C_{12}^{(i)} + C_{22}^{(i)}} \]

The radial, shear and tangential stress distributions of i-th layer are then obtained through strain-displacement relationship and stress-strain relationship:

\[ c_i^{(1)}(r, \theta) = \left( C_{11}^{(i)} + C_{12}^{(i)} \right) F_{01}^{(i)} \left( C_{11}^{(i)} - C_{12}^{(i)} \right) F_{02}^{(i)} \xi^{-2} \]

where \( F_{01}^{(i)}, F_{02}^{(i)}, F_{j}^{(i)}(\cos \theta) + F_{j}^{(i)}(\sin \theta) \)

\[ c_i^{(2)}(r, \theta) = \sum_{j=1}^{3} \beta_{jm}^{(i)} F_{j}^{(i)} \left( C_{11}^{(i)} - C_{12}^{(i)} \right) F_{02}^{(i)} \xi^{-2} \]

\[ c_{ii}^{(2)}(r, \theta) = \sum_{m=2}^{\infty} \sum_{j=1}^{4} \beta_{jm}^{(i)} F_{j}^{(i)} \left( C_{11}^{(i)} + C_{22}^{(i)} \right) F_{02}^{(i)} \xi^{-2} \]

\[ c_{ii}^{(3)}(r, \theta) = \sum_{m=2}^{\infty} \sum_{j=1}^{4} \beta_{jm}^{(i)} F_{j}^{(i)} \left( C_{11}^{(i)} + C_{22}^{(i)} \right) F_{02}^{(i)} \xi^{-2} \]

In order to solve the problem of the N-layers thick cylinders, the stress and displacement continuity conditions between i-th and (i+1)-th layers are applied at \( r = r_{i+1} \):

\[ u_i^{(1)}(r_{i+1}, \theta) = u_{i+1}^{(1)}(r_{i+1}, \theta) \]

\[ c_i^{(1)}(r_{i+1}, \theta) = c_{i+1}^{(1)}(r_{i+1}, \theta) \]

Through matching harmonic terms with same orders and applying orthogonality, the relationship of unknown coefficients in Eqs. (4), (7) is established for

(1) \( n = 0 \) terms:

\[ A_i^{(1)} F_i = B_i^{(1)} F_{i+1}^{(1)} \]

(2) \( n = 1 \) terms:

\[ A_i^{(1)} F_i = B_i^{(1)} F_{i+1}^{(1)} \]

(3) \( n \geq 2 \) terms:

\[ A_m^{(1)} F_m = B_m^{(1)} F_{m+1}^{(1)} \]
Finally, the boundary loading conditions are applied at the inner periphery ($r = a$) of the innermost layer #1:

$$c_i^{(1)}(r_i, \theta) = (C_i^{(1)} + C_i^{(1)} F_i^{(1)}) r_i \theta - (C_i^{(1)} - C_i^{(1)} F_i^{(1)}) + \sum_{m=1}^{M} \sum_{j=1}^{4} P_{m}^{(j)} r_i \theta \cos \theta + (C_i^{(1)} + C_i^{(1)} F_i^{(1)}) r_i \theta \sin \theta - (C_i^{(1)} - C_i^{(1)} F_i^{(1)}) = 0$$

and at the outer periphery ($r = b$) of the outermost layer #N:

$$c_i^{(N)}(r, \theta) = (C_i^{(N)} + C_i^{(N)} F_i^{(N)}) r \theta - (C_i^{(N)} - C_i^{(N)} F_i^{(N)}) + \sum_{m=1}^{M} \sum_{j=1}^{4} P_{m}^{(j)} r \theta \cos \theta + (C_i^{(N)} + C_i^{(N)} F_i^{(N)}) r \theta \sin \theta = 0$$

By combining Eqs. (11, 13, 15, 16), a system of equations is established by matching the $m$-th ($m \geq 1$) order harmonic terms for the N-layer composite cylinders:

$$\begin{bmatrix}
P_{m}^{(0)} & 0 & \cdots & 0 \\
0 & P_{m}^{(1)} & \cdots & 0 \\
R_{m}^{(1)} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & P_{m}^{(N)} \\
p_{m}^{(0)} & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
p_{m}^{(0)} \\
p_{m}^{(1)} \\
p_{m}^{(2)} \\
\vdots \\
p_{m}^{(N)}
\end{bmatrix} = 0$$

(17)

where

$$P_{m}^{(0)} = P_{m}^{(0)}(r_i/a)^{m-1}$$

and

$$R_{m}^{(1)} = R_{m}^{(1)}(r_i/a)^{m-1}$$

3. Validation

3.1. A homogeneous thick cylinder

The present derivation is firstly validated through generating the stress distributions in a homogeneous thick cylinder, modeled as 10 layers under constant internal pressure $p_{in} = -100$ MPa. The analytical stress expressions are already given for a single homogeneous cylinder [8]:

$$\sigma_{int}^{(N)}(r) = p_{in} a^2 (r^2 - b^2)/(r^2 (a^2 - b^2))$$

$$\sigma_{out}^{(N)}(r) = p_{out} a^2 (r^2 + b^2)/(r^2 (a^2 - b^2))$$

(18)

The inner and outer radii of the cylinder are $a = 50$ mm and $b = 100$ mm, respectively. It should be noted that the stress distributions for homogeneous cylinder are not directly related with the material properties, Eq. (18). The existing stress distributions $\sigma_{int}$ and $\sigma_{out}$ are compared between the present solutions and analytical expressions in Fig. 2, where good agreement is obtained.

Then an extended study is conducted for the same 10 layered cylinders by applying internal and external pressures expressed using harmonic terms, which has not been used before:

$$p_{in} = p_{in} \sin 4\theta + p_{in} \cos 4\theta$$

$$p_{out} = p_{out} \sin 2\theta + p_{out} \cos 2\theta$$

(19)

which are also treated as the analytical solutions at boundaries for comparisons, and $p_{in} = -100$ MPa.

First of all, the stress distributions $\sigma_{int}$ are compared at the inner and outer perimeters of the homogeneous cylinder in Fig. 3, where open circles are generated from Eq. (19). It can be easily observed that good agreement is obtained between the present analysis and analytical solutions, where 4th order and 2nd order harmonic terms are applied at inner and outer peripheries, respectively. The offset magnitude at the inner periphery is caused by the constant pressure applied, Eq. (19). In addition, the shear stress distributions $\sigma_{out}$ are also generated (not shown), and practical zeros are obtained along the tangential directions.

3.2. Functionally-graded thick hollow cylinders

3.2.1. Validation

To the best knowledge of authors, there is still not an effective finite-element (FE) based stress or deformation analysis of FG hollow
cylinders under non-axisymmetric loading conditions. One of the best solutions in this situation is the one obtained by Nie and Batra [9], who proposed analytical solutions for a single layer hollow cylinder with linear or power-law FG, under non-axisymmetric loading, by solving complicated stress functions. In the 3rd example of their work, for linear FG, the Young’s modulus and Poisson’s ratio are varied in assumed functions along the radial direction that can be expressed as:

\[ E(r) = E_0 (r/b)^{\lambda_1}, \nu(r) = \nu_0 (r/b) \]  

(20)

where \( E_0 = 200 \text{ GPa}, \nu_0 = 0.3 \). The inner and outer radii of the cylinder are \( a = 0.2 \) and \( b = 1 \), respectively. In addition, the cylinder is subjected to three types of external loading conditions:

\[ p_{\text{out}} = p^0 \cos m\theta \]  

(21)

in which \( m = 0, 2, 4 \), respectively, and \( p^0 = -1 \text{ MPa} \).

Here we compared the present solutions for 80 layered cylinders with the same sizes and same gradation of material properties, as well as under same loading conditions, with the ones obtained by Nie and Batra [9] in Fig. 4. The stress components \( \sigma_{rr}, \sigma_{\theta\theta} \) are illustrated along the radial direction of cylinders at \( \theta = 0 \), while the shear stress \( \sigma_{r\theta} \) is generated at \( \theta = \pi/4 \) for loading cases when \( m = 0, 2 \) and at \( \theta = \pi/8 \) for loading case \( m = 4 \). It should be easily noticed that the present solutions for layered FG cylinders are accurately validated by the results from Nie and Batra [9], even though a few minor errors are caused from the digitization of the data.

It should also be noted is that the FE based analyses were also conducted on similar types of cylinders or spheres, but mostly focused on axisymmetric mechanical-thermal-physical loading conditions [12–15]. Vedeld and Sollund [13] conducted the FE analyses using commercial package Abaqus, where the 8-node brick element C3D8R was used for a six layered off-shore pipeline. As was recorded in their work, a total of 226,800 solid elements, corresponding to approximately 850,000 degrees of freedom were employed. It can be easily seen that for the 80 layered FG cylinders under the similar loading conditions as Eq. (21), many more elements need to be refined to match the continuities between adjacent layers, let alone to satisfy the arbitrary (inner and outer) boundary conditions along the tangential directions. What’s more, the pre- and post-processing is always time consuming and laborious. What is in contrast is the present analytical technique that just requires solving systems of equations for the unknowns of \( m \)-th order terms, Eq. (17). Each set of the equations requires 0.0525 s of computational time on a normal PC, leading to the final accurate solutions with only about 0.0525 \( \times (m + 1) \)-second execution of programs. Additionally, the good agreement between the present solutions and other analytical solutions under general loading conditions gives us more confidence about the accuracy and efficiency of this technique. Since Nie and Batra [9] already derived the solutions for FG cylinders with varied material properties using linear and power-law functions, here we show that the present solution can be used to conduct the stress analysis with arbitrary variation of material properties and different types of loading conditions, which will be illustrated in next section.

### 3.2. Numerical results

The deformations and stresses of a hollow cylinder with continuously exponentially graded material properties in radial direction was derived by Xiang et al. [7]. Similar functions are adopted to describe the variation of the stiffness coefficients along the radial direction of the FG cylinder

\[ C_{11}(r) = C_{11}^0 e^{\lambda r/b}, C_{12}(r) = C_{12}^0 e^{\lambda r/b} \]  

(22)

except here we intend to change the exponential constant \( \lambda \) at different radial locations to show the superiority of the present solution, which

Fig. 3. Comparisons of radial stress \( \sigma_r \) at inner and outer peripheries of the 10-layer cylinder, to mimick a thick homogeneous cylinder.
cannot be archived by other methods in prior literature:

\[
\lambda = \begin{cases} 
3 \text{ m}^{-1} & a \leq r < (3a + b)/4 \\
2 \text{ m}^{-1} & (3a + b)/4 \leq r < (a + b)/2 \\
1 \text{ m}^{-1} & (a + b)/2 \leq r < (a + 3b)/4 \\
0 \text{ m}^{-1} & (a + 3b)/4 \leq r \leq b 
\end{cases}
\]

and \( C_{11}^0 \) and \( C_{12}^0 \) are initial stiffness with \( C_{11}^0 = 81.707 \text{ GPa} \) and \( C_{12}^0 = 31.775 \text{ GPa} \). The FG cylinder has been modeled as \( N \)-layered cylinders (\( N \) is a multiple of 4), with each layer having constant material properties. The material properties of each layer are obtained by picking the values at the mid-point of the local layers, following the gradation in Eq. (22). Another type of loading is applied at the inner radius and the outer pressure is zero:

\[
p_{\text{in}} = p_0, \quad p_{\text{out}} = 0, \quad \rho = -2.379 \text{ m}^{1/2},
\]

where \( p_0 = -100 \text{ MPa} \).

The effect of the number of layers on the stiffness coefficient \( C_{11} \), radial, shear and tangential stress components is tested in Fig. 5 along the radial direction at \( \theta = \pi/4 \), where not much difference is observed for \( \sigma_{rr} \) and \( \sigma_{r\theta} \) (except for \( N = 4 \)), but significantly reduced discontinuities are obtained for \( \sigma_{\theta\theta} \) with the increase of the number of layers. It can be easily seen that a comparatively smooth curve is generated for 100 layers (\( N = 100 \)) along the radial direction. The present functional gradation can be generalized to any arbitrary forms of material properties for the FG cylinders, which have far more meaning in real engineering problems than any other specific functions.

Furthermore, by degenerating constant loading pressure \( p_{\text{in}} = p_0, \quad p_{\text{out}} = 0 \) and letting \( \lambda = 2.379 \text{ m}^{-1} \), the midpoint interpolated stress components are investigated for the 40 layer cylinder and compared with analytical solutions of continuously functionally graded structure [7] in Fig. 6. The radial stress \( \sigma_{rr} \) is still continuous along the radial direction and well matched with the analytical result. The tangential stress \( \sigma_{\theta\theta} \) should be discontinuous in this situation between layers (Fig. 5), but using stress interpolation at middle points of local layers (dash line) still recovers the analytical results for a continuously functionally graded structure. Based on the comparison results, it can be safely concluded that the stress interpolation among the midpoints of layered composites generates exactly the same stress distributions as the analytical solutions of continuously FG structures. Thus, the interpolated stress distributions can be employed to illustrate further numerical characteristics in this presentation. What should also be mentioned is that the shear stress components \( \sigma_{\theta\theta} \) becomes significant under harmonic loading conditions, making itself an important factor in the consideration of delaminations of the composite structures. Thus, a careful examination of shear stress is necessary.

### 3.3. Exponential function vs power law function in a FG cylinder

Several descriptive functions have been used in the past, to describe the FG material properties of cylinders (discs) or spheres in the radial direction. Exponentially-varying functions were adopted by Tutuncu [8], leading to the complicated expressions in the form of Frobenius method; Power law functions were employed by a larger group of researchers [5–6,39–40] mainly because a simpler Navier’s equation can be obtained. Linear functions are sometimes applied but with much less popularity. It has been reported that the effect of the variation of Poisson’s ratio on stress distributions can be neglected and a constant Poisson’s ratio is thus usually assumed along the radial direction as in the literature [5,19,41].
In order to show the effects of the functions used to describe the material gradation, Fig. 7 generates the varying modulus $C_{11}$ of a 20-layer FG cylinder using both exponential and power functions with parameters of different values. Same dimensions and loading conditions are employed here as the in last section, while different values of parameter $\lambda$ are chosen. It could be easily noted that the exponential functions change the modulus with much larger magnitude, which may not be easily achieved in real practice through reinforcements. However, the power law model provides a much more reasonable interpretation of modulus in the radial direction, along with the simpler Navier’s equation, making it a more popular choice.

Also, radial stresses $\sigma_r$ of composite cylinders with modulus expressed in exponential and power gradient functions are generated in Fig. 8 with the constant loading conditions $p_{in} = -100$ MPa, $p_{out} = 0$. Greater differences are noticed for power law function with different parameters, facilitating its applications in the design process.

4. Mori-Tanaka model for homogenization of properties

4.1. M-T model for effective material parameters

Mori and Tanaka [33–34] proposed the exact elastic homogenization technique in accordance with the Eshelby-type embedding approach [42]. M-T model is generally considered as the classical method that provides accurate homogenized moduli and easily-to-implement expressions [43]. The advantages and shortcomings of M-T model can be referred to [44].

A brief description is provided below of the micromechanical model. Based on the average matrix stress assumption, the fibrous, spherical or disc inclusions are embedded into the matrix phase, and then far-field macroscopic strains were applied to generate strain concentration matrix $A_{eff}$ which is used to relate the averaged fiber strain with the macroscopic average strain becomes

$$\varepsilon = A_{eff} \varepsilon_m$$

For any two-phase composite materials,

$$\varepsilon = \varepsilon_f \varepsilon + \varepsilon_m \Rightarrow \varepsilon = [\varepsilon_f A_{eff} + \varepsilon_m I] \varepsilon_m$$

from which the strain concentration matrix relating the average fiber strain with the macroscopic average strain becomes

$$A_f = A_{eff}^\prime [\varepsilon_f A_{eff} + \varepsilon_m I]^{-1}$$

Then the final elastic stiffness matrix can be expressed as

$$C_m = C_m^0 + \varepsilon_f (C_f^0 - C_m^0) A_{eff}^\prime [\varepsilon_f A_{eff} + \varepsilon_m I]^{-1}$$

from which Young’s modulus and Poisson’s ratio are obtained. The explicit expressions were offered by Tandon and Weng [35] for heterogeneous media reinforced with different shaped inclusions.

4.2. A real application for composite cylinders with spherical inclusions

Thick-walled corundum particulate reinforced FGM plaster tubes, manufactured by a method of centrifugal casting, are considered by Fukui and Yamanaka [29]. The material properties of individual constituents are given in Table 1. The inner and outer radii of the tubes are $a = 30$ mm, $b = 45$ mm, respectively. Controlled parameters for the graded distributions of particles are the mean volume fraction $\tau_f$ (which are 15%, 25%, and 35% in this case) and multiples of gravity $G$, which is the ratio of the centrifugal force to the gravity and chosen as 2.1. Fukui and Yamanaka [29] provided an equation for the distributions of the volume fraction along the radius of cylinders:

$$\tau_f (r) = A (r/b)^3 + B (r/b)^2 + C$$

in which $A$, $B$ and $C$ are functions of the mean volume fraction $\tau_f$ and multiples of gravity $G$.

Fig. 9 presents the recovered distributed volume fractions from Eq. (28) in the radial direction, with three mean volume fractions: 15%, 25%, and 35%. The volume fractions behave in a systematic manner while mean values happen at around $r/b = 0.833$. The corresponding stress components are generated by relating the structural analysis with micromechanical calculations by M-T model under loading conditions:

$$p_{in} = p_\theta^0 + p_\theta^0 \sin 2\theta + p_\theta^0 \cos 2\theta$$

$$p_{out} = p_\theta^0 / 2 + p_\theta^0 / 2 \sin 2\theta + p_\theta^0 / 2 \cos 2\theta$$

where $p_\theta^0 = -100$ MPa.

Fig. 10 shows the distributions of $\sigma_r$, $\sigma_\theta$ and $\sigma_\phi$ along the radial direction at $\theta = \pi/4$ for three different volume fraction distributions dictated by Fig. 9. Surprisingly, nearly invisible difference appears in the three stress distributions, which means stress distributions are not significantly affected by the magnitudes of volume fraction.
distributions of similar patterns under the present loading condition. One of the main reasons for the similar stress distributions is due to the similar modulus functions generated from the same volume fraction distribution in Eq. (27) with different unknown coefficients. The stress distributions generated in Fig. 10 show engineers and designers of minimum effect of fiber volume fraction for future guidance. However, more general case needs to be investigated for further guidance.

4.3. Effect of reinforcements on the overall structures

The micromechanical details may affect the stress distributions at the structural level. Thus, 40 layered aluminum composite cylinders with inner and outer radii $a = 30$ mm, $b = 100$ mm are investigated when subjected to same loading conditions as the last section. Each layer of the cylinders is reinforced using SiC aligned fibrous ($\alpha = 10$), spherical ($\alpha = 1$), or disc ($\alpha = 0.1$) inclusions, respectively. The following simple linear function is used to describe the volume fraction along the radial direction

$$v_f (r) = \frac{(r-a)}{(b-a)} \quad (30)$$

where the volume fraction each layer is fixed by picking the value at mid-point. The effective properties of each layer are generated through M-T model [33]. For the unidirectionally aligned composite materials, the isotropy of the inplane material properties still holds, even if the out-of-plane properties may be different for fibrous or disc-shaped inclusions. Thus, the derivations in Section 2 are still valid in the present case based on plane strain (or stress) assumption. Then the homogenized properties are employed as in Section 2 to analyze the stress and displacement fields at the structural level. The stiffness element $C_{11}$ is firstly shown in Fig. 11 along the radius of the cylinders, where $C_{11}$ converges to around 100 GPa at inner radius ($v_f = 0$) and 460 GPa at outer radius ($v_f = 1$), which are the essentially the modulus of matrix and fiber, respectively. It can be easily observed that $C_{11}$ is the smallest for particle reinforcement while progresses interchangeably between two other reinforcements. Fig. 12 investigates the radial, shear and tangential stresses $\sigma_{rr}$, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$, as well as radial displacement $u_r$ along the thickness of the cylinders at $\theta = \pi/4$. Nearly same radial stress component $\sigma_{rr}$ is generated for composite cylinders with different aspect ratios, while shear stress $\sigma_{r\theta}$ shows minor distinctions: Fibrous reinforcement generates higher shear stress at locations where radius is
smaller while disc reinforcement produces higher magnitude at larger-radius location, Fig. 12b. Less systematical patterns happen for tangential stress \( \sigma_{\theta\theta} \): the three types of reinforcements lead to curves with different trends, Fig. 12c. The reinforcements also play roles in affecting the radial displacement distributions \( u_r \). Fibers and discs generate larger and smaller magnitude of displacements, respectively, while spherical reinforcement produces displacement between them, Fig. 12d. The effects of reinforcements with different aspect ratios on stress and displacement components naturally promote the idea of tailoring the parameters for better manipulations of structures. Next section we introduce particle swarm optimization technique to fulfill this purpose.

5. Design of composite cylinders using particle swarm optimization

Optimization techniques could be extraordinarily useful in the design and applications of FG composite materials, providing the inverse calculations of structural properties as well as structural designs of FG materials. Particle swarm optimization has received more and more acceptance in the realm of FG materials since its proposition in 1995 [37]. Fereidoon et al. [45] used particle swarm-based algorithms to search for the optimal volume fractions of FGMs, trying to minimize the peak residual stresses and maximize the safety factor. Kou et al. [46] use PSO to optimize the generic gradation patterns. The applications in FG materials have proved that the PSO is a stable and efficient technique that is recommended for further calculations. Wang et al. [47,48] combined the PSO with elasticity-based homogenization technique to search for the optimal parameters to minimize the local microstructural stresses in stress analyses.

The technique combines the concept of social interaction with solving boundary value problems, and constitutes a swarm of particles keeping searching the target (best solution) by continuously updating the previous experience. First, a swarm of particles with different dimensions of variables are initiated. The position of \( i \)-th particle is denoted as

\[
X^i = (x_{i1}, x_{i2}, \ldots, x_{iD})
\]

where \( D \) stands for the dimension of variables. The velocity of the same particle is

\[
V^i = (v_{i1}, v_{i2}, \ldots, v_{iD})
\]

\( p_{\text{Best}} \) and \( g_{\text{Best}} \) denote the best experience of \( i \)-th particle and other particles, respectively, and they are used to update the positions continuously. The updating algorithms are expressed as [37,47]

\[
v^{i+1} = \omega v^i + a_1 \text{rand}_1^i (g_{\text{Best}} - x^i) + a_2 \text{rand}_2^i (p_{\text{Best}} - x^i)
\]

in which superscript \( k \) and \( k + 1 \) are the iteration numbers. \( \text{rand}_1^i () \) and \( \text{rand}_2^i () \) are random numbers with uniform distributions in the interval \([0,1]\), and \( a_1, a_2 \) are acceleration constants. The parameter \( \omega \) is the inertia weight parameter defined in terms of its initial and final values, \( \omega_{\text{max}} \) and \( \omega_{\text{min}} \) and the current and maximum iteration numbers \( k \) and \( k_{\text{max}} \):

\[
\omega = \omega_{\text{max}} - k (\omega_{\text{max}} - \omega_{\text{min}}) / k_{\text{max}}
\]

The constriction factor \( \chi \) is introduced to ensure convergence:

\[
\chi = \frac{2}{2 - \varphi - \sqrt{\varphi^2 - 4\varphi}}
\]
where $\varphi = a_1 + a_2$ such that $\varphi > 4$. The optimized results are iterated until the desired accuracy or maximum iteration is reached.

It should be noted that the several improved versions of PSO has been proposed for different optimization purposes. Here we list the one with the consideration of gradient decent scheme in the Appendix, which makes the optimization faster with fewer iteration steps. However, the gradient based PSO requires the analytical explicit expressions for the gradient functions, which may not be available to most of the engineering problems.

### 5.1. Inverse calculation of particle volume fraction

PSO is first used in this paper, in conjunction with the Mori-Tanaka model to generate the optimal volume fractions that produce the desired properties of composite materials. Aluminum composite spheres (15 layers) reinforced with SiC particles (aspect ratio $\alpha = 1$) in each layer are considered in this scenario. Typically employed as the radial variation function of Young's modulus in the literature, power law model is used here for the calculation. The inner and outer radii of the composite spheres are $a = 50$ mm, $b = 100$ mm, respectively. The function of the Young's modulus can be expressed as

$$E(r) = E^0(r/b)^{\lambda}$$

in which $\lambda$ is the parameter interpolating the Young’s modulus, which is picked as $\lambda = 3$ here. Initial Young’s modulus is $E(0) = E^0 = 420$ GPa. The target function of PSO is expressed as

$$F_1 = \text{minimize} \left\{ \frac{(E^M - T(v_f)) - E(r)}{E(r)} \right\}$$

where $E^M(v_f)$ is the Young’s modulus generated using M-T model by searching for the best $v_f$ that is matched with $E(r)$ in Eq. (34) at the midpoint of each layer. 20 particles and 20 maximum iterations are designed. Total 3.03 s are used to generate the converged results at 15 locations along the radius, due to the efficient PSO execution and M-T model. Fig. 13 (a) shows the convergence of the PSO after each iteration step, and practically zero is obtained after the 9th iteration, indicating well-matched results between optimized and target modulus. The optimized volume fractions are plotted and the corresponding Young’s modulus is recovered in Fig. 13 (b).

### 5.2. Design of composite cylinders by tailoring parameters

As is already mentioned in previous sections, the shear stresses in the transverse directions of composite cylinders are important in determining the delamination of structures. In addition, several factors play roles in determining the stress and displacement distributions, such as thickness of the cylinders and radial varying functions. Thus,
two corresponding parameters are designed for the structural optimization of layered hollow cylinders with spherical reinforcement, which are inner and outer radius ratio $a/b$ and the parameter $\lambda$ in the function of varying volume fraction $\nu_h = (r/b)^\lambda$. The designated upper and lower bounds of the parameters are set as $a/b \in [0.1; 0.9]$ and $\lambda \in [0;10]$ while the target function is to designed to minimize the shear stresses along the thickness of the cylinders:

$$F_2 = \text{minimize}(\sigma_{\theta\theta}^{\text{max}}(r))$$

(36)

40 particles and 40 maximum iterations are designed at the first step. The final optimized results are obtained after about 20.47 min using HP Intel(R) Core(TM) i7 CPU. The converged parameters are $a/b = 0.501$, $\lambda = 5.140$, and the final result is $\sigma_{\theta\theta}^{\text{max}}(r) = 0.1401$ MPa. Fig. 14 illustrates the initial distributions and final converged distributions of particles in the optimization process. A group of initial particles (open circles) are spared within the designated range, after about 23th iteration, the particles converge to a certain point (solid particles) by keeping updating their previous experiences. A closer observation of the stress distributions along the radial direction is investigated by changing one of the parameters $a/b$ with several values in Fig. 15, it can be easily found that the optimized parameters (0.501) give the minimum absolute values of shear stress $|\sigma_{\theta\theta}^{\text{max}}|$, which are almost zeros along the radius of the cylinders. In addition, the tangential stress $\sigma_{\theta\theta}$ also converges to minimum values along the thickness with the optimized parameter, while the radial stresses $\sigma_r$ are more systematic and smaller (absolute) stresses happen for thicker layered structures. Interpretation of other parameters generates similar situations.

6. Conclusions

Most of the published research work in prior literature is focused on the stress and deformation investigations of continuously functionally graded hollow cylinders (discs) or spheres under axisymmetric loading conditions. Herein a unified framework is established in the structural and microstructural analyses and designs on much more sophisticated topics, by combining structural derivations, micromechanical homogenization and optimization techniques. The composite $N$-layered hollow cylinders (discs) with each layer being homogeneous are studied by applying harmonic loading conditions, the solutions of which are validated against the analytical solutions of boundary expressions and continuously graded elastic heterogeneous structures. It is proved that solutions in $N$-layered structures with each layer being homogeneous can generate the same stress distributions as in continuously graded structures, except with much simplicity. The significant shear stress distributions are also investigated under different harmonic loading cases, which are not present in the constant loading conditions. The Mori-Tanaka model has been briefly introduced and employed in generating homogenized moduli for $N$ composite layers reinforced with different shaped inclusions with designated volume fractions. The accuracy of Mori-Tanaka model has been validated with easily implementable expressions. Several numerical examples also study the effects of the shape and volume fraction of inclusions on the stress distributions. Finally, the well known particle swarm optimization is employed in the present work to not only describe inverse calculations from homogenized moduli by altering volume fractions, but also more importantly to provide design procedures for many potential possibilities for structures by material property gradation or geometrical parameters. Both the quick executions and accurate predictions make the present procedures to be very simple for the analysis and design of continuously functionally graded thick cylinders. In the future work, the multiscale framework can be combined with more sophisticated computational micromechanics techniques that provide not only effective properties but also accurate localized stress concentrations, helping indentifying the possible damage initiations and propagations [50–51]. In addition, the interlayer shear stresses play important roles in the design of the functionally graded structures, especially when considering the delamination phenomenon. Thus, an investigation of shear stress...
Fig. 13. Errors of PSO at each iteration (a), and the optimized volume fractions and corresponding Young’s modulus (b).

Fig. 14. Initial and final particle distributions before and after the optimization.

Fig. 15. Radial, shear and tangential stress distributions using different geometrical parameters $a/b$ compared with the optimized results.
components in other structures, such as (composite spheres), will be necessary and will be presented in our future work.

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Appendix

The improvement of the Particle Swarm Optimization is introduced by Noel [49] based on the classical gradient descent scheme [52]. In the improved version PSO, a point is chosen randomly in the search space and small steps are made in the direction of the negative of the gradient, which makes the new updating algorithm [49]

\[ x_{id}^{k+1} = x_{id}^k - \eta \nabla V(C(x_{id}^k)) \]

(A1)

where \( \eta \) is the step size and can be chosen between [0,0.5]. \( \nabla V(C(x_{id}^k)) \) is the gradient of the analytical function evaluated at point \( x_{id}^k \). The gradient is the row vector composed of the first-order partial derivatives of the analytical function with respect to \( x_{id}^k \).

References