PATH-INDEPENDENT INTEGRALS, ENERGY RELEASE RATES, AND GENERAL SOLUTIONS OF NEAR-TIP FIELDS IN MIXED-MODE DYNAMIC FRACTURE MECHANICS

T. NISHIOKA* and S. N. ATLURI**

Center for the Advancement of Computational Mechanics, School of Civil Engineering, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.

Abstract—In this paper the following topics are addressed: (i) the physical meaning of path-independent integrals for elastodynamically propagating cracks introduced earlier by Atluri, Bui and Kishimoto et al. (ii) the relation of these integrals to the energy release rates, for propagating cracks and (iii) the relation between these integrals and the time-dependent stress-intensity factors $K_I(t)$, $K_{II}(t)$ and $K_{III}(t)$ in general mixed mode dynamic crack propagation. Finally, a new path-independent integral which has the meaning of energy-release-rate for a propagating crack, is introduced.

1. INTRODUCTION

Dynamic Fracture mechanics can be broadly defined as the mechanics of solid bodies containing stationary or propagating cracks, wherein the effects of inertia are accounted for. In such cases, a knowledge of time-dependent asymptotic stress and displacement fields near the crack-tip is essential in understanding the process and nature of fast fracture of solids. Once such asymptotic fields near the tip of a propagating crack are determined, other important parameters of relevance in dynamic fracture mechanics, such as the dynamic stress-intensity factors and dynamic energy release rates can be determined.

On the other hand, since the work of Eshelby[1] and Rice[2], the subject of the so-called path-independent integrals has received much attention due to its several attractive features in application. The well-known $J$-integral of Eshelby, and Rice (which is in fact the component, of a vector integral $J$, along the crack axis), and which is limited to linear and nonlinear elastostatics, has the physical meaning of energy-release per unit self-similar growth of a crack in a loaded cracked-body. Recently Atluri[3] has derived some very general conservation laws for both finite-elastic solids, as well as those described by rate-sensitive or rate-insensitive incremental constitutive laws, wherein body forces, inertia and arbitrary crack-face conditions (traction as well as deformation) were accounted for. On the basis of these conservation laws, Atluri[3] also investigated path-independent integrals in the case of dynamic crack propagation in elastic as well as inelastic solids. In the case of elasto-dynamic crack-propagation, it was found[3] that the path-independent integral (which in fact also involves an integral over the domain between the crack-tip and the chosen far-field path) derived in[3] does not have the physical meaning of energy release per unit dynamic crack-extension. Instead, the path-independent integral in[3] was shown[3] to have the physical meaning of the rate of change of the Lagrangean, of the solid in dynamic motion, per unit crack extension. This conclusion of[3] is at variance with that of Kishimoto et al.[5], who give the physical interpretation of an energy-release rate to their path-independent integral for dynamically propagating cracks. It should, however, be pointed out that the path-independent integrals of Bui[4] and Kishimoto et al.[5] are slightly different from those of Atluri[3]. The principal difference between those of Atluri[3] and Bui[4] mainly stems from the fact that: (i) Atluri[3] uses a path that is fixed in space (thus, in the case of a finite body, the path is chosen such that a propagating crack-tip is at all times surrounded by this path which is fixed in space), while (ii) Bui considers a path of fixed shape that is traversing with the crack-tip (thus, to an observer moving with the crack-tip, the path appears to be fixed). On the other hand, the difference in the integrals of[3] and [5] is only in the nature of the mathematical form (as will be seen later).

Among the objectives of the present paper are: (i) to directly show the relation between the path-independent integral of[3] and the dynamic energy release rate for a propagating crack, (ii) to derive explicit expressions for both the path-independent integral of[3] as well as the energy release rate

*Visiting Assistant Professor.
**Regents' Professor Mechanics.
in terms of the time-dependent stress-intensity factors $K_I(t)$, $K_{II}(t)$, and $K_{III}(t)$ for a general mixed mode problem, and (iii) to show that the integrals in [5], even though slightly different in form as compared to those in [3], still do not have the meaning of energy-release rates for propagating cracks; and (iv) since the relations between the integral in [3], on the one hand and those in [4] and [5] on the other are explicitly stated, it then follows that the integrals in [4] and [5] can be related (but not equal) to the energy-release rate, or the dynamic $K$-factors.

In this paper we introduce a new path-independent integral which does have the meaning of energy-release-rate per unit crack extension for dynamically propagating crack. To start out with, in the present paper, we first summarize the general (eigen) solutions of the asymptotic fields near the tip of an elasto-dynamically propagating crack under combined Mode I, II and III conditions. In doing this, Radok’s [6] complex variable formulation of the dynamic plane-elasticity equation is slightly modified and also extended to the Mode III case. The eigen solutions, including the singular stress and the corresponding displacement fields, are explicitly expressed in a moving coordinate system which appears fixed to an observer moving with the crack-tip. Simple formulae for determining the stress-intensity factors from the complex potentials are also given.

Next the dynamic energy release rates for all the three Modes I, II and III, are determined, in terms of the stress-intensity factors, using the singular stresses and the corresponding displacements, and the concept of crack-closure energy.

Then, the path-independent integrals and the energy release rates are directly evaluated from the expressions given in [3], using the singular stress and the corresponding displacement fields obtained earlier. Useful formulae relating the path-independent integrals of [3] to the energy release rate on the one hand, and the dynamic $K$-factors on the other, are established. Discussions of the apparent contradictions in the physical interpretations of the path independent integrals, for elasto-dynamically propagating cracks, as given in Refs. [3, 5] are presented.

2. GENERAL SOLUTIONS FOR ELASTODYNAMICALLY PROPAGATING CRACKS

We consider the dynamic propagation of a crack with a constant crack-tip velocity $C$ in a linear elastic isotropic planar body. Let $X$ and $Y$ be spatially fixed cartesian coordinates in the plane of the body, and $Z$ be the thickness coordinate of the body such that $Y = 0$ defines the plane of the crack. We assume that the fields of elastic displacement and stresses are independent of $Z$. Now we introduce the moving coordinate system $x$, $y$, $z$, which remains fixed with respect to the moving crack tip, such that $x = X - Ct$ (see Fig. 1). It is now possible to reduce the boundary value problems of elastodynamics to problems of the complex variable method. As a result, the following expressions for the stresses and displacements can be derived [see, for instance, 6]

\[ \sigma_x = -\text{Re} \left[ (1 + 2\beta^2 - \beta^2)\Phi(z_1) + \Psi(z_2) \right] \]  
\[ \sigma_y = \text{Re} \left[ (1 + \beta^2)\Phi(z_1) + \Psi(z_2) \right] \]  
\[ \sigma_{xy} = \text{Im} \left[ 2\beta^2\Phi(z_1) + \frac{1 + \beta^2}{\beta^2} \Psi(z_2) \right] \]  
\[ \sigma_z = \text{Re} \left[ \Omega(z_2) \right] \]  
\[ \sigma_{zz} = -\text{Im} \left[ \beta \Omega(z_2) \right] \]  
\[ \sigma_z = \begin{cases} 0 : \text{plane stress} \\ \nu(\sigma_x + \sigma_y) : \text{plane strain} \end{cases} \]  
\[ \mu u = -\text{Re} \left[ \Phi(z_1) + \Psi(z_2) \right] \]  
\[ \mu v = \text{Im} \left[ \beta \Phi(z_1) + \frac{1}{\beta} \Psi(z_2) \right] \]  
\[ \mu w = \text{Re} \left[ \Omega(z_2) \right] \]
where $\Phi(z_j)$, $\Psi(z_j)$ and $\Omega(z_j)$ are complex potentials which are functions of the complex variables $z_j = x + i\beta_j y = \eta \, e^{i\theta_j} (j = 1, 2)$ where $i = \sqrt{-1}$ and $\beta_j^2 = 1 - C_j^2/C_d^2$; $\beta_j^2 = 1 - C_\gamma^2/C_d^2$ and $C_d$ and $C_\gamma$ are the dilatational and shear wave speeds, respectively. The wave speeds depend on the material constants, and are given in terms of the shear modulus $\mu$, Poisson’s ratio $\nu$ and the mass density $\rho$ by

$$C_d^2 = \frac{\kappa + 1}{\kappa - 1} \frac{\mu}{\rho}, \quad C_\gamma^2 = \frac{\mu}{\rho}$$

(3a)

and

$$\kappa = \begin{cases} 
\frac{(3 - \nu)(1 + \nu)}{3 - 4\nu} & \text{plane stress} \\
3 & \text{plane strain}.
\end{cases} \quad (3b)$$

The complex potentials $\Phi$ and $\Psi$ are related to the inplane motion of the crack which are separable into Mode I (opening mode) and Mode II (inplane sliding mode) crack problems. The complex potential $\Omega$ is related to only the antiplane motion of the crack or the so-called Mode III (antiplane mode) problem.

The expressions related to the complex potentials $\Phi$ and $\Psi$ were originally obtained by Radok[6]. However, in the present paper, in order to simplify the formulation, Radok’s equations are modified as $\Psi = 1/2(1 + \beta_j^2)\psi$. Thus, the expressions for the inplane motion may be referred to as the modified Radok equations.

Now we seek the general solutions (eigen function solutions) for all the three modes, which satisfy the stress free condition $\sigma_r = \sigma_{\theta\theta} = \sigma_{\phi\phi} = 0$, on the crack surface ($\theta = \pm \pi$). This stress free condition, in terms of the complex potentials, can be expressed by

$$\sigma_r + i\sigma_{\theta\phi} = D_1 \Phi' + D_2 \Phi^* + D_3 \Psi' + D_4 \Psi^* = 0$$

at $\theta = \theta_1 = \theta_2 = \pm \pi$ \hspace{1cm} (4)

$$\sigma_{\theta\phi} = -\frac{\beta_j^2}{2i} (\Omega - \Omega^*) = 0 \text{ at } \theta = \theta_2 = \pm \pi$$

(5)

where ( ) denotes the complex conjugate. The constants $D_j (j = 1, 2, 3, 4)$ are given by

$$D_1 = 1 + \beta_j^2 + 2\beta_j, \quad D_2 = 1 + \beta_j^2 - 2\beta_j^2$$

$$D_3 = 2 + \frac{(1 + \beta_j^2)}{\beta_j^2}, \quad D_4 = 2 - \frac{(1 + \beta_j^2)}{\beta_j^2}$$

(6)

We assume the complex potentials to be in a power series form, as:

$$\Phi(z_j) = \sum_n A_n z_j^n = \sum_n (A_n^0 + iA_n^*) r_j \, e^{in\phi_j}$$

(7)

$$\Psi(z_j) = \sum_n B_n z_j^n = \sum_n (B_n^0 + iB_n^*) r_j \, e^{in\phi_j}$$

(8)
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\[ \Omega(z) = \sum_{n} C_n z^n = \sum_{n} (C_n^0 + iC_n^*) r^2 e^{i\lambda_n r^2} \]  

(9)

where \( \lambda_n \) are real eigen values which will be determined later. \( A_n, B_n \) and \( C_n \) are the undetermined complex constants, and \( A_n^0 \) and \( A_n^* \), etc. denote, respectively, the real and imaginary parts of the complex constant \( A_n \). Introducing eqns. (7)-(9) into eqns (4) and (5) one may find the following equations for the inplane motion

\[ e^{i\lambda_n r^2}(D_1 A_n + D_2 B_n) + e^{-i\lambda_n r^2}(D_2 \tilde{A}_n + D_4 \tilde{B}_n) = 0 \]  

(10a)

\[ e^{-i\lambda_n r^2}(D_1 A_n + D_2 B_n) + e^{i\lambda_n r^2}(D_2 \tilde{A}_n + D_4 \tilde{B}_n) = 0 \]  

(10b)

and for the antiplane motion

\[ e^{i\lambda_n r^2} C_n - e^{-i\lambda_n r^2} \tilde{C}_n = 0 \]  

(11a)

\[ e^{-i\lambda_n r^2} C_n - e^{i\lambda_n r^2} \tilde{C}_n = 0 \]  

(11b)

For a non-trivial solution to exist, the determinant of the coefficient matrices for both the cases must be zero. Thus, we have

\[ \begin{vmatrix} e^{i\lambda_n r^2} & e^{-i\lambda_n r^2} \\ e^{-i\lambda_n r^2} & e^{i\lambda_n r^2} \end{vmatrix} = 0. \]  

(12)

The above equation can be reduced to

\[ \sin 2\pi \lambda_n = 0. \]  

(13)

This leads to the eigen values of \( \lambda_n = n/2 \) \((n = 0, \pm 1, \pm 2, \ldots)\). Since the negative eigen values give infinite displacements at the crack tip, the resulting plausible eigen values are

\[ \lambda_n = n/2 \quad (n = 0, 1, 2, 3, \ldots). \]  

(14)

The eigen value \( \lambda_n = 1/2 \) gives the singular stress field of order of \( 1/\sqrt{r} \), which is well known in the linear elastic fracture mechanics. It is also noted that the zero eigen value gives the rigid body motion. Incorporating these eigen values into eqns. (10) and (11), one can find the following relations for the complex constants:

\[ B_n^0 = -\frac{D_3 + (-1)^n D_2}{D_3 + (-1)^n D_2} A_n^0, \quad B_n^* = -\frac{D_3 - (-1)^n D_2}{D_3 - (-1)^n D_2} A_n^* \]  

(15a, b)

\[ \tilde{C}_n = (-1)^n C_n. \]  

(16)

The above relations are rearranged as

\[ B_n^0 = -h(n) A_n^0, \quad B_n^* = -h(n) A_n^* \]  

(17a, b)

\[ C_n = \begin{cases} -iC_n^+ : n \text{ odd} \\ C_n^- : n \text{ even} \end{cases} \]  

(18)

where

\[ h(n) = \begin{cases} 2\beta_1 \beta_2 (1 + \beta_2^2) : n \text{ odd} \\ \frac{1}{2} (1 + \beta_2^2) : n \text{ even} \end{cases} \]  

(19)

and \( \bar{n} = n + 1 \), and \( C_n^+ \) are real undetermined constants.
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The dynamic stress intensity factors can be defined by the following equations

\[
K_I = \lim_{r \to 0} \sqrt{2\pi r} \sigma_x \bigg|_{a=0}
\]
(20)

\[
K_{II} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{xy} \bigg|_{a=0}
\]
(21)

\[
K_{III} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_z \bigg|_{a=0}
\]
(22)

Using the above equations, we obtain the following relations between the dynamic stress intensity factors and the constants.

\[
K_I = -\sqrt{2\pi} \frac{4B_I \beta_2 - (1 + \beta_2^2)}{2(1 + \beta_2^2)} A_1^0
\]
(23)

\[
K_{II} = \sqrt{2\pi} \frac{4B_I \beta_2 - (1 + \beta_2^3)}{4\beta_2} A_1^*
\]
(24)

\[
K_{III} = \sqrt{2\pi} \frac{\beta_2^2}{2} C_1^+
\]
(25)

Employing the new functions of the crack speed \( B_I, B_{II}, \) and \( B_{III}, \) the constants \( A_n^0, A_n^* \) and \( C_n^+ \) are normalized as follows

\[
A_n^0 = \frac{(n + 1)}{\sqrt{2\pi}} B_I(C) K_n^0
\]
(26)

\[
A_n^* = \frac{(n + 1)}{\sqrt{2\pi}} B_{II}(C) K_n^*
\]
(27)

\[
C_n^+ = \frac{(n + 1)}{\sqrt{2\pi}} B_{III}(C) K_n^+
\]
(28)

such that the coefficients with \( n = 1 \) give the dynamic stress intensity factors \((K_1^0 = K_I, K_1^* = K_{II}, \) and \( K_1^+ = K_{III})). \) The functions \( B_M(C) : M = I, II, III \) are defined by

\[
B_I(C) = \frac{(1 + \beta_2^2)}{D}
\]
(29)

\[
B_{II}(C) = \frac{2\beta_2}{D}
\]
(30)

\[
B_{III}(C) = \frac{1}{\beta_2}
\]
(31)

where \( D(C) = 4B_I \beta_2 - (1 + \beta_2^2)^2. \) The equation \( D(C) = 0 \) is well known as the Rayleigh equation[7] and has the roots of \( C = 0 \) and \( C_R \) (Rayleigh wave speed).

Substituting eqns (26)–(28) into eqns. (7)–(9), the first derivatives of the complex potentials, which are needed in the evaluation of stresses, can be expressed as

\[
\Phi'(z) = \sum_{\pi=0}^{\infty} \frac{n(n+1)}{2\sqrt{2\pi}} \left[-B_I(C)K_n^0 + iB_{II}(C)K_n^*\right] z_1^{(n+1)}
\]
(32)
The relations between the dynamic stress intensity factors and the complex potentials can be easily obtained through the observation of eqns. (32)-(34). Thus,

\[ -K_B j(C) + iK_B j(C) = \lim_{z_i \to 0} \sqrt{2\pi z} \Phi(z_i) \]  

\[ K_{III} j(C) = \lim_{z_i \to 0} \sqrt{2\pi z} i\Omega(z_i). \]

These forms may be convenient to determine the dynamic stress intensity factors when the complex potentials have been determined.

Substituting eqns (32)-(34) into eqns (1) and (2), the general solutions (eigen solutions) can be obtained as

\[ \sigma_{in} = \frac{K_n^o B_1(C) n(n + 1)}{\sqrt{2\pi}} \left\{ \frac{1}{2} \left[ (1 + 2\beta r_1^{(n+1)-1}) \cos \left( \frac{n}{2} - 1 \right) \theta_1 - 2h(n) r_2^{(n+1)-1} \cos \left( \frac{n}{2} - 1 \right) \theta_2 \right] \right. \]

\[ + \left. \frac{K_n^o B_1(C) n(n + 1)}{\sqrt{2\pi}} \left( \frac{1}{2} \left[ (1 + 2\beta r_1^{(n+1)-1}) \sin \left( \frac{n}{2} - 1 \right) \theta_1 - 2h(n) r_2^{(n+1)-1} \sin \left( \frac{n}{2} - 1 \right) \theta_2 \right] \right) \right\} \]

\[ \sigma_{in} = \frac{K_n^o B_2(C) n(n + 1)}{\sqrt{2\pi}} \left\{ \frac{1}{2} \left[ (1 + 2\beta r_1^{(n+1)-1}) \cos \left( \frac{n}{2} - 1 \right) \theta_1 + 2h(n) r_2^{(n+1)-1} \cos \left( \frac{n}{2} - 1 \right) \theta_2 \right] \right. \]

\[ + \left. \frac{K_n^o B_2(C) n(n + 1)}{\sqrt{2\pi}} \left( \frac{1}{2} \left[ (1 + 2\beta r_1^{(n+1)-1}) \sin \left( \frac{n}{2} - 1 \right) \theta_1 + 2h(n) r_2^{(n+1)-1} \sin \left( \frac{n}{2} - 1 \right) \theta_2 \right] \right) \right\} \]

\[ \sigma_{in} = \frac{K_n^o B_3(C) n(n + 1)}{\sqrt{2\pi}} \left\{ \frac{1}{2} \left[ (1 + 2\beta r_1^{(n+1)-1}) \cos \left( \frac{n}{2} - 1 \right) \theta_1 - \frac{1}{\beta_2} h(n) r_2^{(n+1)-1} \cos \left( \frac{n}{2} - 1 \right) \theta_2 \right] \right. \]

\[ + \left. \frac{K_n^o B_3(C) n(n + 1)}{\sqrt{2\pi}} \left( \frac{1}{2} \left[ (1 + 2\beta r_1^{(n+1)-1}) \sin \left( \frac{n}{2} - 1 \right) \theta_1 - \frac{1}{\beta_2} h(n) r_2^{(n+1)-1} \sin \left( \frac{n}{2} - 1 \right) \theta_2 \right] \right) \right\} \]

\[ \sigma_{z} = \frac{K_n^o B_3(C) n(n + 1)}{\sqrt{2\pi}} \left\{ \frac{1}{2} \left[ \beta r_1^{(n+1)-1} \cos \left( \frac{n}{2} - 1 \right) \theta_1 \right] \right. \]

\[ + \left. \frac{K_n^o B_3(C) n(n + 1)}{\sqrt{2\pi}} \left( \frac{1}{2} \left[ \beta r_1^{(n+1)-1} \sin \left( \frac{n}{2} - 1 \right) \theta_1 \right] \right) \right\} \]

\[ u_n = \frac{K_n^o B_1(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n + 1) \left\{ r_1^{n/2} \cos \theta_1 - h(n) r_2^{n/2} \cos \theta_2 \right\} \]

\[ + \frac{K_n^o B_2(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n + 1) \left\{ r_1^{n/2} \sin \theta_1 - h(n) r_2^{n/2} \sin \theta_2 \right\} \]
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\[ v_n = \frac{K_n^0 B_1(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ \beta_1 n_1^2 \sin \frac{n_1}{2} \theta_1 + \frac{h(n)}{\beta_2} r_2^{n_2} \sin \frac{n_2}{2} \theta_2 \right\} \]
\[ + \frac{K_n^* B_{11}(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) \left\{ \beta_1 n_1^2 \cos \frac{n_1}{2} \theta_1 - \frac{h(n)}{\beta_2} r_2^{n_2} \cos \frac{n_2}{2} \theta_2 \right\} \]  

(38b)

\[ w_n = \frac{K_n^* B_{111}(C)}{2\mu} \sqrt{\frac{2}{\pi}} (n+1) r_2^{n_2} \begin{cases} 
\sin \frac{n_2}{2} \theta_2 : n \text{ odd} \\
\cos \frac{n_2}{2} \theta_2 : n \text{ even}
\end{cases} \]  

(38c)

The Mode I eigen solution which is related to the constant \( K_n^0 \) has been derived by Malluck\[8\] following the scheme employed by Rice\[9\]. However, it is noted that the present expressions for Mode I eigen solution differ from those of Malluck\[8\] in the coefficients including the number \( n \).

For Mode I problems, Nilsson\[10\] has shown that both the differential equations and the boundary conditions for an arbitrarily moving crack coincide with those for the problem of steady growth. From this it was concluded that the angular distribution of the singular stress field is only dependent on instantaneous crack velocity. This conclusion can be easily drawn for the other fracture modes. Therefore, the general solutions given in eqns (37) and (38) are valid for all elastodynamic crack problems if we use the instantaneous values of crack velocity and coefficients including the stress intensity factors.

The singular stress field and the corresponding displacement field are given by the \( n = 1 \) terms:

\[ \sigma_x = \frac{K_n B_1(C)}{\sqrt{2\pi}} \left\{ \cos \frac{1}{2} \theta_1 \frac{1}{r_1^{1/2}} - \frac{4\beta_1 \beta_2}{1 + \beta_2^2} \cos \frac{1}{2} \theta_2 \frac{1}{r_1^{1/2}} \right\} \]
\[ + \frac{K_n B_{111}(C)}{\sqrt{2\pi}} \left\{ - (1 + \beta_2^2) \frac{\sin \frac{1}{2} \theta_1}{r_1^{1/2}} + (1 + \beta_2^2) \frac{\sin \frac{1}{2} \theta_2}{r_1^{1/2}} \right\} \]  

(39a)

\[ \sigma_y = \frac{K_n B_1(C)}{\sqrt{2\pi}} \left\{ - (1 + \beta_2^2) \frac{\cos \frac{1}{2} \theta_1}{r_1^{1/2}} + \frac{4\beta_1 \beta_2}{1 + \beta_2^2} \cos \frac{1}{2} \theta_2 \frac{1}{r_1^{1/2}} \right\} \]
\[ + \frac{K_n B_{111}(C)}{\sqrt{2\pi}} \left\{ (1 + \beta_2^2) \frac{\sin \frac{1}{2} \theta_1}{r_1^{1/2}} - (1 + \beta_2^2) \frac{\sin \frac{1}{2} \theta_2}{r_1^{1/2}} \right\} \]  

(39b)

\[ \sigma_{xy} = \frac{K_n B_1(C)}{\sqrt{2\pi}} \left\{ \frac{\sin \frac{1}{2} \theta_1}{r_1^{1/2}} \cos \frac{1}{2} \theta_1 \frac{1}{r_1^{1/2}} - \frac{\sin \frac{1}{2} \theta_2}{r_1^{1/2}} \cos \frac{1}{2} \theta_2 \frac{1}{r_1^{1/2}} \right\} \]
\[ + \frac{K_n B_{111}(C)}{\sqrt{2\pi}} \left\{ \cos \frac{1}{2} \theta_1 \frac{1}{r_1^{1/2}} - \frac{1 + \beta_2^2}{2\beta_1} \cos \frac{1}{2} \theta_2 \frac{1}{r_1^{1/2}} \right\} \]  

(39c)

\[ \sigma_x = \frac{K_n B_{111}(C)}{\sqrt{2\pi}} \left\{ - \frac{\sin \frac{1}{2} \theta_2}{r_2^{1/2}} \right\} \]  

(39d)

\[ \sigma_y = \frac{K_n B_{111}(C)}{\sqrt{2\pi}} \left\{ \frac{\cos \frac{1}{2} \theta_2}{r_2^{1/2}} \right\} \]  

(39e)
Although the complete singular stress and the corresponding displacement fields for all the three fracture modes are shown in a unified fashion in the present paper, many works concerned with the derivation of the stress field in individual modes can be found in literature.

For Mode I crack propagation the present expressions for singular stress field coincide with those obtained by Rice [9], and the corresponding displacements agree with those obtained by Malluck [8]. For Mode II crack propagation, so far only the expressions for $\sigma_x$ and $\sigma_y$ have been presented in literature, by Freund [11]. The present expressions for $\sigma_x$ and $\sigma_y$ coincide with those in Ref. [11]. The expressions for the Mode III stress and displacement field coincide with those obtained by Burgers [12]. Achenbach and Bazant [13] have investigated the singular stresses and displacements for all the three modes. However the solutions for stresses and displacements have not been given explicitly in Ref. [13].

The general solutions expressed by eqns (37) and (38) contain the zero stress and rigid body motions ($n = 0$), the singular stresses and corresponding displacements ($n = 1$), the constant stresses and linear displacements ($n = 2$), and the higher order terms ($n \geq 3$). The general solutions can be incorporated in moving singular elements as was successfully done by the present authors [14–18]. Also the general solutions are useful for determining the stress intensity factors by fitting the general solutions to finite element solutions. This was done by Malluck [8] and the present authors [18].

3. THE DYNAMIC ENERGY RELEASE RATES

Since the complete general solutions for dynamically propagating cracks under general mixed mode conditions are known, as shown in the previous section, the relation between the energy release rate and stress intensity factor can be easily derived through the formula for the crack-closure energy [19]:

$$G = \lim_{\delta a \to 0} \frac{1}{\delta a} \int_0^{\delta a} (\sigma_x v + \sigma_y u + \sigma_z w) \, dx. \quad (41)$$

Substituting eqns (39) and (40) into eqn (41), and decomposing the energy release rate into the corresponding fracture modes, the following relation may be obtained as

$$G = G_I + G_{II} + G_{III} \quad (42)$$
where

\[ G_M = \frac{K_M^2}{2\mu} A_M(C) : M = I, II, III \]  

and

\[ A_I(C) = \beta_1(1 - \beta_0^2)/D(C) \]  

\[ A_{II}(C) = \beta_1(1 - \beta_0^2)/D(C) \]  

\[ A_{III}(C) = 1/\beta_2. \]  

The variations of the functions with crack speed are shown in Figs. 2–4 along with the effect of Poisson’s ratio. For the limiting case as \( C \to 0 \), the functions become: \( A_I(0) = (\kappa + 1)/4 \), \( A_{II}(0) = (\kappa + 1)/4 \) and \( A_{III}(0) = 1 \). Thus, when the crack speed becomes zero, the relations expressed by eqns (42)–(43) reduce to those for a stationary crack. Similar results can be found in many separate works[10, 20–22]. Nilsson[10] has shown that the \( G_I \) vs \( K_I \) relation is completely general for all elastodynamic crack problems. Since the only contributions to the integral in eqn (41) come from the singular part of stresses, the relations for all the three fracture modes shown in eqns (42)–(44) are valid for all general elastodynamic crack problems if we insert the instantaneous crack velocity.

4. PATH INDEPENDENT INTEGRALS

Recently Atluri[3] has derived a very general conservation law for elastic and inelastic solids. On the basis of this conservation laws, many types of path independent integrals, of relevance in fracture mechanics, were obtained in Ref.[3]. The following material constitutive properties were included in Ref.[3]: (i) finite and infinitesimal elasticity, (ii) rate-independent incremental flow theory of elastoplasticity and (iii) rate-sensitive behavior including elastoviscoplasticity, and creep. In each case, finite deformations are considered, along with the effect of material acceleration, body forces and arbitrary traction and displacement condition on the crack face. The physical interpretations of each of the integrals were also explored.

In the case of propagating cracks in elastodynamic fields, it was found[3] that the path independent integral given in[3] does not have the physical meaning of the energy release rate, but does still have the physical meaning of rate of change of the Lagrangean, of the solid in dynamic motion, per unit crack growth.

Now we focus our attention to the problem of dynamic crack propagation in a linear elastic solid. The path independent integral derived by Atluri[1] is given by†

\[ J_k = \lim_{e \to 0} \int_{\Gamma_c} \left[ (W - T)n_k - t_i u_{i,k} \right] ds \]  

\[ = \lim_{e \to 0} \int_{\Gamma_c} \left[ (W - T)n_k - t_i u_{i,k} \right] ds + \int_{V_c} \frac{d}{dt} (\rho \dot{u}_{i,k}) dV \]  

where \( W \) and \( T \) denote the strain energy density and the kinetic energy density, respectively, \( n_k \) are direction cosines of the unit outward normal, \( t_i(n_k n_j) \) is the surface traction and the definitions of the paths \( \Gamma_c \), \( V_c \) and the volumes \( V, V_c \) are shown in Fig. 5.

For convenience in the calculation of \( J_k \), the last term of eqn (45b) can be rewritten as

\[ I_k = \int_{V} \frac{d}{dt} (\rho \dot{u}_{i,k}) dV \]  

\[ = \int_{V} \rho \ddot{u}_{i,k} dV + \int_{\partial V} T_{n_k} ds. \]  

†The index notations are used in this section for convenience. Thus the following identities should be noted: \( X_1 = X, X_2 = Y, X_3 = Z, u_1 = u, u_2 = v, u_3 = w \) and \( \sigma_{11} = \sigma_x, \sigma_{12} = \sigma_{xy}, \) etc. where \( (X_1, X_2, X_3) \) are the spatially fixed coordinates.

†In Eq. (45a) and henceforth \( \frac{d}{dt} \) implies \( \frac{d}{dt} \) etc. where \( (X_1, X_2, X_3) \) are the spatially fixed coordinates.
When eqn (46) is substituted in eqns. (45a, b) we obtain,

\[ \tilde{f}_k = \lim_{\epsilon \to 0} \int_{\Gamma_0} (Wn_k - t_1 u_{1,k}) \, ds \]  

(47a)

\[ = \lim_{\epsilon \to 0} \int_{\Gamma - \Gamma_0} (Wn_k - t_1 u_{1,k}) \, ds + \int_{V - V_0} \rho \tilde{u}_i u_{1,k} \, dV. \]  

(47b)
Equation (47b) is the "path-independent" integral given by Kishimoto et al. [5], even though its fundamental definition and equivalence to the limiting integral (47a) does not appear to have been stressed.

It is worthwhile to study the limits of the integrals in eqns (47a) and (47b) in the limit $\varepsilon \to 0$. In the integral in (47a), both $W$ and $t_i u_{i,k}$ vary as $(1/r)$ near the crack-tip. Thus the limit of the integral in (47a) is clearly seen to exist, if one considers $\Gamma_0$ to be a circle of radius $\varepsilon$. For then, $W$ and $t_i u_{i,k}$ vary as $1/\varepsilon$ on $\Gamma_0$, and $d\zeta = \varepsilon d\theta$ on $\Gamma_0$. For similar reasons, the limit of the first integral in (47b) exists. On the other hand, since $\dot{u}_i$ (material acceleration) varies as $(r^{-3/2})$ and $u_{i,k}$ varies as $(r^{-1/2})$, the integrand in the second integral of (47b) is $0(r^{-1})$. Thus, on first sight the limit of this integral does not seem to exist. Clearly since the integral in (47a) is equal to the sum of the integrals in (47b), the limit of the second integral in
(47b) must exist. The only way for this to happen is that the angular variation (i.e. with $\theta$) of the term $\tilde{u}_i u_{i,k}$ is such that

$$
\lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \left[ \int_0^r (\rho \tilde{u}_i u_{i,k}) r \, dr \right] d\theta \to 0.
$$

This has been verified by the present authors analytically using eqn (40) and some unpleasant algebra.

It should be noted that the contour of the far-field integral appearing in eqn (45b), as originally given in[3], is fixed in space, and the crack-tip moves into this space-fixed contour. On the other hand, Bui[4] considers the far field contour $\Gamma$ to be a rigid path surrounding the crack-tip and in translation at the same velocity $v$ as the crack-tip, and $V$ is the surface surrounded by $\Gamma$. With this in mind, the first component of the path-independent integral vector given in[4] can be written:

$$
J_1* = \int_{\Gamma} \left[ W_n - \sigma_{i,k} u_{i,1} - \frac{1}{2} \rho \tilde{u}_i u_{i,1} - \rho \tilde{u}_i u_{i,1}, C n_1 \right] ds + \frac{D}{Dt} \int_V \rho \tilde{u}_i u_{i,1} dV.
$$

Noting that $V$ is the area between the crack-tip and $\Gamma$, and that $\tilde{u}_i u_{i,1}$ is $O(r^{-1})$ near the crack-tip, it can be shown† that:

$$
\frac{D}{Dt} \int_V \rho \tilde{u}_i u_{i,1} = \lim_{\epsilon \to 0} \int_{-\epsilon \rightarrow -\epsilon} \frac{d}{dt} (\rho \tilde{u}_i u_{i,1}) \, dV + \int_{\Gamma} \rho \tilde{u}_i u_{i,1} C n_1 \, ds
$$

The last integral on the rhs of (48b) can be evaluated from the asymptotic (singular field near the propagating crack-tip):

$$
\dot{u}_i = - C u_{i,1} - \rho \tilde{u}_i u_{i,1}, C n_1 = n_1 \rho \tilde{u}_i \dot{u}_i = 2 T_n_1.
$$

Using (48b) and (48c) in (48a), and comparing with (45b) we obtain:

$$
J_1* = \int_{\Gamma} \left[ W_n - \sigma_{i,k} u_{i,1} - \frac{1}{2} \rho \tilde{u}_i u_{i,1}, C n_1 \right] ds + \frac{D}{Dt} \int_{\Gamma} 2 T_n_1 \, ds
$$

$$
= J_1 + \lim_{\epsilon \to 0} \int_{\Gamma} 2 T_n_1 \, ds.
$$

Atluri[3] has also shown, from the first principles, that the energy release rate can be expressed by

$$
G = (C_k/C) C_k \text{ and } G_k = \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \left[ (W + T)n_k - \rho \tilde{u}_i u_{i,1} \right] ds
$$

where $C_k$ denotes the component of the crack velocity in the $X_k$ direction. Comparing eqns (45a, 47a, and 48d) and (49b) it is seen that

$$
J_k = G_k - \lim_{\epsilon \to 0} 2 \int_{\Gamma_\epsilon} T_n_1 \, ds; \dot{J}_k = G_k - \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} T_n_1 \, ds; J_1* = G_1.
$$

For stationary cracks in dynamic elastic fields the kinetic energy density $T = 1/2 \rho \tilde{u}_i \dot{u}_i$ is nonsingular since $\dot{u}_i$ is of the order of $O(r^{-1})$. Thus, the last term of eqns (50-52) vanishes as $\Gamma_\epsilon$ shrinks into the crack tip.

On the other hand, for a dynamically propagating crack the kinetic energy density becomes singular since $\dot{u}_i$ has the order of $O(r^{-1/2})$ near the crack tip. This is easily shown from the evaluation of

†This result is due to Bui[27].
the total time derivative $\dot{u}_i = (\partial u_i / \partial t) - C(\partial u_i / \partial x_i)$ for a crack propagating in the $X_1$ direction, as derived in eqn (40). Since the last term of eqn (50) remains finite, it is clearly seen that $J_k \neq G_k$ for dynamic crack propagation. This finding [3] is different from the conclusion drawn in the currently reported studies [5, 23] in literature. The false conclusion drawn by the other investigators [5, 23] can be attributed to the fact that they used the procedure by Achenbach [24] which was originally developed for the evaluation of the energy release rate in terms of the stress intensity factors.

Now we present a path-independent integral $J_k'$ which is identically equal to $G_k$, and hence has the meaning of energy-release rate in dynamic crack propagation. To this end, we first observe that

\[ \int_{V_{-V}} \frac{\partial}{\partial X_k} \left( \frac{1}{2} \rho \dot{u}_i \right) \, dV = \int_{V_{-V}} \rho \dot{u}_i u_{ik} \, dV = \int_{\Gamma_1^{+\Gamma_1}} T_{nk} \, ds - \int_{\Gamma_1^{+\Gamma_1}} T_{nk} \, ds. \]  

Likewise

\[ \int_{V_{-V}} \left( \frac{\partial}{\partial X_k} T \right) \, dV = \int_{\Gamma_2^{+\Gamma_2}} T_{nk} \, ds - \int_{\Gamma_3^{+\Gamma_3}} T_{nk} \, ds. \]

Thus, from eqns (47a, b and 53a) one may define a path-independent integral, such that,

\[ J_k = G_k = \lim_{\epsilon \to 0} \int_{\Gamma_1^{+\Gamma_1}} [(W + T) n_k - \sigma_{ik}] \, ds \]

\[ = \lim_{\epsilon \to 0} \int_{\Gamma_1^{+\Gamma_1}} [(W + T) n_k - t_i u_{ik}] \, ds + \int_{V_{-V}} [\rho \dot{u}_i u_{ik} - T_{ik}] \, dV. \]  

Path-independence of the extreme rhs term of eqn (54) is evident, since, it can be easily verified from the elasto-dynamic equilibrium condition and eqn (53b) that:

\[ \int_{\Gamma_2^{+\Gamma_2}} [(W + T) n_k - t_i u_{ik}] \, ds + \int_{V_{-V}} [\rho \dot{u}_i u_{ik} - T_{ik}] \, dV \]

\[ - \int_{\Gamma_3^{+\Gamma_3}} [(W + T) n_k - t_i u_{ik}] \, ds = 0. \]

Now we seek the relations between the $J_k$ integrals and the stress intensity factors. Suppose we consider a 2-dimensional problem wherein $V_\epsilon$ is a circular domain of radius $\epsilon$ centered at the crack tip (see Fig. 5). From eqns (45a) and (49b), the components of $J$ integral and strain energy release rate, respectively become:

\[ J_k = \lim_{\epsilon \to 0} \int_{-\pi}^{+\pi} [(W - T) n_k - t_i u_{ik}] \, d\theta \]  

\[ G_k = \lim_{\epsilon \to 0} \int_{-\pi}^{+\pi} [(W + T) n_k - t_i u_{ik}] \, d\theta \]

in which the direction cosine of the unit outward normal can be expressed as $n_1 = \cos \theta$ and $n_2 = \sin \theta$. For the 2-dimensional problems the strain energy density takes the form

\[ W = \frac{1}{16\mu} \left[ (\kappa + 1)(\sigma_{11}^2 + \sigma_{22}^2) - 2(3 - \kappa)\sigma_{11}\sigma_{22} + 8\sigma_{12}^2 \right] \]

where the material constant $\kappa$ is defined in eqn (3b). Putting eqns (39) and (40) into eqns (56) and (57) and
carrying out the integration gives the relations

\[ G_1 = \frac{1}{2\mu} \{ K_i^2 A_I(C) + K_i^2 A_{II}(C) + K_i^2 A_{III}(C) \} \] (59a)

\[ G_2 = \frac{K_i K_{II}}{\mu} A_{IV}(C) \] (59b)

\[ J_1 - \frac{1}{2\mu} \{ K_i^2 F_I(C) + K_i^2 F_{II}(C) + K_i^2 F_{III}(C) \} \] (60a)

\[ J_2 = \frac{K_i K_{II}}{\mu} F_{IV}(C). \] (60b)

Further details of the above derivations are given in Appendix 1. Equation (59a) (computed directly from eqn (57)) coincides with the \( G vs K \) relation (computed from crack-closure energy eqn (41)) as given in eqns (42) and (43). The following relations may also be convenient:

\[ J_{1M} = G_{1M} E_{M}(C) : M = I, II, III \] (61)

\[ J_2 = G_2 E_{IV}(C) \] (62)

where

\[ E_{M}(C) = F_{M}(C)/A_{M}(C) (M = I, II, III \text{ and } IV). \] (63)

The functions \( A_{M}(M = I, II, III) \) are given by eqn (44). The other functions of crack speed are listed below:

\[ F_I(C) = \frac{\beta_1(1 - \beta_2^2)}{(D(C))^2} \left\{ 4\beta_1 - \frac{1}{\beta_1} (1 + \beta_2^2) - 4(\beta_1 - \beta_2) \frac{(1 + \beta_2^2)}{\sqrt{(1 + \beta_1)(1 + \beta_2)}} \right\} \] (64a)

\[ F_{II}(C) = \frac{\beta_2(1 - \beta_2^2)}{(D(C))^2} \left\{ 4\beta_2 - \frac{1}{\beta_2} (1 + \beta_2^2) - 4(\beta_2 - \beta_1) \frac{(1 + \beta_2^2)}{\sqrt{(1 + \beta_1)(1 + \beta_2)}} \right\} \] (64b)

\[ F_{III}(C) = \frac{1}{\beta_2} \] (64c)

\[ E_I(C) = \frac{1}{D(C)} \left\{ 4\beta_1 - \frac{1}{\beta_1} (1 + \beta_2^2) - 4(\beta_1 - \beta_2) \frac{(1 + \beta_2^2)}{\sqrt{(1 + \beta_1)(1 + \beta_2)}} \right\} \] (65a)

\[ E_{II}(C) = \frac{1}{D(C)} \left\{ 4\beta_2 - \frac{1}{\beta_2} (1 + \beta_2^2) - 4(\beta_2 - \beta_1) \frac{(1 + \beta_2^2)}{\sqrt{(1 + \beta_1)(1 + \beta_2)}} \right\} \] (65b)

\[ E_{III}(C) = \frac{1}{\beta_2} \] (65c)

\[ \Lambda_{IV}(C) = \frac{(\beta_1 - \beta_2)(1 - \beta_2^2)}{(D(C))^2} \left\{ \frac{[4\beta_1 \beta_2 + (1 + \beta_2^2)(2 + \beta_2 + \beta_1)]}{2\sqrt{(1 + \beta_1)(1 + \beta_2)}} - 2(1 + \beta_2^2) \right\} \] (66)

\[ F_{IV}(C) = \frac{4\beta_1 \beta_2 + (1 + \beta_2^2)(1 - \beta_2^2)(\beta_1^2 - \beta_2^2)}{2(D(C))^2 \sqrt{(1 + \beta_1)(1 + \beta_2)}} \] (67)

The variations of the functions \( F_{M}(C) (M = I, II, III) \) are shown in Figs. 6-8. The functions \( E_{M}(C) (M = I, II, III) \) are shown in Figs. 9, 10 and 4, respectively. Also the functions \( \Lambda_{IV}(C) \), \( F_{IV}(C) \), and \( E_{IV}(C) \) are shown in Figs. 11-13, respectively.
Path-independent integrals, energy release rates

Fig. 6. Crack speed function $F_i(C)$; plane stress.

$$J_{II} = \frac{K}{2\mu} F_i(C)$$
$$F_i(0) = \frac{\sigma + 1}{4}$$

Fig. 7. Crack speed function $F_\Pi(C)$; plane stress.

$$J_{I\Pi} = \frac{K}{2\mu} F_\Pi(C)$$
$$F_\Pi(0) = \frac{\sigma + 1}{4}$$
Fig. 8. Crack speed function $F_m(C)$.

Fig. 9. Crack speed function $E_t(C)$; plane stress.
Path-independent integrals, energy release rates

\[ J_{II} = G_{II} \cdot \mathcal{E}_d(c) \]

Fig. 10. Crack speed function \( E_d(c) \): plane stress.

Fig. 11. Crack speed function \( A_{fr}(c) \): plane stress.
Fig. 12. Crack speed function $F_{cr}(C)$; plane stress.

Fig. 13. Crack speed function $E_{cr}(C)$; plane stress.
Path-independent integrals, energy release rates

For a stationary crack \((C = 0)\), the above functions take the form:

\[
F_I(0) = F_{II}(0) = \frac{\kappa + 1}{4}, \quad F_{III}(0) = 1 \tag{68a}
\]

\[
E_I(0) = E_{II}(0) = E_{III}(0) = 1 \tag{68b}
\]

and

\[
A_{IV}(0) = F_{IV}(0) = \frac{\kappa + 1}{4}, \quad E_{IV}(0) = 1. \tag{68c}
\]

Substituting eqns (68a–b) into eqns (60a) and (60b) respectively, we obtain

\[
J_1 = G_1 = \frac{\kappa + 1}{8\mu} (K_I^2 + K_{II}^2) + K_{III}^2 \tag{71a}
\]

\[
J_2 = G_2 = -\frac{\kappa + 1}{4\mu} K_I K_{II}. \tag{71b}
\]

The above relations are the well known relations for stationary cracks in dynamic as well as static fields. The relation between \(J_k\) and \(K_I\), \(K_{II}\), and \(K_{III}\) are given in Appendix 2. Finally, it should be mentioned that the equivalence \(J_k' = G_k\) has also been verified numerically and these results will be reported in a companion brief note.

The path independent integrals developed by Atluri [3] have also been successfully used in the studies of non-steady creep crack growth [25–26].

5. CONCLUDING REMARKS

For all the three fracture modes, the general solutions (eigen solutions) of near-tip fields for dynamically propagating cracks have been established using the complex potential method. As verified in Refs. [10, 13], if we use the instantaneous crack velocity, the general solutions obtained in this study are valid in the vicinity of the crack tip for steady-state and transient crack propagation, and for propagation along straight and curved paths. The general solutions contain the zero stress and rigid body motion \((n = 0)\), the singular stress and corresponding displacement fields \((n = 1)\), the constant stresses and linear displacements \((n = 2)\), and the higher order terms \((n \geq 3)\).

The complete expressions for the relations between the energy release rates and stress intensity factors were derived using the singular stresses and the corresponding displacements for all the three fracture modes.

The path independent integrals \(J_k\) derived by Atluri [3] for dynamically propagating cracks can be very useful in dynamic fracture mechanics, for the numerical evaluation of the energy release rates and the stress intensity factors. In the present paper, the formulas relating to the path independent integrals \(J_k\) to the energy release rates on the one hand, and to the stress intensity factors on the other, were obtained through the singular stresses and the corresponding displacement field for all the three fracture modes. The application of the path independent integrals \(J_k\) [3] to the finite element analysis of dynamic crack propagation will be published elsewhere.

We have also introduced a new path-independent integral \(J_k'\) which is strictly equivalent to the energy release rate \(G_k\). Thus, as an alternative to using \(J_k\) one may directly use \(J_k'\). The essential difference is only one of numerical convenience.

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APPENDIX 1

The singular term of the order $O(r^{-1})$ in the kinetic energy density can be expressed by

$$T = \frac{1}{2} \rho C^2 \left( \frac{\partial u}{\partial x} \right)^2 as r \to 0.$$ 

(A1)

Thus, the integrations in the $J_i$ integrals and the strain energy release rates expressed by eqns (56) and (57) require the estimation of the derivatives $\partial u/\partial x$. One of the advantages of using complex potential method is that the estimation of the derivative can be easily performed using the Cauchy-Riemann relations. From eqn (2), we have

$$\frac{\partial u}{\partial x} = -\frac{1}{\mu} \text{Re} [\Phi + \Psi], \quad \frac{\partial u}{\partial y} = \frac{1}{\mu} \text{Im} [\beta_1 \Phi + \beta_2 \Psi]$$

$$\frac{\partial \tau}{\partial x} = \frac{1}{\mu} \text{Im} \left[ \beta_1 \Phi + \frac{1}{\beta_2} \Psi \right], \quad \frac{\partial \tau}{\partial y} = \frac{1}{\mu} \text{Re} [\beta_2 \Phi + \Psi]$$

$$\frac{\partial \omega}{\partial x} = -\frac{1}{\mu} \text{Re}[\Omega], \quad \frac{\partial \omega}{\partial y} = -\frac{1}{\mu} \text{Im}[\beta_2 \Omega].$$

(A2)

The complex potentials for singular stresses can be obtained from eqns (32)-(34):

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \left( -K_0 B_l + iK_0 B_0 \right) z^{1/2}$$

$$\Psi(z) = \frac{1}{\sqrt{2\pi}} \left( \frac{2 \beta_1 \beta_2 z}{1 + \beta_2^2} K_{0B_l} - i \left( 1 + \beta_2^2 \right) \frac{1}{2} K_{0B_0} \right) z^{1/2}$$

$$\Omega(z) = -\frac{1}{\sqrt{2\pi}} K_{0B_l B_0} z^{-1/2}. $$

(A3)
Using eqn (A3) in eqn (A2), the first derivatives are obtained as

\[
\frac{\delta u}{\delta x} = \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_1^{-1/2} \cos \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_1^{-1/2} \cos \theta_2 \right\}
\]

\[
+ \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_2^{-1/2} \sin \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_2^{-1/2} \sin \theta_2 \right\}
\]

\[
\frac{\delta v}{\delta y} = \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_1^{-1/2} \cos \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_1^{-1/2} \cos \theta_2 \right\}
\]

\[
+ \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_2^{-1/2} \sin \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_2^{-1/2} \sin \theta_2 \right\}
\]

\[
\frac{\delta w}{\delta x} = \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_1^{-1/2} \sin \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_1^{-1/2} \sin \theta_2 \right\}
\]

\[
+ \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_2^{-1/2} \sin \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_2^{-1/2} \sin \theta_2 \right\}
\]

\[
\frac{\delta w}{\delta y} = \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_1^{-1/2} \cos \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_1^{-1/2} \cos \theta_2 \right\}
\]

\[
+ \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_2^{-1/2} \cos \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_2^{-1/2} \cos \theta_2 \right\}
\]

\[
\frac{\delta w}{\delta z} = \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_1^{-1/2} \cos \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_1^{-1/2} \cos \theta_2 \right\}
\]

\[
+ \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_2^{-1/2} \cos \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_2^{-1/2} \cos \theta_2 \right\}
\]

\[
\frac{\delta w}{\delta x} = \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_1^{-1/2} \sin \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_1^{-1/2} \sin \theta_2 \right\}
\]

\[
+ \frac{K_B}{\mu \sqrt{2\pi}} \left\{ -r_2^{-1/2} \sin \theta_1 - \frac{2\beta_1 \beta_2}{2(1+\beta_1)} r_2^{-1/2} \sin \theta_2 \right\}
\]

From eqns (59), (40) and (A4), all necessary integrations in eqns (56) and (57) can be summarized as follows:

\[
R_j = \int_{-\sqrt{r}}^{\sqrt{r}} h_j(r, \theta) \cos \theta \, dr \, d\theta \quad (j = 1, 2, \ldots, 10)
\]

\[
S_j = \int_{-\sqrt{r}}^{\sqrt{r}} h_j(r, \theta) \sin \theta \, dr \, d\theta \quad (j = 1, 2, \ldots, 10)
\]

where \( r = \sqrt{x^2 + y^2} \), \( \sin \theta = \sin \tan^{-1}(y/x) \), and

\[
h_1 = \frac{1}{r_1} \cos \theta_1, \quad h_2 = \frac{1}{r_2} \cos \theta_2
\]

\[
h_3 = \frac{1}{r_1} \sin \theta_1, \quad h_4 = \frac{1}{r_2} \sin \theta_2
\]

\[
h_5 = \frac{1}{\sqrt{r_1}} \cos \theta_1, \quad h_6 = \frac{1}{\sqrt{r_2}} \cos \theta_2
\]

\[
h_7 = \frac{1}{\sqrt{r_1}} \sin \theta_1, \quad h_8 = \frac{1}{\sqrt{r_2}} \sin \theta_2
\]

\[
h_9 = \frac{1}{\sqrt{r_1}} \cos \theta_1, \quad h_{10} = \frac{1}{\sqrt{r_2}} \cos \theta_2
\]

\[
h_{11} = \frac{1}{\sqrt{r_1}} \sin \theta_1, \quad h_{12} = \frac{1}{\sqrt{r_2}} \sin \theta_2
\]

The resulting integration values are listed in Table 1.

**APPENDIX 2**

The relations between the path independent integrals \( I_i \) and the stress intensity factors are obtained as follows:

\[
I_i = \frac{K_B^2}{2\mu}, \quad \Phi_i = M = 1, II, III
\]
Table 1. Necessary integration values in the estimation of $J_\alpha$ integrals and energy release rates

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_j$</td>
<td>$H_1$</td>
<td>$H_2$</td>
<td>$-H_1$</td>
<td>$-H_2$</td>
<td>$H_{12}$</td>
<td>0</td>
<td>0</td>
<td>-$H_{12}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
| $s_j$ | 0  | 0  | 0  | 0  | 0  | 0  | $H_{12}$ | $H_{12}$ | 0  | $H_1$ | $H_2$

where $H_1 = \frac{\pi}{(1 + R_2)}$, $H_2 = \frac{\pi}{(1 + R_2)}$, $H_{12} = \frac{\pi}{\sqrt{(1 + R_1)(1 + R_2)}}$

\[ F_1(C) = \frac{\beta_1(1 - \beta_2^2)}{(D(C))^2} \left[ 2\beta_2(1 + \beta_2) - \frac{(1 + \beta_1)(1 + \beta_2^2) - 2(\beta_1 - \beta_2)}{(1 + \beta_1)(1 + \beta_2)} \right] \]

\[ F_{II}(C) = \frac{\beta_1(1 - \beta_2^2)}{(D(C))^2} \left[ 2\beta_2(1 + \beta_2) - \frac{(1 + \beta_1)(1 + \beta_2^2) - 2(\beta_1 - \beta_2)}{(1 + \beta_1)(1 + \beta_2)} \right] \]

\[ F_{IV}(C) = \frac{1 + \beta_2}{2\beta_2} \]

and

\[ J_2 = \frac{K_\alpha K_\mu}{\mu} F_{IV}(C) \]

\[ F_{IV}(C) = \frac{(\beta_1 - \beta_2)(1 - \beta_2^2)}{(D(C))^2} \left[ \frac{4\beta_1\beta_2 + (1 + \beta_2^2)(1 + \beta_1 + \beta_2)}{2(1 + \beta_1)(1 + \beta_2)} \right] \]

The path independent integrals $\hat{J}_\alpha$ also correlate to the energy release rates as follows:

\[ J_{IV} = G_{12} E_{IV}(C) : M = I, II, III \]

\[ E_1(C) = \frac{1}{D(C)} \left[ 2\beta_2(1 + \beta_2) - \frac{(1 + \beta_1)(1 + \beta_2^2) - 2(\beta_1 - \beta_2)}{(1 + \beta_1)(1 + \beta_2)} \right] \]

\[ E_{II}(C) = \frac{1}{D(C)} \left[ 2\beta_2(1 + \beta_2) - \frac{(1 + \beta_1)(1 + \beta_2^2) - 2(\beta_1 - \beta_2)}{(1 + \beta_1)(1 + \beta_2)} \right] \]

\[ E_{IV}(C) = \frac{1 + \beta_2}{2\beta_2} \]

and

\[ J_2 = G_2 E_{IV}(C) \]

\[ E_{IV}(C) = \frac{4\beta_1\beta_2 + (1 + \beta_2^2)(1 + \beta_1 + \beta_2) - 2(1 + \beta_2^2)(1 + \beta_1)(1 + \beta_2)}{4(\beta_1\beta_2 + (1 + \beta_2^2)(2 + \beta_1 + \beta_2) - 4(1 + \beta_2^2)(1 + \beta_1)(1 + \beta_2)} \]

The following relation can also be found:

\[ F_{IV}(C) = E_{IV}(C) \cdot A_{IV}(C) : M = I, II, III, and IV. \]