A FULL TANGENT STIFFNESS FIELD-BOUNDARY-ELEMENT FORMULATION FOR GEOMETRIC AND MATERIAL NON-LINEAR PROBLEMS OF SOLID MECHANICS

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SUMMARY

The field-boundary-element method naturally admits the solution algorithm in the incompressible regimes of fully developed plastic flow. This is not the case with the generally popular finite-element method, without further modifications to the method such as reduced integration or a mixed method for treating the dilatational deformation. The analyses by the field-boundary-element method for geometric and material non-linear problems are generally carried out by an incremental algorithm, where the velocities (or displacement increments) on the boundary are treated as the primary variables and an initial strain iteration method is commonly used to obtain the state of equilibrium. For problems such as buckling and diffused tensile necking, involving very large strains, such a solution scheme may not be able to capture the bifurcation phenomena, or the convergence will be unacceptably slow when the post-bifurcation behaviour needs to be analysed. To avoid this predicament, a full tangent stiffness field-boundary-element formulation which takes the initial stress-velocity gradient (displacement gradient) coupling terms accurately into account is presented in this paper. Here, the velocity field both inside and on the boundary are treated as primary variables. The large strain plasticity constitutive equation employed is based on an endochronic model of combined isotropic/kinematic hardening plasticity using the concepts of material director triad and the associated plastic spin. A generalized mid-point radial return algorithm is presented for determining the objective increments of stress from the computed velocity gradients. Numerical results are presented for problems of diffuse necking, involving very large strains and plastic instability, in initially perfect elastic-plastic plates under tension. These results demonstrate the clear superiority of the full tangent stiffness algorithm over the initial strain algorithm, in the context of the integral equation formulations for large strain plasticity.

INTRODUCTION

The boundary-element method (BEM) is based upon classical integral equation formulations of boundary-value/initial-value problems. Although such formulations were originally thought to be primarily of theoretical interest, the early engineering applications of this method were in linear elastostatics and potential problems. Later this method was applied to a wide range of time dependent and vibration problems, and those involving fluid flow and material non-linearities. Recently, this method has been extended to solve material as well as geometrical non-linear problems.

In the boundary-element method, the trial and the test function spaces are quite different from each other. The test functions correspond to fundamental solutions, in infinite space, of the differential operator. If the fundamental solution can be derived for the entire differential operator of the problem, the integral representations involve only boundary integrals. Thus, the boundary-
element method is obtained. However, in geometric and material non-linear problems of solid mechanics, if the fundamental solutions to only the highest-order linear differential operators of the problem are used as test functions, the integral representations involve not only boundary integrals but also domain integrals. Discretizations of these integral equations lead to the so-called *field-boundary-element method*.

The analysis of the field-boundary-element method for geometric and material non-linear problems in solid mechanics is generally carried out by an incremental algorithm, where the solution methodology employed is first to obtain velocities (or displacement increments) through an integral relationship. The velocity gradients on the boundary are then obtained through an integral relationship or a boundary stress-strain rate (see Okada et al.\textsuperscript{15,16}) algorithm. Once the boundary variables are completely determined, the velocities and velocity gradients at the interior are evaluated by taking the source point in their respective integral equations to the desired interior location. Such an initial strain iteration algorithm is, in general, insensitive to the initial stress-velocity gradient coupling terms (in the domain integrals, which play a dominant role in problems which involve bifurcation phenomena (buckling, diffused necking, etc.) in its solution path. Hence, the convergence may be unacceptably slow or the bifurcation phenomena may be completely ignored when initial strain type iteration methods were to be employed to solve such problems.

The initial stress-velocity gradient coupling terms (i.e. $\tau_{ij}v_{i,j}$ terms) need to be properly accounted for in the calculation of velocities (or displacement increments) for problems which involve bifurcation phenomena. For this purpose, a full tangent stiffness field-boundary-element formulation is presented in this paper. Unlike in the initial strain approach, in the full tangent stiffness approach, the velocity field in the entire domain of the body is assumed as a primary variable; and the velocity gradients are expressed through the differentiated shape functions and nodal velocities in the domain elements. Hence a system of equations involving all non-linear effects in terms of velocities in the form of a tangent stiffness matrix is obtained.

In this paper, a tangent stiffness field-boundary-element formulation is presented for elasto-plastic solids undergoing large strains. A generalized mid-point radial return algorithm is presented for determining the objective increments of stress from the computed velocity gradients. Moreover, a mid-point evaluation of the generalized Jaumann integral is used to determine the material increments of stress. The constitutive equation employed (the one chosen is for illustrative purposes only) for the analysis is based on an endochronic model of combined isotropic/kinematic hardening finite plasticity using the concepts of a material director triad and the associated plastic spin. The problem of diffused necking instability of a plate subjected to uniform tension is analysed using the initial strain and the full tangent stiffness field-boundary-element algorithms. Both *initially perfect* plates, as well as those containing initial imperfections, are analysed. The superiority of the full tangent stiffness algorithm, in capturing the bifurcation phenomena is demonstrated when the plastic instabilities are analysed. It should also be noted that this is the first numerical solution for such a class of problems, using the *field-boundary-element method*.

**KINEMATICS**

We adopt a rate form of updated Lagrangian formulation. Such formulations can be seen in References 19–21. For simplicity, a Cartesian co-ordinate system is employed here. Let $x_i$ be the current spatial co-ordinates of a material particle, and $\dot{s}_{ij}$ be the Truesdell stress rate referred to the current configuration. The linear and momentum balance laws are given in the rate form, as follows,
BE FORMULATION FOR NON-LINEAR PROBLEMS

\[
\left( \dot{s}_{ij} + \tau_{ik} \frac{\partial v_i}{\partial x_k} \right) + b_j = 0
\]  
\[
\dot{s}_{ij} = \dot{\varepsilon}_{ij}
\]

where \(v_j\) is the velocity of the material particle, \(b_j\) is the rate of body force per unit volume and \(\dot{\varepsilon}( )_i\) denotes \(\partial( )/\partial x_i\). The Kirchhoff stress tensor \(\tau_{ij}\) is defined by

\[
\tau_{ij} = J \sigma_{ij}
\]

where \(J\) is the Jacobian and \(\sigma_{ij}\) is the Cauchy stress tensor.

The stretching (velocity-strain) tensor \(D_{ij}\) and spin tensor \(W_{ij}\) are expressed by the following equations (in terms of velocity gradients \(v_i,j\)) as

\[
D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})
\]

\[
W_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})
\]

Using the Jaumann rate of Kirchhoff stress \(\dot{\tau}_{ij}\), the Truesdell stress rate \(\dot{s}_{ij}\) is expressed as

\[
\dot{s}_{ij} = \dot{\tau}_{ij} - D_{ik} \tau_{kj} - \tau_{ik} D_{kj}
\]

Here, \(\dot{\tau}_{ij}\) is defined as

\[
\dot{\tau}_{ij} = \dot{\varepsilon}_{ij} + \tau_{ik} W_{kj} - W_{ik} \tau_{kj}
\]

Substituting the expression of equation (6) into equation (1), the linear momentum balance law is given by the following formula:

\[
\left\{ \frac{\tau_{ij} - \tau_{ik} W_{kj} + W_{ik} \tau_{kj} - \frac{\partial v_i}{\partial x_k} \tau_{kj}}{\partial x_k} \right\} + b_j = 0
\]

The angular momentum balance law (2) is satisfied identically provided \(\dot{\tau}_{ij}\) is defined to be symmetric through an objective constitutive law.\(^{19,20}\)

ELASTIC-PLASTIC CONSTITUTIVE EQUATION

Here, we consider a general type of elastic-plastic constitutive model, which includes the isotropic, the kinematic and the combined isotropic/kinematic hardening behaviour of the solid at large strains. It has been pointed out by several authors (see, for instance, References 22-24) that in a kinematic hardening large strain plasticity model, if the evolution equations for the Jaumann rates of the Kirchhoff stress and of the back stress, respectively, are simply taken to be linear functions of the plastic component of the velocity-strain, certain anomalous consequences, such as an oscillatory stress response of the material in finite simple shear, may result. More general evolution equations, especially to account for the non-coaxiality of the Cauchy stress, and the Cauchy-like back stress, in shear and non-proportional loadings, have been attempted\(^{23,24}\) to suppress the above physically unacceptable oscillatory stress responses. Although these methods based on formal continuum mechanics were quite successful for the simple shear case, the physics and micromechanics of finite plastic flow indicates that a more consistent large strain elastoplastic constitutive law should involve an evolution equation for the plastic component of the spin tensor. Such an elastic-plastic constitutive model is developed, for instance, in Im and Atluri,\(^{25,26}\) which is the finite strain version of the endochronic constitutive model of Watanabe and Atluri.\(^{27,28}\)

\(^{1}\) Some experimental work is under way at Georgia Tech to confirm the validity of these theories.
Here, the concept of a material director triad is introduced and the relaxed intermediate configuration is chosen to be isoclinic. The plastic spin tensor is defined through internal time. Such an endochronic constitutive model (for large strain elasto-plasticity) employed here, can be summarized as follows.

Let \( N_{ij} \) be the normal to the yield surface in the deviatoric Kirchhoff stress space. When the stress is on the yield surface and \( N_{ij}D_{ij} \geq 0 \), the process is a plastic process.

\[
N_{ij} = (\tau'_{ij} - r_{ij})/\|\tau_{kl} - r_{kl}\| \tag{9}
\]

\[
\zeta = D_{ij}N_{ij}/C \tag{10}
\]

\[
D_{ij}^p = N_{ij}^\zeta \tag{11}
\]

\[
W_{ij}^p = \Omega_{ij}^\zeta \tag{12}
\]

\[
\Omega_{ij} = \left\{ \frac{m_1(r_{ik}\tau'_{kj} - \tau_{ik}r_{kj})}{\tau_{ij}^2 f^2(\zeta)} + \frac{m_2(r_{ik}r_{kl}\tau'_{ij} - \tau_{ik}r_{ij}r_{kl})}{\tau_{ij}^2 f^3(\zeta)} 
+ \frac{m_3(r_{ik}\tau'_{kj} - \tau_{ik}\tau_{ki}r_{ij})}{\tau_{ij}^2 f^2(\zeta)} \right\} \tag{13}
\]

\[
\zeta_{ij} = \lambda(D_{kk})\delta_{ij} + 2\mu(D_{ij} - D_{ij}^p) - W_{ik}^p\tau_{kj} + \tau_{ik}W_{ij}^p \tag{14}
\]

\[
\tau_{ij}^p = 2\mu\rho_1 D_{ij}^p \frac{\alpha r_{ij}(D_{ij}^pD_{ij}^p)^{1/2}}{f(\zeta)} - W_{ik}^p\tau_{kj} + \tau_{ik}W_{ij}^p \tag{15}
\]

where \( r_{ij} \) is the back stress, \( \tau_y \) is the effective yield stress in shear and \( \tau_{ij}' \) is the deviatoric part of Kirchhoff stress \( \tau_{ij} \). \( f(\zeta) \) and \( r_{ij} \) represent the expansion and translation of the von Mises type yield surface. \( D_{ij}^p \) and \( W_{ij}^p \) are the rate of plastic strain and the plastic spin, respectively. \( \zeta \) represents the internal time variable. It is seen that \( \Omega_{ij} \) accounts for the non-coaxiality of the tensors \( \tau_{ij} \) and \( r_{ij} \).

The reader is referred to References 25 and 26 for further details of the constitutive model.

### FIELD-BOUNDARY-ELEMENT EQUATION FORMULATION

A boundary-interior integral equation for large strain elastoplasticity is presented in this section. Such an integral equation is seen in Okada et al.\(^{15,16}\) Here, the weighted residual method is used to formulate the integral equation. Let \( v_i \) be the trial function for velocity, and \( \tilde{v}_i \) be the test function. The weak form of equation (8) (linear momentum balance) is written as

\[
\int_{\Omega} \left\{ \left[ \tau_{ij} - \tau_{ik}W_{kj} + W_{ik}\tau_{kj} - \frac{\partial v_i}{\partial x_k}\tau_{kj} \right]_{ij} + b_j \right\} \tilde{v}_j \, d\Omega = 0 \tag{17}
\]

where \( \Omega \) represents the domain of the body considered here.

Substituting the elastic-plastic constitutive equations discussed in the previous section into equation (17) and integrating twice by parts, we obtain the following expression:

\[
-\int_{\Omega} (E_{ijkl}\tilde{v}_j,i)dv_k \, d\Omega = \int_{\Omega} \{ \tilde{i}_j\tilde{v}_j - n_i(E_{ijkl}\tilde{v}_j,i)u_k \} \, d(\partial \Omega) + \int_{\Omega} b_j\tilde{v}_j \, d\Omega \\
+ \int_{\Omega} \{ E_{ijkl}D_{ij}^p + \tau_{ik}(W_{kj} - W_{kj}^p) - (W_{ik} - W_{ik}^p)\tau_{kj} + v_i,k\tau_{kj} \} \tilde{v}_j,i \, d\Omega \tag{18}
\]
where $E_{ijkl}$ represents the (presently considered isotropic) elasticity tensor, and $t_j$ is the rate of the surface traction vector per unit area in the current configuration $[t_j = n_i(s_{ij} + \sigma_{ik}v_{j,k})]$.

Now, the test function $\bar{v}_i$ is chosen to be the solution in infinite space (Kelvin's solution) of the equation

$$\left(E_{ijkl}\bar{v}_{k,l}\right)i + \delta(x_m - \xi_m)\delta_{jp}e_p = 0 \quad (19)$$

Here, $e_p$ is the unit load vector along the $x_p$ axis at the point $\xi_m$, and $\delta(x_m - \xi_m)$ is the Dirac delta function. The test function $\bar{v}_i$, therefore, turns out to be

$$\bar{v}_i = v^*_j(x_m, \xi_m)e_p$$
$$n_iE_{ijkl}\bar{v}_{k,i} = n_iE_{ijkl}v^*_j(x_m, \xi_m)e_p$$
$$= t^*_j(x_m, \xi_m)e_p \quad (20)$$

Here, $v^*_j$ and $t^*_j$ are called the displacement and traction kernels respectively. Substituting equation (20) into equation (18), we obtain the following integral representation for $v_q$ as,

$$C_{pq}v_q(\xi_m) = \int_{\partial\Omega} \left\{ t_jv^*_j(x_m, \xi_m) - t^*_j(x_m, \xi_m)v_j \right\} d(\partial\Omega)$$
$$+ \int_{\Omega} b_jv^*_j(x_m, \xi_m) d\Omega$$
$$+ \int_{\Omega} \left\{ E_{ijkl}D^p_{kl} + \tau_{ik}(W_{kj} - W^p_{kj}) \right\} d\Omega$$
$$- (W_{ik} - W^p_{ik})\tau_{kj} + v_{i,k}\tau_{kj} \} v^*_j(x_m, \xi_m) d\Omega \quad (21)$$

Here, when $\xi_m$ is a smooth boundary point ($\xi_m \in \text{smooth } \partial\Omega$), $C_{pq} = \frac{1}{2}\delta_{pq}$ and when $\xi_m$ is an interior point ($\xi_m \in \Omega$), $C_{pq} = \delta_{pq}$.

It is interesting to briefly examine the nature of the integral equation (21). The first term on the right-hand side of equation (21), involving only a boundary integral, corresponds to the linear elastic behaviour of the material in its current state. The second term, involving a domain integral, accounts for the body forces, but still corresponds to linear elastic behaviour in the current state. The third term on the right-hand side of (21), involving a domain integral, contains the crux of the finite strain elastic-plastic problem: (i) the term $E_{ijkl}D^p_{kl}$ accounts for the effects of plastic strain rate, (ii) the terms such as $\tau_{ik}W^p_{kj}$, etc., account for the effects of plastic spin, and (iii) the term $v_{i,k}\tau_{kj}$ accounts for the coupling for initial stresses in the current configuration with the additional deformation from the current configuration (this term plays a dominant role in problems of buckling, instability, etc.). Only the first term on the right-hand side is used in any discretization process, and the domain integral terms are used only in an iterative process, one is led to the so-called 'initial strain' iteration approach, based on the so-called 'linear elastic stiffness matrix'. Such an iterative technique is shown in the present paper to fail in the case of plastic instability problems. If all the terms on the right-hand side of equation (18) are simultaneously used in a discretization process, one is led to the so-called 'full tangent stiffness' method. It is also seen that this tangent stiffness method involves trial functions $v_l$ not only at the boundary $\partial\Omega$ but also in the domain $\Omega$. These concepts of 'initial strain' and 'tangent stiffness' field-boundary-element methods were discussed in Reference 29.
The $v_{ip}^*$, $t_{ip}^*$ and $v_{ip, i}$ kernels for the case 2D plane strain are defined as follows:

$$
\begin{align*}
    y_i &= x_i - \xi_i, \\
    r &= \sqrt{(x_i - \xi_x)(x_i - \xi_x)} \\
    v_{ip}^*(x_m, \xi_m) &= \frac{1}{8\pi\mu(1-\nu)} \left\{ (3-4\nu) \ln \left( \frac{1}{r} \right) \delta_{ip} + \frac{y_i y_p}{r^2} \right\} \\
    t_{ip}^*(x_m, \xi_m) &= -\frac{1}{4\pi(1-\nu)r} \left[ n_k y_k \left\{ (1-2\nu) \delta_{ip} + 2 \frac{y_i y_p}{r^2} \right\} \\
                        &\quad - (1-2\nu) \left( \frac{y_p}{r} n_i - \frac{y_i}{r} n_p \right) \right] \\
    v_{ip, i}(x_m, \xi_m) &= \frac{1}{8\pi\mu(1-\nu)r} \left\{ - (3-4\nu) \frac{y_i}{r} \delta_{ip} \left( \delta_{ip} + \frac{y_p}{r} \right) - 2 \frac{y_i y_p}{r^3} \right\}
\end{align*}
$$

(22)

Here, $n_k$ are the components of the unit outward normal to the boundary at $x_m$, and $\mu$ and $\nu$ are the shear modulus and the Poisson's ratio respectively.

It is seen that the test functions $v_{ip}^*$ and $t_{ip}^*$ remain unchanged during the entire elastic-plastic deformation path, and correspond to the initial elastic state of the solid. Further, $v_{ip}^*$ remains symmetric under $i \leftrightarrow p$ interchange, while $t_{ip}^*$ has a skew-symmetric component under $i \leftrightarrow p$ interchange. It is seen that if the material is incompressible even in its initial elastic state ($\nu = \frac{1}{2}$), the kernel $t_{ip}^*$ in fact becomes simpler, in as much as its skew-symmetric part tends to zero. Also, as plastic flow fully develops, the rate of deformation is nearly incompressible, i.e. $v_i$ satisfy the condition that $v_{i,i} = 0$. From equation (21) it can be examined that this potential constraint condition on $v_i$ does not pose any 'locking' problems. This is not the case in symmetric Galerkin type finite-element formulations, wherein, in a pure displacement approach for finite strain plasticity, the appropriate functionals (and weak forms) involve terms of the type $(\lambda \delta_{ik} v_k v_i)$ where $\lambda$ is the bulk modulus.\(^{19,20}\)

As seen from References 19 and 20, the functional in governing the rate problem of finite strain plasticity, from which the finite-element equations are derived, would involve terms of the type

$$
E_{ijkl}^* v_{k,i} v_{i,j} + \frac{1}{2} \delta_{ik} v_{k,k} - \tau_{ij} v_{i,k} v_{j,k}
$$

where $E_{ijkl}^*$ are the 'tangent' elastic-plastic moduli. In a typical material stress–strain relation at large plastic strains, especially when the stress level reaches a saturation at large values of strain, the magnitude of $E_{ijkl}^*$ is several orders smaller than the modulus of $\lambda$. Also, the magnitude of $\tau_{ij}$ is much smaller than that of $\lambda$, but not as small as the magnitude of $E_{ijkl}^*$. Thus, in a finite-element formulation (when the trial functions $v_i$ do not a priori satisfy the condition $v_{i,i} = 0$), the term $\lambda \delta_{ik} v_{k,k}$ totally dominates the other terms, if 'full' quadrature is used to integrate this term, leading to the so-called 'locking'. To avoid this, either reduced integration of $\lambda \delta_{ik} v_{k,k}$ term, or a mixed-variational treatment of this term\(^{19,20}\) is used. On the other hand, from equation (21) it is seen that the term $\lambda \delta_{ik} v_{k,k}$ does not appear in the present integral relation. However, for a typical material, the magnitude of the elastic moduli, $E_{ijkl}$ in equation (21), is expected to be much larger than that of $\tau_{ij}$ (about 2 orders of magnitude). However, this difference is not as severe as that between $E_{ijkl}^*$ and $\lambda$, as in the finite-element formulation. This is the primary reason as to why 'locking' does not occur in the field-boundary-element method. However, the term $\tau_{ij} v_{i,k}$ plays a central role in equation (21) in altering stiffness of the structure at bifurcation.
NUMERICAL IMPLEMENTATION OF A FULL TANGENT STIFFNESS FIELD-
BOUNDARY-ELEMENT METHOD

In the ‘initial strain’ type field-boundary-element method, the velocities and boundary traction
rates are solved first through an integral equation for velocities on the boundary (i.e. by
discretizing the first term on the right-hand side of equation (21) only, assuming that the body
forces are zero). Then the velocity gradients on the boundary are solved either by the boundary
stress-strain algorithm or by an integral representation (see Okada \textit{et al.}\textsuperscript{16}). The velocity and
velocity gradients in the interior are solved next by integral representations. Then, the rate of
plastic strain and other material parameters are determined. An iterative initial strain type
algorithm is carried out till all field parameters converge. Such an algorithm is not very sensitive to
the initial stress-velocity gradient coupling (i.e. the $\tau_{ik}v_{i,k}$ term) and will not be suitable for
problems involving bifurcation phenomena.

To avoid such a predicament, in the ‘tangent stiffness’ field-boundary-element method, we
assume the velocity field as a primary variable not only on the boundary but also in the interior of
the body. The velocity gradients can be expressed by differentiating the shape functions associated
with the nodal velocities in the interior elements. Thus, all material and geometric non-linear
effects are directly accounted for in the field-boundary integral equation for velocities throughout
the domain. Furthermore, a tangent stiffness matrix can be obtained with known or unknown
surface traction rates, as well as velocities throughout the domain. (See Reference 30 for an
analogous method in small strain plasticity.)

In the following sections, numerical implementations of the integral equations for the present full
tangent stiffness field-boundary-element method are discussed for a 2D plane strain case.

\textit{Boundary integrals}

Three noded quadratic isoparametric boundary elements are used to discretize the surface of the
body. Let $N_{Y}^{P}$ ($I = 1, 2, 3$) be a shape function associated with the $I$th nodal point in a boundary
element. Let $v_{i}^{f}$ and $t_{p}^{f}$ ($I = 1, 2, 3$) be a nodal velocity and traction vector at the $I$th nodal point in a
boundary element. Thus, the boundary integral in equation (21) is given by the following
discretization:

$$
\int_{\partial \Omega} (v_{i}^{f}t_{j}^{f} - t_{j}^{f}v_{i}^{f}) d(\partial \Omega)
$$

$$
= \sum_{IB=1}^{NBE} \left\{ \sum_{I=1}^{3} \left\{ i_{j}^{f} \int_{\partial \Omega_{e}} N_{Y}^{P} v_{j}^{f} d(\partial \Omega_{e}) - v_{j}^{f} \int_{\partial \Omega_{e}} N_{Y}^{P} t_{j}^{f} d(\partial \Omega_{e}) \right\} \right\} \tag{23}
$$

where $NBE$ is the total number of boundary elements.

\textit{Domain integrals}

Eight noded isoparametric elements are used here. Stress components, plastic strain and plastic
spin evolution equations are evaluated at four points (sampling points) in each domain element. (See Figure 1.)

Thus, the stress components, plastic strain and plastic spin evolution equations are interpolated
to any other location from their sampling points, by using the Lagrange interpolation functions. Let $\tilde{N}^{I}$ ($I = 1, 2, 3, 4$) be the Lagrange interpolation function associated with the $I$th stress point in
where \( C_{ijkl}(\xi, \eta) \) represents the tensor in the plastic strain evolution equation

\[
\tau_{ij}(\xi, \eta) = \sum_{l=1}^{4} \tilde{N}^l(\xi, \eta) \tau_{ij}(\xi_l, \eta_l)
\]

(24)

Also, by using the shape functions \( N_I (I = 1, 2, \ldots, 8) \) of eight noded isoparametric elements, the velocity and the velocity gradients can be expressed as follows:

\[
v_I(\xi, \eta) = \sum_{I=1}^{8} N_I^P(\xi, \eta) v_I(\xi_I, \eta_I)
\]

\[
\frac{\partial v_I(\xi, \eta)}{\partial x_j} = \sum_{I=1}^{8} \left( \frac{\partial \xi}{\partial x_j} \frac{\partial N_I^P}{\partial \xi} + \frac{\partial \eta}{\partial x_j} \frac{\partial N_I^P}{\partial \eta} \right) v_I(\xi_I, \eta_I)
\]

(25)

The domain integral in equation (21) is therefore expressed by the following expressions (in the absence of body force):

\[
D.I. = \left\{ \int_{\Omega} \left[ E_{ijkl} D_{kl}^P + \tau_{ij}(W_{kj} - W_{kj}^p) - (W_{ik} - W_{ik}^p) \tau_{kj} + v_{i,k} \tau_{kj} \right] \Gamma_{jp,k} v_{jp,i} \, d\Omega \right\}
\]

\[
= \int_{\Omega} A_{ijkl} v_{k,i} \Gamma_{jp,i} \, d\Omega
\]

\[
= \sum_{I=1}^{NDE} \sum_{J=1}^{4} A_{ijkl}(\xi_J, \eta_J) v_k(\xi_I, \eta_I)
\times \int_{\Omega_e} \tilde{N}_J(\xi, \eta) \left( \frac{\partial \xi}{\partial x_i} \frac{\partial N_I^P}{\partial \xi} + \frac{\partial \eta}{\partial x_i} \frac{\partial N_I^P}{\partial \eta} \right) v_{jp,i} \, d\Omega_e
\]

\[
= \sum_{I=1}^{NDE} \left\{ \sum_{p=1}^{8} K_{kp} v_k(\xi_I, \eta_I) \right\}
\]

(26)
where $A_{ijkl}$ (in the absence of plastic spin, for simplicity) is written as

$$A_{ijkl} = E_{ijmn} \left( \frac{1}{C} N_{mn} N_{kl} \right) + \frac{1}{2} \tau_{im} \delta_{jm} (\delta_{mk} \delta_{nl} - \delta_{nk} \delta_{ml})$$

$$- \frac{1}{2} \delta_{in} (\delta_{mk} \delta_{nl} - \delta_{nk} \delta_{ml}) \tau_{mj} + \delta_{ik} \tau_{lj}$$

(27)

where NDE is the total number of domain elements. $K^l_{ip}$ is termed the domain element stiffness matrix for the source point $\xi_C$. It is important to note that, unless the geometry is updated, we need not recompute domain integrals, because the initial stress and the plastic strain evolution equation are given pointwise, outside of the integral (equation (26)).

Matrix formulation

A system of linear equations can be obtained through the discretized equations (equations (23) and (26)) given in the previous section. Using the process of collocation, and by taking each nodal point to be a source point $(\xi_m)$, a sufficient number of linear equations is generated. These are shown in matrix form as

$$[T^\bullet] \{v\} = [U^\bullet] \{t\}$$

(28)

Let $N$ be the total number of nodal points (both on the boundary and in the interior), and let $M$ be the total number of boundary nodal points only. Then $\{v\}$ is the vector of nodal velocities $(N \times 1)$, and $\{t\}$ is the vector of nodal tractions $(M \times 1)$. $[T^\bullet]$ and $[U^\bullet]$ are the coefficient matrices, which depend on the initial stresses, etc. (see equations (23) and (26)).

To solve for unknown nodal values, known and unknown variables are rearranged appropriately such that one is led to a system of equations

$$[K] \{x\} = \{y\}$$

(29)

where $[K]$ is the tangent stiffness matrix, and $\{x\}$ and $\{y\}$ are the unknown and the load vector respectively. The unknown vector is obtained by

$$\{x\} = [K]^{-1} \{y\}$$

(30)

Numerical quadrature scheme

Since isoparametric elements are considered, closed form evaluation of the integrals in equation (21) is not possible. Hence, numerical quadrature schemes which accurately evaluate both boundary and domain integrals of equation (21) are presented below.

As for the boundary integrals, the 10 Gauss point quadrature formula is employed for all non-singular cases, and the logarithmic weighted 7 Gauss point formula is used for logarithmic singular cases (see Stroud). The $1/r$ singular integral associated with the Cauchy principal value is evaluated by the use of the rigid body modes.

The kernel functions $v^f_{ip}$ that are present in the domain integral have the structure $(1/r)f(\cos \theta, \sin \theta)$. Here, $r$ is the distance between a field point $x_m$ and a source point $\xi_m$, and $\theta$ is the angle made by the line joining $x_m$ and $\xi_m$ with the $x_1$ axis $(\theta = \tan^{-1} [(x_2 - \xi_2)/(x_1 - \xi_1)])$. When the field point falls within a set of field elements immediately surrounding the source point, the $(1/r)$ singularity in the integrand is cancelled with the $(r dr d\theta)$ term of the Jacobian and mapping the elements appropriately. On the other hand, the $f(\cos \theta, \sin \theta)$ part is considered to have a steeper variation

*Here on we consider only the case of isotropic hardening, for simplicity, and without loss of any generality.
than the $1/r$ part, when the distance between a source point and a field element is small compared with the element size. Thus, for this group of field elements, we choose a numerical quadrature based on the maximum angular variation in a domain element, which is determined as shown in Figure 2.

The numerical quadrature schemes employed for the domain integral are shown in Table I, with respect to the maximum angular variation in an element. Also, it is noted that the non-product formulas (Stroud [32]) as well as the standard product Gauss formulas are employed for the domain integral.

### Table I. The quadrature rules employed for the domain integral

<table>
<thead>
<tr>
<th>Quadrature rule</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.9 \leq \theta_{\text{max}}$</td>
<td>$10 \times 10$ Gauss</td>
</tr>
<tr>
<td>$0.6 \leq \theta_{\text{max}} &lt; 0.9$</td>
<td>$8 \times 8$ Gauss</td>
</tr>
<tr>
<td>$0.42 \leq \theta_{\text{max}} &lt; 0.6$</td>
<td>$6 \times 6$ Gauss</td>
</tr>
<tr>
<td>$0.31 \leq \theta_{\text{max}} &lt; 0.42$</td>
<td>$5 \times 5$ Gauss</td>
</tr>
<tr>
<td>$0.21 \leq \theta_{\text{max}} &lt; 0.31$</td>
<td>Non-product 20 points</td>
</tr>
<tr>
<td>$0.12 \leq \theta_{\text{max}} &lt; 0.21$</td>
<td>Non-product 12 points</td>
</tr>
<tr>
<td>$\theta_{\text{max}} &lt; 0.12$</td>
<td>$3 \times 3$ Gauss</td>
</tr>
</tbody>
</table>

$\theta_{\text{max}}$: The maximum angular variation in a domain element

### SOLUTION ALGORITHM

A non-iterative full tangent stiffness method is employed in the analysis of finite strain elastoplasticity, with the mid-point radial return algorithm and an objective stress integration scheme. The mid-point radial return algorithm and the objective stress integration scheme, introduced in this section, are quite effective for the non-iterative finite strain elastic-plastic calculation. These algorithms can prevent unreasonably small time increments, generally required in non-iterative elastic–plastic calculations. Moreover, they can also give a more accurate solution to the analysis. The details of both algorithms are described in Im and Atluri, Atluri, Rubinstein and Atluri and Reed and Atluri. The computational algorithms are briefly described here.

The mid-point radial return algorithm for determining $\dot{t}_{ij}$ (Jaumann rate of Kirchhoff stress) from $D_{ij}$ (rate of strain) is summarized as follows.

1. Compute $D_{ij}$.
2. Check if

\[
\{\dot{t}_{ij} + 2\mu D_{ij}\Delta t)(\dot{t}_{ij} + 2\mu D_{ij}\Delta t)\} \geq R_N^2
\]
where $R_N$ is the current radius of the yield cylinder, and $\Delta t$ is the time increment. If $R \geq R_N$, then the process is plastic. Go to step 3.

3. Define a \textit{generalized mid-point normal} to the yield solution

$$N_{\theta ij} = \frac{(\tau'_{ij} + 2\mu \theta D_{ij} \Delta t) - r_{ij}}{\| (\tau'_{ij} + 2\mu \theta D_{ij} \Delta t) - r_{ij} \|}$$  \hspace{1cm} (32)

where $0 \leq \theta \leq 1$ and $\theta \Delta t$ shows a time point in the current time step $(t_N \rightarrow t_N + \Delta t)$.

4. Define $\xi = (1/C_{\theta})(N_{\theta ij} D_{ij})$.

5 Compute the Jaumann stress rate

$$\tau_{ij}^1 = 2\mu \left[ D_{ij} - \frac{1}{C_{\theta}} N_{\theta mn} D_{mn} N_{\theta i} \right]$$

and

$$\tau_{kk}^1 = (2\mu + 3\lambda) D_{kk}$$

6. Compute the rate of back stress

$$r_{ij}^1 = 2\mu \frac{\rho_i}{C_{\theta}} (N_{\theta kl} D_{kl}) N_{\theta ij} - \alpha r_{ij} \frac{N_{\theta kl} D_{kl}}{C_{\theta}}$$

$$- (\Omega_{m} r_{kj} - r_{ik} \Omega_{kj}) \frac{N_{\theta mn} D_{mn}}{C_{\theta}}$$

The objective stress integration scheme to determine the material stress increment from the Jaumann stress rate evaluated through the mid-point radial return algorithm is summarized as follows.

Let $Q_{ij}(t)$ be the rotation of the material particle with respect to the reference time $t_N$, which is the beginning of the current time step. The Kirchhoff stress $\tau_{ij}$ at the time $t_N + \Delta t$ can be given by the objective stress integral,

$$\tau_{ij}(t_N + \Delta t) = J^{-1}(t_N + \Delta t) Q_{ki}(t_N + \Delta t) \tau_{kl}(t_N) Q_{ij}(t_N + \Delta t)$$

$$+ \int_{t}^{t+\Delta t} J^{-1}(\xi) \cdot Q_{ki}(t + \Delta t) Q_{km}(\xi) \delta_{mn}(\xi)$$

$$\times Q_{ln}(\xi) Q_{ij}(t + \Delta t) d\xi$$  \hspace{1cm} (35)

where $\delta_{mn}$ is the Jaumann rate of Cauchy stress. The above expression is approximated for the finite time step $t_N \rightarrow t_N + \Delta t$ as

$$\tau_{ij}(t_N + \Delta t) = J^{-1}(t_N + \Delta t) Q_{ki}(t_N + \Delta t) \tau_{kl}(t_N) Q_{ij}(t_N + \Delta t)$$

$$+ J^{-1}(t_N + \frac{1}{2} \Delta t) Q_{ki}(t_N + \Delta t) Q_{km}(t_N + \frac{1}{2} \Delta t)$$

$$\times \delta_{mn}(t_N + \frac{1}{2} \Delta t) Q_{ln}(t_N + \frac{1}{2} \Delta t) Q_{ij}(t_N + \Delta t) \Delta t$$  \hspace{1cm} (36)

Here, $Q_{ij}(t_N + \theta \Delta t)$ (0 $\leq \theta \leq 1$) is derived by

$$Q_{ij}(t_N + \theta \Delta t) = \delta_{ij} + \frac{\sin(\omega \theta \Delta t)}{\omega} W_{ij}$$

$$+ \frac{1 - \cos(\omega \theta \Delta t)}{\omega^2} W_{ik} W_{kj}$$  \hspace{1cm} (37)
where

$$\omega = \frac{1}{2} W_{ij} W_{ij}$$

**NUMERICAL EXAMPLE: DIFFUSED NECKING OF A TENSILE PLATE**

The plastic instability problem of diffuse necking of a tensile bar is analysed as a numerical example. Here, rectangular elastic-plastic plates are subjected to tensile deformation (in plane strain) with shear free end conditions, as shown in Figures 3 (a) and (b). Two different solution techniques are employed here. One is the present full tangent stiffness method, and the other is an initial strain iteration method. The results obtained by these methods are compared (for the load and the diffused necking bifurcation point) with the direct analytical method. The details of the direct method are shown in the Appendix.

The boundary-value problems shown in Figures 3 (a) and (b) are considered here. As shown in Figure 3, the case (1) and the case (2) have their initial aspect ratios as \((H_0/L_0)^{1/4}\) and \(1\), respectively. Shear free displacement boundary conditions are specified at both ends of the tensile plate. Mesh discretizations and boundary conditions for the field-boundary-element analysis are also shown in Figure 3. From symmetry considerations, one-fourth of the actual problem is analysed. Material properties used in the analysis are given in Table II. A power law hardening elastic-plastic

![Diagram](image-url)

**Figure 3.** Specimens for analyses of diffused necking and their field-boundary-element mesh discretizations

<table>
<thead>
<tr>
<th>Table II. Material properties employed to the analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus (E)</td>
</tr>
<tr>
<td>Poisson's ratio (\nu)</td>
</tr>
<tr>
<td>Yield stress (\sigma_Y)</td>
</tr>
<tr>
<td>One dimensional stress-strain relationship.</td>
</tr>
<tr>
<td>(\varepsilon = \sigma/E)</td>
</tr>
<tr>
<td>(\varepsilon = \sigma^n/(E\sigma^n_{Y-1}))</td>
</tr>
<tr>
<td>(N = 8)</td>
</tr>
</tbody>
</table>
material is considered here. The analysis is carried out under plane strain conditions.

In Figures 4 and 5, the solutions obtained by an initial strain iteration field-boundary-element method\textsuperscript{16} are first presented for the case (1), wherein the initial aspect ratio is $\frac{1}{3}$. Two different types of tensile bars are considered here: one has a perfect initial geometry, and the other one has an initial geometric imperfection of one per cent in the width of the tensile plate at its midlength. The

![Figure 4](image1.png)

Figure 4. Normalized load-engineering strain curve for the case 1. (Analysed by a conventional initial strain iteration field-boundary-element algorithm, without initial geometric imperfections)

![Figure 5](image2.png)

Figure 5. Normalized load-engineering strain curve for the case 1. (Analysed by a conventional initial strain iteration field-boundary-element algorithm, with 1 per cent of geometric initial imperfection)
normalized load engineering strain curves for those two cases of geometrical imperfection are shown in Figures 4 and 5, respectively. The solution for the case without initial imperfections traces over on the fundamental homogeneous deformation path obtained by the direct method. In the case with an initial geometric imperfection, the solution (by the initial strain iteration method) fails to converge after the bifurcation point. These numerical results clearly indicate that either the plastic instability is ignored or the convergence fails, when an initial strain type iterative algorithm is used.

The same problem is successfully analysed by using the present full tangent stiffness field-boundary-element approach. The analysis is carried out without any initial geometric imperfections when using the tangent stiffness method. The normalized load-engineering strain curves are shown in Figures 6 and 7 for the cases 1 and 2, wherein the initial aspect ratios are \( \frac{1}{4} \) and \( \frac{1}{3} \), respectively. It is seen that the load vs engineering strain curve obtained by the full tangent stiffness method traces over that of the fundamental homogeneous deformation path up to the bifurcation point, and drops gradually thereafter. This type of behaviour in the load-engineering strain curves corresponds to the diffused necking deformation mode. The displacement increment \( \Delta \) in each incremental time step is controlled carefully, as shown in the Figures 4-7, with respect to the level of engineering strain.

The deformed mesh configurations are shown in Figures 8 (a) and (b) at the engineering strains of 15.3 and 20.4 per cent, respectively, for the case 1 (i.e. of initial aspect ratio \( \frac{1}{4} \)). It is seen that the deformation is homogeneous up to the bifurcation point (engineering strain 15.3 per cent), and then diffused necking develops thereafter. The deformed mesh configurations at the engineering strains of 17.2 and 20.7 per cent are shown in Figures 9 (a) and (b), for the case 2 (i.e. of initial aspect ratio \( \frac{1}{3} \)). The same trend is observed in the solution for the case 2 (engineering strain at the bifurcation point is 17.1 per cent).

A further discussion of the analysis (by the full tangent stiffness method) is illustrated with the distributions of relative rate of equivalent plastic strain and equivalent plastic strain. The relative
M: Maximum load point.
B: Bifurcation point.

Homogeneous deformation path.

\[ F' = (\gamma L_0) \]

\[ 0 \leq \gamma \leq 1 \]

\[ \frac{\Delta}{L_0} = 0.001 \quad 0.000 < \frac{\Delta}{L_0} < 0.030 \]
\[ \frac{\Delta}{L_0} = 0.003 \quad 0.030 < \frac{\Delta}{L_0} < 0.050 \]
\[ \frac{\Delta}{L_0} = 0.01 \quad 0.050 < \frac{\Delta}{L_0} < 0.131 \]
\[ \frac{\Delta}{L_0} = 0.001 \quad 0.131 < \frac{\Delta}{L_0} < 0.168 \]
\[ \frac{\Delta}{L_0} = 0.005 \quad 0.168 < \frac{\Delta}{L_0} < 0.187 \]
\[ \frac{\Delta}{L_0} = 0.0005 \quad 0.187 < \frac{\Delta}{L_0} \leq 0.197 \]

\( \star \) : B.E.M. Solution.

Figure 7. Normalized load-engineering strain curve for the case 2 (Analysed by present full tangent stiffness field-boundary-element approach)

Figure 8. Deformed mesh configurations for the case 1, at the engineering strain (a) 15.3% and (b) 20.4% per cent
rate of plastic strain \( \dot{\varepsilon}_R^p \) is defined by,

\[
\dot{\varepsilon}_R^p = \frac{\dot{\varepsilon}_R^p}{\sqrt{\frac{2}{3} (\varepsilon_{ij} \varepsilon_{ij})^{1/2}}}
\]

(38)

and

\[
\varepsilon^p = \sqrt{\frac{2}{3} (\varepsilon_{ij} \varepsilon_{ij})^{1/2}}
\]

(39)

Here, \( \varepsilon_{ij} \) is the rate of plastic strain, \( l \) is the current length of the plate and \( \dot{l} \) is its time derivative. When a rigid-plastic plate is undergoing homogeneous stretching under plane strain conditions, the relative rate of equivalent plastic strain \( \varepsilon_R^p \) is always equal to one.

The equivalent plastic strain is defined as

\[
\bar{\varepsilon}^p = \sqrt{\frac{2}{3} (\varepsilon_{ij} \varepsilon_{ij})^{1/2}}
\]

(40)

where \( \varepsilon_{ij} \) is plastic strain.

The distribution of relative rate of plastic strain at the engineering strains of 14.5, 15.3, 17.8 and 19.3 per cent is shown in Plates 1(a), (b), (c) and (d), for the case 1. These figures show the development of the diffused necking deformation mode and the elastic unloading zone (\( \varepsilon_R^p = 0 \)) in the plate. At an engineering strain of 19.3 per cent (as the deformation proceeds) the distribution of \( \varepsilon^p \) seems to concentrate at the portion of necking (Plate 2). Moreover, \( \dot{\varepsilon}_R^p \) and \( \varepsilon^p \) at the engineering strain of 19.3 per cent (Plates 1(c) and 11) seem to indicate high concentration of plastic
Plate 1. Distributions of the relative rate of equivalent plastic strain for the case 1, at the engineering strains (a) 14.5, (b) 15.3, (c) 17.8 and (d) 19.3 per cent

Plate 2. Distribution of equivalent plastic strain for the case 1, at the engineering strain 19.3 per cent
Plate 3. Distributions of relative rate of equivalent plastic strain for the case 2, at the engineering strain (a) 16.6, (b) 17.2 and (c) 20.7 per cent.

Plate 4. Distribution of equivalent plastic strain for the case 2, at the engineering strain 20.7 per cent.
deformation around a 50 degree section across the diffused necking portion of tensile plate. This behaviour corresponds to a shear band type deformation.

The same trend is seen in the solution for the case 2 whose initial aspect ratio is \( \frac{1}{3} \), as shown in Plates 3 (a), (b) and (c) wherein the distribution of \( \varepsilon^p \) is illustrated at the engineering strains of 16.6, 17.2 and 20.7 per cent. In Plate 4, the distribution of \( \varepsilon^p \) at the engineering strain of 20.7 per cent is shown.

When the finite-element method was used for a problem of this class, geometric imperfections were often introduced into the problem to lead the analysis automatically into a diffused necking mode of deformation. As for the finite element method, when the initial imperfections are considered to be absent, a superposition of a small magnitude of the necking mode (with elastic unloading at the top of the bar) onto the homogeneous deformation mode was found to be necessary to lead the tensile bar into the region of diffused necking. But, in the present full tangent stiffness field-boundary-element analysis, the initial geometric imperfection is not introduced into the problem. Since all the effects of non-linearities in the problem are properly and directly accounted for in the calculation of the velocity (displacement) field, the present full tangent stiffness field-boundary-element analysis is quite sensitive to numerical instabilities. The diffused necking instability contained in the problem is propagated by the effect of a small amount of numerical error produced in numerical integrations, which makes the present algorithm switch its solution path from the fundamental homogeneous mode to the diffused necking mode by itself.

The analysis is quite sensitive to the time increment in the incremental elastic-plastic analysis, when the numerical instabilities are propagating in the problem. Hence, this necessitates the time increment in each time step to be controlled very carefully around the bifurcation point. If the time increment is too large, then the analysis would ignore the numerical instabilities around the bifurcation point, and trace over the fundamental homogeneous deformation path.

As described above, the present full tangent stiffness field-boundary-element method is superior to the conventional approach using an initial strain iteration algorithm in capturing plastic instabilities. Moreover, as shown in this section, the present approach is capable of not only capturing the plastic instability but also in analysing the post-bifurcation behaviour.

CONCLUSION

This paper presents the first successful attempt at solving bifurcation problems involving large elastic-plastic strains (such as buckling, diffused necking, etc.) by the field-boundary-element method. The solution algorithm presented here is based on a full tangent stiffness field-boundary-element method where the velocity field both inside and on the boundary of the body is taken as the primary variable. Such a scheme takes the initial stress-velocity gradient coupling terms accurately into account (unlike the conventional initial strain type algorithm). This is of paramount importance to these geometric and material non-linear instability problems. It has also been shown that such an algorithm is also capable of analysing the post-bifurcation behaviour.

ACKNOWLEDGEMENTS

The support of this work by the U.S. Office of Naval Research and the encouragement of Dr Y. Rajapakse is gratefully acknowledged. The authors are also indebted to Dr M. Kikuchi of Science

*It is well known that the determinant of the tangent stiffness matrix changes its sign when the deformation path passes through the bifurcation necking point. Thus, the determinant of the tangent stiffness is carefully maintained at each increment. If a change in its sign is detected the size of the increment is reduced, and then carefully controlled, to precisely define the bifurcation point, as well as the post-bifurcation necking solution.
University of Tokyo for lending his linear elasticity BEM program. It is also a pleasure to thank Ms Deanna Winkler for her assistance in the preparation of this manuscript.

APPENDIX: THE CALCULATION OF THE FUNDAMENTAL HOMOGENEOUS DEFORMATION PATH AND THE BIFURCATION POINT

1. The calculation of the fundamental homogeneous deformation path

The fundamental homogeneous deformation path is calculated by an incremental algorithm, assuming the material to be incompressible and incrementally linear (piecewise linear).

The displacement increment in an incremental time step is given at the top end of the plate, shown as Figure 10. Let \( l \) and \( \Delta l \) be the current length of the plate and its increment, respectively. Thus, the strain increment is given by

\[
\begin{align*}
\varepsilon_{11} \Delta t &= \Delta \varepsilon_{11} = \Delta l / l \\
\end{align*}
\]

From the assumptions of the plane strain condition and the incompressible material,

\[
\begin{align*}
\Delta \varepsilon_{33} &= 0 \quad (\dot{\varepsilon}_{33} = 0) \\
\Delta \varepsilon_{11} + \Delta \varepsilon_{22} &= 0 \quad (\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} = 0)
\end{align*}
\]

By the plastic strain evolution equation,

\[
\Delta \varepsilon_{11}^p = \frac{1}{C} N_{11} (N_{11} \Delta \varepsilon_{11} + N_{22} \Delta \varepsilon_{22})
\]

\[
= \frac{1}{C} N_{11} (N_{11} - N_{22}) \Delta \varepsilon_{22}
\]

\[
\Delta \varepsilon_{22}^p = -\Delta \varepsilon_{11}^p, \quad \Delta \varepsilon_{33}^p = 0
\]

The constitutive equation is written as

\[
\begin{align*}
\Delta \sigma_{11} &= 4 \mu (\Delta \varepsilon_{11} - \Delta \varepsilon_{11}^p) \\
\Delta \sigma_{22} &= 0 \\
\Delta \sigma_{33} &= 2 \mu (\Delta \varepsilon_{11} - \Delta \varepsilon_{11}^p)
\end{align*}
\]

Figure 10. A rectangular plate subjected to tensile deformation.
The increment of first Piola–Kirchhoff stress $\Delta T_{ij}$, which refers to the current configuration, is expressed by

$$
\Delta T_{11} = \Delta \sigma_{11} - \Delta \epsilon_{11} \sigma_{11} \\
\Delta T_{22} = 0 \\
\Delta T_{33} = \Delta \sigma_{33}
$$

(A5)

Therefore, the increment of applied force $\Delta F$ is

$$
\Delta F = H \cdot \Delta T_{11}
$$

(A6)

where $H$ is the current width of the plate.

The stress, strain and load in the fundamental homogeneous deformation path are obtained by repeating the procedures of equations (A1)-(A6) in each incremental time step.

2. The estimation of the bifurcation point

The numerical estimation of the bifurcation point is discussed based on a theory developed by Hill and Hutchinson. They considered the bifurcation from a state of homogeneous in-plane tension and assume the material to be incompressible and incrementally linear. The closed form equations for the estimation of the bifurcation point had been made by Reed and Atluri for isotropic power law hardening elasto-plasticity based on Reference 39, and also by Murakawa and Atluri for isotropic linear hardening elasto-plasticity based on the Onat–Cowper solution.

The closed form equation for the endochronic constitutive model which contains isotropic, kinematic and combined isotropic/kinematic hardening models would be quite complicated. On the other hand, using the method to calculate the fundamental homogeneous deformation path (introduced in the previous section), a numerical method for the estimation of the bifurcation point can be constructed quite easily, without restrictions for elastic-plastic constitutive equations.

From equations (A2) and (A3), the constitutive equation is also expressed by

$$
\dot{\sigma}_{11} = 4\mu (\dot{\epsilon}_{11} - \dot{\epsilon}^{p}_{11}) \\
= 4\mu^{*} \dot{\epsilon}_{11}
$$

$$
4\mu^{*} = \mu \left\{ 1 - \frac{1}{C} N_{11} (N_{11} - N_{22}) \right\}
$$

(A7)

The bifurcation criterion given by Hill and Hutchinson is

$$
0 = -\frac{\sigma_{11}}{4\mu^{*}} + \frac{\gamma}{\sin 2\gamma} \cdot \frac{2\mu^{*}}{\mu} \left\{ \left( \frac{\gamma}{\sin \gamma} \right)^3 (1 + \cos \gamma) \right\} \\
0 = -\frac{1}{8} \cdot \frac{\gamma}{\sin^2 \gamma} \left( \frac{1}{4} + \frac{1}{3} \gamma^2 \right)
$$

(A8)

and

$$
\gamma = m\pi H/l
$$

(A9)

where $l$ and $H$ are the current length and width of the plate. $m(= 1, 2, \ldots)$ are the bifurcation modes and $m=1$ corresponds to the diffused necking bifurcation. The bifurcation point is computed by analysing the sign change in the right-hand side of equation (A8).
Table III. A comparison of the estimation for bifurcation

<table>
<thead>
<tr>
<th>$l_0/H_0$</th>
<th>N (Power)</th>
<th>$(l-l_0)/l_0$ at bifurcation ($m=1$)</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>0.393</td>
<td>0.399</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.334</td>
<td>0.334</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
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<td>0.315</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>0.234</td>
<td>0.237</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>0.172</td>
<td>0.176</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>0.154</td>
<td>0.157</td>
</tr>
</tbody>
</table>

Material properties

- Young's modulus: $E = 6.895 \times 10^4$ MPa
- Poisson's ratio: $\nu = 0.5$
- Yield stress: $\sigma_y = 344.75$
- Hardening law: $\dot{\varepsilon} = \sigma^n/[E\sigma_y^{\alpha-1}]$

The results obtained by present algorithm are quite analogous to those obtained by Reed and Atluri\(^{18}\) and Burke and Nix,\(^{41}\) as shown in Table III where a power law hardening material is considered.

REFERENCES


