AN ANALYSIS OF, AND REMEDIES FOR, KINEMATIC MODES IN HYBRID-STRESS FINITE ELEMENTS: SELECTION OF STABLE, INVARIANT STRESS FIELDS

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In this paper a study of the existence of spurious kinematic modes in hybrid-stress finite elements, based on assumed equilibrated stresses and compatible boundary displacements, and the resulting rank-deficiency of the element stiffness matrix, is presented. A method of selection of least-order, stable, invariant, stress fields is developed so as to ensure the prevention of kinematic modes. A 20-node cubic element, a 8-node cubic element and a 4-node square, based on assumed equilibrated stresses within the element and compatible displacements at the boundary of the element, are discussed for purposes of illustration. Comments are made on the generality of the present method, which is based on group theoretical arguments.

0. Introduction

In the context of the present paper, by ‘hybrid’ we mean an element formulation wherein certain constraints, such as displacement compatibility or traction reciprocity at interelement boundaries is enforced a posteriori through Lagrange multipliers; by ‘mixed’ we mean an element formulation wherein certain constraints within an element are enforced a posteriori, and as such may involve discretization of more than one field variable within an element. We do not mean these to be absolute definitions, but, rather, explanations of our terminology.

Of the several hybrid methods we deal here with one: the method based on a complementary energy principle in linear solid mechanics where, in an application to a finite element scheme, an equilibrated stress field is assumed in each element, but interelement traction reciprocity is enforced through a Lagrange multiplier which turns out to be the interelement boundary displacement field. This boundary displacement is unique for two elements that have a common boundary, and as such displacements are continuous at the interelement boundary, but traction reciprocity is satisfied only in an average sense in the discrete formulation. It is well known that this hybrid method (better known through literature as hybrid-stress method) leads to the usual concept of an element stiffness matrix [1, 2]. Fraeijs de Veubeke [3] aptly calls these elements “statically admissible and weakly diffusive” elements (but “strongly conforming”). The so-called equilibrium models of de Veubeke [3], also based on a complementary energy principle and also leading to a stiffness matrix, are, on the other hand, strongly diffusive but weakly conforming. Unfortunately, despite their strong theoretical
appeal, both the 'hybrid-stress' and 'equilibrium' model approaches have not yet been fully exploited, due in part to the following reasons.

(i) Considerably more ingenuity is required to develop a 'good' element based on these approaches as compared to standard displacement methods.

(ii) Even though the methods lead to a stiffness matrix it is not necessarily well behaved unless the stress interpolation is 'proper'. By this we mean that unless 'due care' is exercised in the choice of element stresses and boundary displacements, the element will have spurious mechanism (or kinematic' or 'zero-energy') modes in addition to the rigid body modes.

(iii) The method involves the inversion of a matrix at the element level which, from the point of view of researchers and developers concerned solely with 'computational operation-counts' and 'cost', is enough of a reason to abandon it altogether.

While functional analysts have studied the problem of stability and convergence of discrete variational problems with Lagrange multipliers, and the so-called Ladyzhenskaya–Babuska–Brezzi (LBB) conditions [4, 5] that are necessary and sufficient for stability and convergence of such finite element methods are reasonably well known, unfortunately, often times, these conditions serve only as checks on a formulation, but do not necessarily provide useful guidance in the element formulation stage. By this 'guidance', we mean, for instance, that the LBB condition per se does not tell one how to choose a stable invariant stress field in each element in the hybrid-stress method such that the element has no spurious zero-energy modes other than the pure rigid-body modes. In this paper we report on a modest success of attempts of a formal theory to analyze the kinematic modes in the hybrid-stress and equilibrium methods, and suggest remedies to avoid such modes at the element formulation stage. This formal theory provides guidance to the developer in the proper choice of stable (non-existence of kinematic modes) as well as invariant (no preferred directions) stress interpolation in each element. A key ingredient in this formal theory is the theory of groups [6]. We present here certain elementary notions of group theory as well to make the paper more readable.

1. Stable, invariant, stress-spaces for hybrid-stress finite element method

1.1. The method and a diagnosis of its delicacy

Consider a linear elastic solid wherein (i) geometry is described by Cartesian coordinates $x_i$; (ii) a partial derivative w.r.t. $x_i$ denoted by $(\cdot)_i$; (iii) the tensor of stress is $\sigma_{ij}$; (iv) the tensor of strain is denoted by $\varepsilon_{ij}$; (v) the prescribed surface tractions are $t_{ij}$ at the external surfaces $S_i$; (vi) the prescribed displacement is $u_i$ at the external surface $S_a$; (vii) the body forces per unit volume are $F_i$; and (viii) the complementary energy density is $B(\sigma_{ij})$ per unit volume, such that $\varepsilon_{ij} = \partial B/\partial \sigma_{ij}$. Consider the solid to be discretized into finite elements $V_m$ ($m = 1, 2, \ldots$) each with a boundary $\partial V_m$. Note that in general, $\partial V_m = \rho_m + S_m + S_{am}$, where $\rho_m$ is the interelement boundary. We assume for simplicity that tractions and displacements are prescribed only at the external boundary of the solid, so that for most of the elements $\partial V_m = \rho_m$, save for those whose boundaries coincide with the external boundary of the solid.

Suppose now that in each $V_m$ one assumes $\sigma_{ij}$ such that (LMB) $\sigma_{ij} + \bar{F}_i = 0$ (henceforth let $\bar{F}_i = 0$ for simplicity) and (AMB) $\sigma_{ij} = \sigma_{ji}$ are satisfied. But the traction reciprocity, i.e., $(n_i \sigma_{ij})^+ + (n_i \sigma_{ji})^- = 0$ at $\rho_m$ is to be enforced a posteriori, through a Lagrange multiplier $\tilde{u}_i$ at
R. Rubinstein et al., Kinematic modes in hybrid-stress finite elements

\[ \rho_m. \] Further, if \( \tilde{u}_i \) is chosen so that \( \tilde{u}_i = \bar{u}_i \) at \( S_{um} \), and also enforces the traction b.c., i.e., \( (n_i \sigma_{ji}) = \bar{t}_i \) at \( S_m \) through a Lagrange multiplier \( \bar{u}_i \), then one can show [1, 2] that the appropriate variational functional, whose stationary value leads a posteriori to (i) compatibility in \( V_m \), i.e., \( \partial B / \partial \sigma_{ij} = \frac{1}{2} (\sigma_{ij} + \sigma_{ji}) \); (ii) traction reciprocity at \( \rho_m \), (iii) traction b.c. at \( S_{um} \), and (iv) displacement b.c. at \( S_{um} \), is given by

\[
HS(\sigma_{ij}, \tilde{u}_i) = \sum_m \left\{ \int_{V_m} -B(\sigma_{ij})dV + \int_{\partial V_m} n_i \sigma_{ji} \tilde{u}_i ds - \int_{S_{um}} \bar{t}_i \tilde{u}_i ds \right\}. \tag{1.1a}
\]

For a linear elastic material

\[
B(\sigma_{ij}) = \frac{1}{2} \sigma_{ij} C_{ijkl} \sigma_{kl}
\]

where \( C_{ijkl} \) are the material's compliance coefficients. For convenience we denote \( \sigma_{ij} \) by a \((6 \times 1)\) vector and \( u_i \) by a \((3 \times 1)\) vector, \( t_i = n_i \sigma_{ji} \) by a \((3 \times 1)\) vector, and \( C_{ijkl} \) by a \((6 \times 6)\) matrix. Thus (1.1a) can be written as

\[
HS(\sigma, \tilde{u}) = \sum_m \left\{ \int_{V_m} -\frac{1}{2} \sigma^{i'} \sigma \ dV + \int_{\partial V_m} t^i \cdot \tilde{u} ds - \int_{S_{um}} \bar{t}^i \cdot \tilde{u} ds \right\}. \tag{1.1b}
\]

Let us introduce the element approximations

\[
\sigma = A \beta \quad \text{in} \ V_m \tag{1.2}
\]

and

\[
\tilde{u} = L q \quad \text{at} \ \partial V_m \tag{1.3}
\]

where \( \beta \) are undetermined stress parameters and \( q \) are nodal displacements. Thus, the functional (1.1b) takes the form

\[
HS(\beta, q) = \sum_m \left\{ -\frac{1}{2} \beta^{i'} H \beta + \beta^{i'} G q - Q^i \cdot q \right\} \tag{1.4}
\]

with

\[
H = \int_{V_m} A^{i'} CA \ dV \tag{1.5}
\]

and

\[
G = \int_{\partial V_m} R^i L \ dV \tag{1.6}
\]

where

\[
t = R \beta \quad \text{at} \ \partial V_m . \tag{1.7}
\]

Since tractions \( t_i = \sigma_{ij} n_j \) at \( \partial V_m \) were derived from an equilibrated stress field, the divergence theorem leads to the result

\[
G = \int_{V_m} A^{i'} \cdot B \ dV \tag{1.8}
\]
where, if the $\tilde{u}_i$ from $\partial V_m$ are extended into $V_m$, one may define the corresponding strain field as $\varepsilon(\tilde{u}) = Bq$. We do recognize that, in problems with $C^1$ continuity requirements for displacements, this extension is nontrivial, but defer such problems to a later report. Thus (1.8) can also be written as:

$$\beta'Gq = \int_{V_m} \sigma_{ij} E_{ij}(\tilde{u}_k) dV.$$  \hfill (1.9)$$

Since $\beta$ is independent for each element, one obtains the element level equations

$$H\beta = Gq \quad \text{or} \quad \beta = H^{-1}Gq.$$ \hfill (1.10a,b)

Using (1.10) in (1.4) one identifies the element stiffness matrix $k$ as

$$k = G'H^{-1}G.$$ \hfill (1.11)

Let the number of stress parameters per element be $s$, the number of generalized nodal displacements per element be $d$, and the number of rigid-body modes of each element $r$ (i.e., $r = 3$ for plane elements and $6$ for 3-D elements). The following is then noted.

(i) From (1.5) it is seen that, since $B(\sigma_{ij})$ is positive definite for all $\sigma_{ij}$, the $s \times s$ matrix $H$ is always positive definite, i.e., has rank $s$.

(ii) Even though (1.11) appears to indicate the need for inverting $H$, in reality, since $H^{-1}G$ appears in (1.11), this term can be evaluated directly from (1.10a) by an equation solver with multiple ‘right-hand sides’. This is in fact much less ‘expensive’ than explicitly finding $H^{-1}$.

(iii) A ‘good’ stiffness matrix should involve all the rigid-body modes of the element. Thus the rank of $k$ should be $d - r$.

(iv) The matrix $G$ is of order $s \times d$. From (1.9) it is seen that, since $\varepsilon_{ij}(\tilde{u}_k) = 0$ for $r$ rigid-body modes, the rank of $G$ is, at best, the minimum of $(s, d - r)$.

(v) In view of (iv) it is seen that the rank of $k$ is, at best, the minimum of $(s, d - r)$. Thus, in view of the requirement (iii), $s \geq d - r$.

(vi) For reasons of simplicity we will consider the case $s = d - r$.

(vii) The central problem then becomes one of assuring, by careful choice of $\sigma_{ij}$ in $V_m$, that the rank of $G$ is $s = d - r$.

The implication of (vii) in the choice of $\sigma_{ij}$ can be viewed as follows. If the formulation is as in (vi), then $G$ is a $(d - r) \times d$ matrix. If the rank of $G$ must be $(d - r)$, it is seen that the equation

$$Gq = 0$$ \hfill (1.12)$$

must have, as its nontrivial solutions, only the vectors $q = q_{rb}$ ($rb = 1, 2, \ldots, r$) i.e. the rigid-body modes. Any other nontrivial solution would be a kinematic mode. More importantly, in (1.9), since $\varepsilon_{ij}(\tilde{u}_k)$ involves only $d - r$ parameters, in order to assure the rank of $G$ to be $d - r$ we must choose $d - r$ equilibrated stress modes $\sigma_{ij}$ in each element such that

$$\int_{V_m} \sigma_{ij} E_{ij}(\tilde{u}_k) dV > 0, \quad \varepsilon_{ij}(\tilde{u}_k) \neq 0.$$ \hfill (1.13)
Obeying (1.13) in the element formulation stage is then the central issue, and one that calls for further insight.

At this point it is worth considering the so-called LBB condition [4] for this formulation. This condition states that, if there exists a constant $\beta > 0$ such that

$$\sup_{(\sigma, u) \in \mathcal{X}} \sum_{k=1}^{N} \int_{\Omega_k} \sigma_i e_{ij}(\bar{u}_k) dV \geq \beta \|u_k\|_{L^1} \quad \forall \, u \in U_0,$$

(1.14)

then the finite element problem has a unique solution. Moreover, if $\beta$ does not depend on the mesh parameter $h$, then convergence is established. Thus (1.14) is a necessary condition for stability. Thus (1.13) is necessary to establish (1.14), even though the condition that $\beta$ is independent of $h$ should be checked separately [7].

1.2. Measures to assure the rank of $G$ to be $d - r$ when $s = d$

The most direct attack on the problem of mechanism modes is to choose as a stress interpolation any complete equilibrated polynomial field having at least as many degrees of freedom as the strains to form the $G$ matrix, and then to compute the rank of $G$ by a procedure such as Gaussian elimination. If the rank proves too small, the next highest-order stresses must be added to the interpolation. When the rank of $G$ equals $d - r$, this elimination process should reveal which stress degrees of freedom are redundant. In practice, this straight-forward method is cumbersome, since the matrix $G$ will be both large and relatively dense. Moreover, it is not easy to eliminate redundant stresses while at the same time preserving the invariance of the stress interpolants.

The requirement of invariance alone suggests the relevance of group representation theory [6] to this problem. In the present we use such a theory and also show that it leads us to a sparse quasi-diagonal $G$ matrix from which we can very easily determine the invariant $d - r$ dimensional stress interpolants, which lead to an element formulation that is free from the curse of kinematic modes.

To begin the analysis we borrow from group theory the concepts of conjugacy class [6, Section 241], of representation [6, Section 208] and of irreducibility [6, Section 197] of Burnside's classic book. We will illustrate the use of these concepts in the context of a 20-noded cubic hybrid-stress element.

The orientation-preserving symmetry group $G$ of the cube contains 24 operations divided into 5 conjugacy classes. We consider the vertices of the cube to be at the eight points in the parent-coordinate system $(\pm 1, \pm 1, \pm 1)$. Then $G$ consists of the following 24 matrices in 5 classes.

\[\text{We did consider the possibility of illustrating the concepts through the simple case of a planar 4-noded square element. However, it was found that this example is deceptive in that the essential features of the problem are masked. Thus we choose to introduce the problem of the 20-noded cube at the outset, and take the rather unusual step of relegating the matter of a 4-noded square to Appendix A!}\]
The above 5 classes can be viewed as a group of permutations of the vertices, edges, faces, or diagonals of the cube. Thus class $C_1$ of $G$ maps $(x, y, z)$ into $(x, y, z)$; and the first matrix of class $C_2$ maps $(x, y, z)$ into $(-y, x, z)$ etc. Since $G$ has 5 classes, it has 5 irreducible representations (see [6, Section 208]). We will exhibit these as transformations of certain sets of homogeneous polynomials. Referring to (1.15) it is seen that the quadratic $(x^2 + y^2 + z^2)$ is an invariant of $G$; each element of $G$ multiplies $(x^2 + y^2 + z^2)$ by unity. In the interest of notation, but with undeniable pedantry, we may state that $G$ transforms the one-dimensional vector space generated by $(x^2 + y^2 + z^2)$ by the $1 \times 1$ matrices

$$
\Gamma_1: \quad C_1 \quad [1], \\
C_2 \quad [1] \quad \text{(all elements)}, \\
C_3 \quad [1] \quad \text{(all elements)}, \\
C_4 \quad [1] \quad \text{(all elements)}, \\
C_5 \quad [1] \quad \text{(all elements)}. 
$$

which defines the irreducible representation $\Gamma_1$, a one-dimensional representation possible for
any group. Less trivially $G$ transforms the cubic (xyz) by the $1 \times 1$ matrices

$$
\Gamma_2: \begin{array}{c}
C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ (all elements)}, \\
C_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ (all elements)}, \\
C_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (all elements)}, \\
C_5 \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ (all elements)},
\end{array}
$$

which defines representation $\Gamma_2$. Likewise $G$ transforms the two-dimensional space generated by $(x^2 - y^2, y^2 - z^2)$ by

$$
\Gamma_3: \begin{array}{c}
C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
C_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \\
C_3 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
C_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
C_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\end{array}
$$

which defines representation $\Gamma_3$. $G$ may be regarded as an irreducible representation of itself through its transformation of the variables $(x, y, z)$ by (1.15); we call this representation $\Gamma_4$. Finally the three quadratics $(xy, yz, zx)$ are transformed by

$$
\Gamma_5: \begin{array}{c}
C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
C_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
C_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\
C_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
C_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\end{array}
$$
which defines the last irreducible representation \( \Gamma_5 \). Let \( n_i \) be the dimension of the space on which \( \Gamma_i \) is defined. Then from (1.15–1.19) it is seen that

\[
(n_1, n_2, n_3, n_4, n_5) = (1, 1, 2, 3, 3) .
\]

In the present analysis we will be interested in representations of \( G \) on stress and strain tensors with homogeneous polynomial components. If \( G \) transforms any \( n \) quantities among themselves, we can find among them \( \gamma_i \) sets of \( n_i \) quantities which transform according to \( \Gamma_i \) for each \( i = 1, \ldots , 5 \). The multiplicities \( \gamma_i \) satisfy (see [6, Section 205])

\[
\sum_{i=1}^{5} n_i \gamma_i = n .
\]

These quantities form irreducible invariant subspaces. To find them we form the operator

\[
K_i = \sum_{j=1}^{5} \chi^{(j)}_i \sum_{A \in C_j} A ,
\]

in which \( \chi^{(j)}_i \) is the trace of any matrix of irreducible representation \( \Gamma_i \) in class \( C_j \), and \( A \)'s are the matrices of representation. The quantities \( \chi^{(j)}_i \), the so-called characters, can be evaluated from (1.15–1.19) and exhibited in a character table as

\[
\begin{array}{c|ccccc}
\Gamma_i & C_1 & C_2 & C_3 & C_4 & C_5 \\
\hline
\Gamma_1 & 1 & 1 & 1 & 1 & 1 \\
\Gamma_2 & 1 & -1 & 1 & 1 & -1 \\
\Gamma_3 & 2 & 0 & 2 & -1 & 0 \\
\Gamma_4 & 3 & 1 & -1 & 0 & -1 \\
\Gamma_5 & 3 & -1 & -1 & 0 & 1 \\
\end{array}
\chi^{(j)}_i \quad (j = 1, \ldots , 5)
\]

Operator \( K_i \) is the projection onto the subspace generated by all quantities which transform under \( G \) by the matrices of \( \Gamma_i \) (see [6, Section 229]).
The decomposition of stress and strain fields into irreducible representations is justified by the following useful orthogonality relation: let \( \sigma_{(p)}, 1 \leq p \leq \gamma_i n_i \) (no sum on \( i \)) and \( \varepsilon_{(q)}, 1 \leq q \leq n_j \gamma_j \) (no sum on \( j \)) be the bases of invariant stress and strain tensors, respectively, which transform by distinct \((i \neq j)\) irreducible representations \( I_i \) (with dimension of bases \( n_i \) and multiplicity \( \gamma_i \)) and \( I_j \) (with dimension \( n_j \) and multiplicity \( \gamma_j \)). Then the matrix

\[
G_{pq} = \int \sigma_{(p)} : \varepsilon_{(q)} dV = 0 \quad (1 \leq p \leq n_i \gamma_i; \quad 1 \leq q \leq n_j \gamma_j).
\]  

(1.24)

In (1.24) the bold-face notations to represent a second-order tensor, and \( A : B = A_{ij} B_{ij} \) are used.

The proof of (1.24) follows from certain established results in group theory [6]. However, in the interest of brevity here we omit the proof; rather we demonstrate that \( G \) matrix for the present 20-noded element is in fact quasi-diagonal in nature due to (1.24). Thus suppose \( G \) is formed with stress and strain degrees of freedom chosen as the natural irreducible invariant subspaces and suppose that \( G \) is partitioned horizontally and vertically so that quantities transforming by \( I_i \) are partitioned. The orthogonality relation (1.24) assures that the blocks \((i, j)\) in \( G \) with \( i \neq j \) are zero. To determine the rank of \( G \) and to identify the redundant stresses one only needs to investigate each non-zero block separately. Moreover, provided one only discards complete irreducible invariant subspaces from the stress interpolation, the interpolation will necessarily remain invariant.

Having outlined our modus operandi we now implement it beginning with the decomposition of the strain space. For a 20-noded element the displacement field can be written as

\[
U = (1, x, y, z, x^2, y^2, z^2, xy, yz, zx, x^2 y, y^2 z, z^2 x, xy^2, yz^2, zx^2, xyz^2, x^2 yz, xyz) X
\]

\[
+ (1, x, y, z, x^2, y^2, z^2, xy, yz, zx, x^2 y, y^2 z, z^2 x, xy^2, yz^2, zx^2, xyz^2, x^2 yz, xyz) Y
\]

\[
+ (1, x, y, z, x^2, y^2, z^2, xy, yz, zx, x^2 y, y^2 z, z^2 x, xy^2, yz^2, zx^2, xyz^2, x^2 yz, xyz) Z.
\]

(1.25)

In the above the familiar dyadic notation \( U = (U_x X + U_y Y + U_z Z) \) is used, and each displacement has 20 degrees of freedom. The above displacement field can be found by inspection to have 13 subspaces which are apparently invariant under \( G \) of (1.15)

\[
U^{(1)} = (X, Y, Z) \quad (3),
\]

\[
U^{(2)} = (xX, yY, zZ) \quad (3),
\]

\[
U^{(3)} = (xY, zZ, xX, zY, zX) \quad (6),
\]

\[
U^{(4)} = (xyZ, yzX, zxY) \quad (3),
\]

\[
U^{(5)} = (xyX, xyY, xzX, xzY, yzY, yzZ) \quad (6),
\]

\[
U^{(6)} = (x^2 X, y^2 Y, z^2 Z) \quad (3),
\]

\[
U^{(7)} = (x^2 Y, x^2 Z, y^2 X, y^2 Z, z^2 X, z^2 Y) \quad (6),
\]

\[
U^{(8)} = (x^2 yZ, y^2 zX, z^2 xY, x^2 zY, y^2 xZ, z^2 yX) \quad (6),
\]

(1.26)
\[ U^{(9)} = (x^2yY, y^2zZ, z^2xX, x^2zZ, y^2xX, z^2yY) \]  
\[ U^{(10)} = (x^2yX, y^2zY, z^2xZ, x^2zX, y^2yY, z^2yZ) \]  
\[ U^{(11)} = (xyzX, xyzY, xyzZ) \]  
\[ U^{(12)} = (xyz^2X, xyz^2Y, xy^2zX, xy^2zY, xz^2X, xyz^2Y) \]  
\[ U^{(13)} = (xyz^2Z, xy^2zY, xz^2yX) \]  

Now we find the projections of \( U \) onto subspaces spanned by \( \Gamma_i \). Consider the monomial \( xX \) as an example. This is transformed by symmetry group \( G \) as follows:

\[
C_1 \quad xX, \\
C_2 \quad yY, zZ, xX, yY, zZ, xX, \\
C_3 \quad xX, xX, xX, \\
C_4 \quad yY, zZ, zZ, yY, zZ, yY, zZ, \\
C_5 \quad yY, yY, zZ, zZ, xX, xX.
\]

Thus, using (1.22) we find, for instance, that for \( xX \),

\[
K_1(xX) = [1(xX) + 1(2yY + 2zZ + 2xX) + 1(3xX) + 1(4yY + 4zZ) \\
+ 1(2xX + 2yY + 2zZ)] \\
= 8[xX + yY + zZ],
\]

\[
K_2(xX) = 0, \quad K_4(xX) = 0, \quad K_5(xX) = 0, \quad K_3(xX) = (8xX - 4yY - 4zZ).
\]

Thus the irreducible subspaces corresponding to \( xX \) are

\[
a(xX + yY + zZ) \quad \text{(transforms by } \Gamma_1),
\]

\[
\{a(2xX - yY - zZ) + b(-xX + 2yY - zZ)\} \quad \text{(transforms by } \Gamma_3).
\]

The strain states corresponding to (1.29) are

\[
\varepsilon_1 = a \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]

\[
\varepsilon_2 = a \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + b \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} = a' \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + b' \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.
\]

We note that the strains \((\varepsilon_{xx}, 2\varepsilon_{xy}, 2\varepsilon_{xz}, 2\varepsilon_{yz}, \varepsilon_{yy}, \varepsilon_{zz})\) result from the operator

\[
\varepsilon = DU = (X\partial/\partial x + Y\partial/\partial y + Z\partial/\partial z)U.
\]
Since $D$ itself is an invariant of $G$, corresponding strains also transform by the same matrices. Applying the above procedure to all monomials in $U^{(\sigma)}$, we find the strain decomposition into irreducible subspaces to be of the form

$$
\Gamma_1: \quad \varepsilon^{(1)}_1 = \mu_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \varepsilon^{(2)}_1 = \mu_2 \begin{bmatrix} y^2 + z^2 & 2xy & 2xz \\ 2xy & x^2 + z^2 & 2yz \\ 2xz & 2yz & x^2 + y^2 \end{bmatrix},
$$

$$
\Gamma_2: \quad \varepsilon^{(1)}_2 = \mu_3 \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}, \quad \varepsilon^{(2)}_2 = \mu_4 \begin{bmatrix} y^2 - z^2 & 0 & 0 \\ 0 & z^2 - x^2 & 0 \\ 0 & 0 & x^2 - y^2 \end{bmatrix},
$$

$$
\varepsilon^{(3)}_2 = \mu_5 \begin{bmatrix} 4xyz & z(x^2 + y^2) & y(x^2 + z^2) \\ z(x^2 + y^2) & 4xyz & x(y^2 + z^2) \\ y(x^2 + z^2) & x(y^2 + z^2) & 4xyz \end{bmatrix};
$$

$$
\Gamma_3: \quad \varepsilon^{(1)}_3 = \mu_6 \begin{bmatrix} 0 & z & 0 \\ z & 0 & -x \\ 0 & -x & 0 \end{bmatrix} + \mu_7 \begin{bmatrix} 0 & 0 & y \\ 0 & 0 & -x \\ y & -x & 0 \end{bmatrix},
$$

$$
\varepsilon^{(2)}_3 = \mu_8 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \mu_9 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},
$$

$$
\varepsilon^{(3)}_3 = \mu_{10} \begin{bmatrix} y^2 & xy & 0 \\ xy & -z^2 & -yz \\ 0 & -yz & 0 \end{bmatrix} + \mu_{11} \begin{bmatrix} 0 & 0 & -xz \\ 0 & z^2 & yz \\ -xz & yz & -x^2 \end{bmatrix},
$$

$$
\varepsilon^{(4)}_3 = \mu_{12} \begin{bmatrix} 0 & xy & 0 \\ xy & x^2 & -yz \\ 0 & -yz & -y^2 \end{bmatrix} + \mu_{13} \begin{bmatrix} z^2 & -xy & xz \\ -xy & -x^2 & 0 \\ xz & 0 & 0 \end{bmatrix},
$$

$$
\varepsilon^{(5)}_3 = \mu_{14} \begin{bmatrix} 0 & y^2z & -yz^2 \\ y^2z & 4xyz & x(y^2 - z^2) \\ -yz^2 & x(y^2 - z^2) & -4xyz \end{bmatrix} + \mu_{15} \begin{bmatrix} 4xyz & z(x^2 - y^2) & x^2y \\ z(x^2 - y^2) & -4xyz & -xy^2 \\ x^2y & -xy^2 & 0 \end{bmatrix};
$$

$$
\Gamma_4: \quad \varepsilon^{(1)}_4 = \mu_{16} \begin{bmatrix} 2y & x & 0 \\ x & 0 & z \\ 0 & z & 2y \end{bmatrix} + \mu_{17} \begin{bmatrix} 0 & y & z \\ y & 2x & 0 \\ z & 0 & 2x \end{bmatrix} + \mu_{18} \begin{bmatrix} 2z & 0 & x \\ 0 & 2z & y \\ x & y & 0 \end{bmatrix},
$$

$$
\varepsilon^{(2)}_4 = \mu_{19} \begin{bmatrix} 4xy & (x^2 - y^2) & 0 \\ (x^2 - y^2) & -4xy & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_{20} \begin{bmatrix} 4xz & 0 & (x^2 - z^2) \\ 0 & 0 & 0 \\ (x^2 - z^2) & 0 & -4xz \end{bmatrix} + \mu_{21} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4yz & (y^2 - z^2) \\ 0 & (y^2 - z^2) & -4yz \end{bmatrix}.$$

\[ \varepsilon_4^{(3)} = \mu_{22} \begin{bmatrix} 2yz^2 & xz^2 & 4xyz \\ xz^2 & 0 & x^2z \\ 4xyz & x^2z & 2x^2y \end{bmatrix} + \mu_{23} \begin{bmatrix} 0 & yz^2 & y^2z \\ yz^2 & 2xz^2 & 4xyz \\ y^2z & 4xyz & 2xy^2 \end{bmatrix} + \mu_{24} \begin{bmatrix} 2y^2z & 4xyz & xy^2 \\ 4xyz & 2x^2z & x^2y \\ xy^2 & x^2y & 0 \end{bmatrix}, \]

\[ \varepsilon_4^{(4)} = \mu_{25} \begin{bmatrix} 0 & -xz & xy \\ -xz & 0 & 0 \\ xy & 0 & 0 \end{bmatrix} + \mu_{26} \begin{bmatrix} 0 & -yz & 0 \\ -yz & 0 & xy \\ 0 & xy & 0 \end{bmatrix} + \mu_{27} \begin{bmatrix} 0 & 0 & -yz \\ 0 & 0 & xz \\ -yz & xz & 0 \end{bmatrix}, \]

\[ \varepsilon_4^{(5)} = \mu_{28} \begin{bmatrix} 0 & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{bmatrix} + \mu_{29} \begin{bmatrix} 0 & x & 0 \\ x & 0 & z \\ 0 & z & 0 \end{bmatrix} + \mu_{30} \begin{bmatrix} 0 & 0 & x \\ y & 0 & 0 \\ 0 & y & 0 \end{bmatrix} + \mu_{31} \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_{32} \begin{bmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix}, \]

\[ \Gamma_5: \varepsilon_5^{(1)} = \mu_{34} \begin{bmatrix} 2y & x & 0 \\ x & 0 & -z \\ 0 & -z & 2y \end{bmatrix} + \mu_{35} \begin{bmatrix} 0 & y & -z \\ y & 2x & 0 \\ -z & 0 & 0 \end{bmatrix} + \mu_{36} \begin{bmatrix} 2z & 0 & x \\ 0 & 2z & -y \\ x & -y & 0 \end{bmatrix}, \]

\[ \varepsilon_5^{(2)} = \mu_{37} \begin{bmatrix} 2yz & xz & xy \\ xz & 0 & 0 \\ xy & 0 & 0 \end{bmatrix} + \mu_{38} \begin{bmatrix} 0 & yz & 0 \\ yz & 2xz & xy \\ 0 & xy & 0 \end{bmatrix} + \mu_{39} \begin{bmatrix} 0 & 0 & yz \\ 0 & 0 & xz \\ yz & xz & 2xy \end{bmatrix}, \]

\[ \varepsilon_5^{(3)} = \mu_{40} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_{41} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \mu_{42} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \]

\[ \varepsilon_5^{(4)} = \mu_{43} \begin{bmatrix} 4xy & (x^2 + y^2) & 0 \\ (x^2 + y^2) & 4xy & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_{44} \begin{bmatrix} 4xz & 0 & (x^2 + z^2) \\ 0 & 0 & 0 \\ (x^2 + z^2) & 0 & 4xz \end{bmatrix} + \mu_{45} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4yz \\ 0 & 0 & (y^2 + z^2) \end{bmatrix}, \]

\[ \varepsilon_5^{(5)} = \mu_{46} \begin{bmatrix} 2yz^2 & xz^2 & 0 \\ xz^2 & 0 & -x^2z \\ 0 & -x^2z & -2x^2y \end{bmatrix} + \mu_{47} \begin{bmatrix} 0 & yz^2 & -y^2z \\ yz^2 & 2xz^2 & 0 \\ -y^2z & 0 & -2xy^2 \end{bmatrix} + \mu_{48} \begin{bmatrix} 2y^2z & 0 & xy^2 \\ 0 & -2x^2z & -x^2y \\ xy^2 & -x^2y & 0 \end{bmatrix} \]
The above are the 54 ‘natural’ strain modes corresponding to the 60 parameter displacement field for the cube. In this case, the multiplicities \( \gamma_i \) in each representation \( \Gamma_i \) are \{2, 3, 5, 6, 7\}, and \( n_i \) as given earlier are \{1, 1, 2, 3, 3\} thus \( \sum n_i \gamma_i = 54 \).

Likewise an equilibrated stress field with constant and linear terms decomposes into the irreducible spaces

\[
\Gamma_1: \quad \sigma_1^{(\ell)} = \delta_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};
\]

\[
\Gamma_2: \quad \sigma_2^{(\ell)} = \delta_2 \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix};
\]

\[
\Gamma_3: \quad \sigma_3^{(\ell)} - \delta_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \delta_4 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},
\]

\[
\sigma_3^{(\ell)} = \delta_5 \begin{bmatrix} 0 & z & 0 \\ z & 0 & x \\ 0 & -x & 0 \end{bmatrix} + \delta_6 \begin{bmatrix} 0 & 0 & y \\ 0 & 0 & -x \\ y & -x & 0 \end{bmatrix};
\]

\[
\Gamma_4: \quad \sigma_4^{(\ell)} = \delta_7 \begin{bmatrix} y \\ 0 \\ y \end{bmatrix} + \delta_8 \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} + \delta_9 \begin{bmatrix} z \\ 0 \\ z \end{bmatrix}, \quad (1.33)
\]

\[
\sigma_4^{(\ell)} = \delta_{10} \begin{bmatrix} 2x & -y & -z \\ -y & 0 & 0 \\ -z & 0 & 0 \end{bmatrix} + \delta_{11} \begin{bmatrix} 0 & -x & 0 \\ -x & 2y & z \\ 0 & -z & 0 \end{bmatrix} + \delta_{12} \begin{bmatrix} 0 & 0 & -x \\ 0 & 0 & -y \\ -x & -y & 2z \end{bmatrix};
\]

\[
\Gamma_5: \quad \sigma_5^{(\ell)} = \delta_{13} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \delta_{14} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \delta_{15} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\sigma_5^{(\ell)} = \delta_{16} \begin{bmatrix} y \\ 0 \\ -y \end{bmatrix} + \delta_{17} \begin{bmatrix} 0 \\ x \\ -x \end{bmatrix} + \delta_{18} \begin{bmatrix} z \\ -z \\ 0 \end{bmatrix},
\]

\[
\sigma_5^{(\ell)} = \delta_{19} \begin{bmatrix} 0 & y & -z \\ y & 0 & 0 \\ -z & 0 & 0 \end{bmatrix} + \delta_{20} \begin{bmatrix} 0 & x & 0 \\ x & 0 & -z \\ 0 & -z & 0 \end{bmatrix} + \delta_{21} \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & -y \\ x & -y & 0 \end{bmatrix}.
\]
Likewise an equilibrated stress field with homogeneous quadratic variation decomposes into

\[
\Gamma_1: \quad \mathbf{\sigma}_1^{(2)} = \delta_22 \begin{bmatrix}
    x^2 & -xy & -xz \\
    -xy & y^2 & -yz \\
    -xz & -yz & z^2 
\end{bmatrix}, \quad \mathbf{\sigma}_1^{(3)} = \delta_{23} \begin{bmatrix}
    y^2 + z^2 \\
    x^2 + z^2 \\
    x^2 + y^2 
\end{bmatrix};
\]

(1.34a)

\[
\Gamma_2: \quad \mathbf{\sigma}_2^{(2)} = \delta_{24} \begin{bmatrix}
    -y^2 + z^2 \\
    x^2 - z^2 \\
    -x^2 + y^2 
\end{bmatrix};
\]

(1.34b)

\[
\Gamma_3: \quad \mathbf{\sigma}_3^{(3)} = \delta_{25} \begin{bmatrix}
    2(y^2 + z^2) & 0 & 0 \\
    0 & -(x^2 + z^2) & 0 \\
    0 & 0 & -(x^2 + y^2) 
\end{bmatrix}

+ \delta_{26} \begin{bmatrix}
    -(y^2 + z^2) & 0 & 0 \\
    0 & 2(x^2 + z^2) & 0 \\
    0 & 0 & -x^2 + y^2 
\end{bmatrix},
\]

(1.34c)

\[
\mathbf{\sigma}_3^{(4)} = \delta_{27} \begin{bmatrix}
    2(y^2 - z^2) & 0 & 0 \\
    0 & -(z^2 - x^2) & 0 \\
    0 & 0 & -(x^2 - y^2) 
\end{bmatrix}

+ \delta_{28} \begin{bmatrix}
    -(y^2 - z^2) & 0 & 0 \\
    0 & 2(z^2 - x^2) & 0 \\
    0 & 0 & -x^2 + y^2 
\end{bmatrix},
\]

(1.34d)

\[
\Gamma_4: \quad \mathbf{\sigma}_4^{(3)} = \delta_{31} \begin{bmatrix}
    2xy & (x^2 - y^2) & 0 \\
    (x^2 - y^2) & -2xy & 0 \\
    0 & 0 & 0 
\end{bmatrix}

+ \delta_{32} \begin{bmatrix}
    2xz & 0 & (x^2 - z^2) \\
    0 & 0 & 0 \\
    (x^2 - z^2) & 0 & -2xz 
\end{bmatrix}

+ \delta_{33} \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 2yz & (y^2 - z^2) \\
    0 & (y^2 - z^2) & -2yz 
\end{bmatrix},
\]

(1.34e)
Thus the complete equilibrated quadratic stress interpolation has multiplicities $\gamma_i = (3, 2, 5, 4, 7)$ with $\sum n_i \gamma_i = 48$. On the other hand it was shown that the strain field for a 20-node cube has multiplicities $(2, 3, 5, 6, 7)$. Keeping in mind the orthogonality relation cited earlier (1.24), since representations $\Gamma_2$ and $\Gamma_4$ occur fewer times among the quadratic stresses than among the strains, the quadratic stress interpolation necessarily contains mechanism modes. To eliminate them we must add cubic stresses. The equilibrated homogeneous cubic stress field decomposes into

$$\Gamma_1: \, \mathbf{\sigma}_1^{(4)} = \delta_{49} \begin{bmatrix} 0 & (y^2-x^2)z & (x^2-z^2)y \\ (y^2-x^2)z & 0 & (z^2-y^2)x \\ (x^2-z^2)y & (z^2-y^2)x & 0 \end{bmatrix} ;$$

$$\Gamma_2: \, \mathbf{\sigma}_2^{(4)} = \delta_{50} \begin{bmatrix} 0 & z^3 & y^3 \\ z^3 & 0 & x^3 \\ y^3 & x^3 & 0 \end{bmatrix} , \quad \mathbf{\sigma}_2^{(4)} = \delta_{51} \begin{bmatrix} -4xyz & z(x^2+y^2) & y(x^2+z^2) \\ z(x^2+y^2) & -4xyz & x(y^2+z^2) \\ y(x^2+z^2) & x(y^2+z^2) & -4xyz \end{bmatrix} ;$$

$$\Gamma_3: \, \mathbf{\sigma}_3^{(4)} = \delta_{52} \begin{bmatrix} 0 & 0 & -y^3 \\ 0 & 0 & x^3 \\ -y^3 & x^3 & 0 \end{bmatrix} + \delta_{53} \begin{bmatrix} 0 & -z^3 & 0 \\ -z^3 & 0 & 0 \\ y^3 & 0 & 0 \end{bmatrix} .$$

$$\mathbf{\sigma}_3^{(4)} = \delta_{54} \begin{bmatrix} 2xyz & z(x^2-y^2) & x^3y \\ (x^2-y^2)z & -2xyz & -y^2x \\ x^3y & -y^2x & 0 \end{bmatrix}$$

$$+ \delta_{55} \begin{bmatrix} 0 & y^2z & -z^2y \\ y^2z & 2xyz & x(y^2-z^2) \\ -z^2y & x(y^2-z^2) & -2xyz \end{bmatrix} ,$$

(1.35c)
\[ \sigma_3^{(6)} = \delta_{s6} \begin{bmatrix} 0 & -z(2x^2+y^2) & y(2x^2+z^2) \\ -z(2x^2+y^2) & 6xyz & x(y^2-z^2) \\ y(2x^2+z^2) & x(y^2-z^2) & -6xyz \end{bmatrix} + \delta_{s7} \begin{bmatrix} -6xyz & z(y^2-x^2) & y(x^2+2z^2) \\ z(y^2-x^2) & 6xyz & -x(y^2+2z^2) \\ y(x^2+2z^2) & -x(y^2+2z^2) & 0 \end{bmatrix}; \]

\[ \Gamma_4: \quad \sigma_4^{(5)} = \delta_{s8} \begin{bmatrix} 0 & x^3 \\ x^3 & \end{bmatrix} + \delta_{s9} \begin{bmatrix} y^3 \\ 0 \end{bmatrix} + \delta_{s6} \begin{bmatrix} z^3 \\ 0 \end{bmatrix}, \]

\[ \sigma_4^{(6)} = \delta_{s1} \begin{bmatrix} yz^2 \\ 0 \\ x^2y \end{bmatrix} + \delta_{s2} \begin{bmatrix} 0 \\ xz^2 \\ xy^2 \end{bmatrix} + \delta_{s3} \begin{bmatrix} zy^2 \\ xz^2 \\ 0 \end{bmatrix}, \]

\[ \sigma_4^{(7)} = \delta_{s4} \begin{bmatrix} 2x^3 & -3x^2y & -3y^2z \\ -3x^2y & 0 & 6xyz \\ -3x^2y & 0 & 6xyz \end{bmatrix} + \delta_{s5} \begin{bmatrix} 0 & -3xy^2 & 6xyz \\ -3xy^2 & 2y^3 & -3y^2z \\ -3xy^2 & 2y^3 & 0 \end{bmatrix} + \delta_{s6} \begin{bmatrix} 0 & 6xyz & -3xz^2 \\ 6xyz & 0 & -3yz^2 \\ -3xz^2 & -3yz^2 & 2z^3 \end{bmatrix}, \]

\[ \sigma_4^{(8)} - \delta_{s7} \begin{bmatrix} x^2y & 0 & -2xyz \\ 0 & 0 & 0 \\ -2xyz & 0 & z^2y \end{bmatrix} + \delta_{s8} \begin{bmatrix} 0 & 0 & 0 \\ 0 & xy^2 & -2xyz \\ 0 & -2xyz & xz^2 \end{bmatrix} + \delta_{s9} \begin{bmatrix} x^2z & -2xyz & 0 \\ -2xyz & y^2z & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ \sigma_4^{(9)} = \delta_{s10} \begin{bmatrix} x(y^2+z^2) & -yz^2 & -y^2z \\ -yz^2 & 0 & 0 \\ -y^2z & 0 & 0 \end{bmatrix} + \delta_{s11} \begin{bmatrix} 0 & -xz^2 & 0 \\ -xz^2 & y(x^2+z^2) & -x^2z \\ 0 & x^2z & 0 \end{bmatrix} + \delta_{s12} \begin{bmatrix} 0 & 0 & -xy^2 \\ 0 & 0 & -x^2y \\ -xy^2 & -x^2y & z(x^2+y^2) \end{bmatrix}, \]

\[ \sigma_4^{(10)} - \delta_{s13} \begin{bmatrix} 0 & 0 & x^3-3xy^2 \\ 0 & 0 & y^3-3x^2y \\ x^3-3xy^2 & y^3-3x^2y & 0 \end{bmatrix} + \delta_{s14} \begin{bmatrix} 0 & y^3-3yz^2 & z^3-3y^2z \\ y^3-3yz^2 & 0 & 0 \\ z^3-3y^2z & 0 & 0 \end{bmatrix} + \delta_{s15} \begin{bmatrix} 0 & x^3-3xz^2 & 0 \\ x^3-3xz^2 & 0 & z^3-3x^2z \\ 0 & z^3-3x^2z & 0 \end{bmatrix}; \]
We have given above the natural irreducible stress representation for an equilibrated cubic order stress field. We consider the least-order stable invariant stress field, such that the rank of the matrix $G$ of (1.8) is 54. Thus, out of the 90 stress modes given in (1.33-1.35) we choose only 54. The orthogonality relation (1.24), which can easily be verified for the strain field of (1.32) and the stress field of (1.33-1.35), insures that it suffices to compute the block of $G$ which involves the trace of stresses and strains belonging to the same irreducible representations $\Gamma_i$. The total multiplicity of the cubic stress representation given in (1.33-1.35) is $(4, 4, 8, 4, 10, 12)$ and that of the strain representation of (1.32) is $(2, 3, 5, 6, 7)$. Recall that the dimension of each of the representations is $(1, 1, 2, 3, 3)$. Thus, the dimensions of each of the non-zero blocks of $G$ are $\{(4 \times 2), (4 \times 3), (16 \times 10), (30 \times 18), (36 \times 21)\}$. If $G$ is written as
Then, each of the diagonal blocks can easily be computed, and these are shown in Table 1(a)–1(e).

We can now read off a least-order (54 parameter) stable, invariant stress interpolation at once, for a 20-noded element, from Tables 1(a)–1(e). From the inspection of the non-zero blocks of $\mathbf{G}$ as given in Tables 1(a)–1(e) we can choose this 54 parameter stress field to

(i) include the constant and linear terms $\sigma_1^{(1)}$, $\sigma_2^{(1)}$, $\sigma_3^{(1)}$, $\sigma_4^{(2)}$, $\sigma_5^{(2)}$, $\sigma_6^{(2)}$, and $\sigma_7^{(2)}$ of (1.33); these involve 21 parameters;

(ii) include either $\sigma_1^{(2)}$ or $\sigma_1^{(3)}$ of (1.34), but not $\sigma_1^{(4)}$ since it does zero work on both $\varepsilon_1^{(1)}$ and $\varepsilon_1^{(2)}$; this involves one more parameter;

(iii) include both $\sigma_2^{(2)}$ and $\sigma_2^{(4)}$, but not $\sigma_2^{(6)}$; this involves two more parameters;

(iv) include any two of $\sigma_3^{(3)}$, $\sigma_3^{(4)}$, and $\sigma_3^{(5)}$, and any one of $\sigma_3^{(7)}$ and $\sigma_3^{(8)}$; this involves 6 more parameters;

(v) include $\sigma_4^{(3)}$ and $\sigma_4^{(4)}$, any one of $\sigma_4^{(7)}$ and $\sigma_4^{(10)}$, and any one of $\sigma_4^{(6)}$ and $\sigma_4^{(9)}$; this involves 12 more parameters;

(vi) include either $\sigma_5^{(6)}$ or $\sigma_5^{(12)}$, and any three out of $\sigma_5^{(4)}$, $\sigma_5^{(5)}$, $\sigma_5^{(6)}$, and $\sigma_5^{(7)}$; this involves 12 more parameters.

It is seen from above that item (ii) involves two choices, item (iv) involves six choices, item (v) involves 4 choices, and item (vi) involves eight choices. Thus there are a total of 384 choices of a 54 parameter stable invariant stress field for a 20-noded element.

Of course, the above analysis includes the case of an 8-noded element as a subset. A separate detailed study of the 8-noded element case was initially presented in [8]. It can be seen from the above development or the details in [8] that for an 8-noded element the natural strain-modes consist of (a) $\varepsilon_1^{(1)}$ belonging to $\Gamma_1$; (b) $\varepsilon_2^{(1)}$ belonging to $\Gamma_2$; (c) $\varepsilon_3^{(1)}$ and $\varepsilon_3^{(2)}$

<table>
<thead>
<tr>
<th>Table 1(a)</th>
<th>$\sigma_1^{(1)}$</th>
<th>$\sigma_2^{(1)}$</th>
<th>$\sigma_3^{(1)}$</th>
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<td>16</td>
<td></td>
</tr>
<tr>
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<td>$-\frac{16}{3}$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_1^{(3)}$</td>
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<td>$\frac{24}{3}$</td>
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<tr>
<td>$\sigma_1^{(4)}$</td>
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<table>
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<th>Table 1(b)</th>
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<th>$\sigma_3^{(1)}$</th>
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<td>$\frac{32}{5}$</td>
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<tr>
<td>$\sigma_2^{(4)}$</td>
<td>$\frac{32}{3}$</td>
<td>0</td>
<td>$-\frac{54}{15}$</td>
</tr>
</tbody>
</table>
Table 1(c)
\[ \sigma(\Gamma_3) : e(\Gamma_3) \rightarrow \]

\[
\begin{array}{|c|cc|cc|cc|}
\hline
& e_3^{(1)} & e_3^{(2)} & e_3^{(3)} & e_3^{(4)} & e_3^{(5)} \\
\hline
\sigma_3^{(1)} & 16 & -8 & \frac{10}{3} & \frac{-8}{3} & \frac{-8}{3} & \frac{10}{3} \\
-8 & 16 & -\frac{8}{3} & \frac{10}{3} & \frac{10}{3} & \frac{-8}{3} \\
\hline
\sigma_3^{(2)} & \frac{7}{2} & \frac{16}{5} & \frac{16}{5} & \frac{16}{5} & \frac{16}{5} \\
\frac{16}{3} & \frac{16}{5} & \frac{16}{5} & \frac{16}{5} & \frac{16}{5} \\
\hline
\sigma_3^{(3)} & 16 & 0 & \frac{112}{15} & \frac{112}{15} & \frac{112}{15} \\
16 & 16 & \frac{112}{15} & \frac{112}{15} & \frac{112}{15} \\
\hline
\sigma_3^{(4)} & 0 & 0 & \frac{22}{3} & \frac{22}{3} & \frac{22}{3} \\
0 & 0 & \frac{22}{3} & \frac{22}{3} & \frac{22}{3} \\
\hline
\sigma_3^{(5)} & \frac{-8}{3} & \frac{16}{3} & \frac{-8}{3} & \frac{-8}{3} & \frac{-8}{3} \\
\frac{16}{3} & \frac{-8}{3} & \frac{-8}{3} & \frac{-8}{3} & \frac{-8}{3} \\
\hline
\sigma_3^{(6)} & \frac{-10}{5} & \frac{-22}{5} & \frac{-10}{5} & \frac{-10}{5} & \frac{-10}{5} \\
\frac{-10}{5} & \frac{-22}{5} & \frac{-10}{5} & \frac{-10}{5} & \frac{-10}{5} \\
\hline
\sigma_3^{(7)} & \frac{10}{5} & \frac{22}{5} & \frac{10}{5} & \frac{10}{5} & \frac{10}{5} \\
\frac{10}{5} & \frac{22}{5} & \frac{10}{5} & \frac{10}{5} & \frac{10}{5} \\
\hline
\sigma_3^{(8)} & \frac{-10}{5} & \frac{10}{5} & \frac{-10}{5} & \frac{-10}{5} & \frac{-10}{5} \\
\frac{10}{5} & \frac{-10}{5} & \frac{-10}{5} & \frac{-10}{5} & \frac{-10}{5} \\
\hline
\end{array}
\]

belonging to \( \Gamma_3 \); (d) \( e_3^{(1)} \) belonging to \( \Gamma_4 \); and (e) \( e_3^{(1)}, e_3^{(2)} \) and \( e_3^{(3)} \) belonging to \( \Gamma_5 \) as given in above (1.32). Thus there are \((1+1+4+3+9=18)\) strain modes. An 18-parameter stable invariant stress field can be easily identified from Tables 1(a)-1(e) as follows.

(i) Include \( \sigma_3^{(1)}, \sigma_3^{(2)}, \sigma_3^{(3)} \) (both terms), \( \sigma_3^{(4)} \) (both terms) and \( \sigma_3^{(5)} \) (all 3 terms).
(ii) Include \( \sigma_3^{(1)} \) (all 3 terms) or \( \sigma_3^{(2)} \) (all 3 terms).
(iii) Include \( \sigma_3^{(3)} \) (all 3 terms) or \( \sigma_3^{(4)} \) (all 3 terms).
(iv) Include \( \sigma_3^{(5)} \) (all 3 terms) or \( \sigma_3^{(6)} \) (all 3 terms).

It is thus seen that there are eight choices for an 18-parameter stable invariant stress field for an 8-noded cubic element.

Of all the possible choices, i.e., 384 for a 20-noded element, and 8 for an 8-noded element, which is "best"? Of course, all of these choices lead to the correct rank of the stiffness matrix. Even though each of these choices is a stable invariant stress field, almost all of these involve incomplete polynomial approximations to each stress component. In the remainder of this paper, we (i) compute the stiffness matrices for all the 8 choices for the 8-noded element, and 8 selected choices for the 20-node element; (ii) compute the eigenvalues of each of the stiffness matrices; (iii) identify the ones with lowest traces of eigenvalues for the 8-noded and 20-noded elements, respectively; and (iv) compare the eigenvalues of the 'best' hybrid-stress elements with those of the standard 'displacement' elements with 8 and 20 nodes, respectively. Before presenting these results.
Table 1(d)
\( \sigma(\Gamma_4) : \varepsilon(\Gamma_4) \rightarrow \)

<table>
<thead>
<tr>
<th>( \sigma^{(1)} )</th>
<th>( \varepsilon_{4}^{(1)} )</th>
<th>( \varepsilon_{4}^{(2)} )</th>
<th>( \varepsilon_{4}^{(3)} )</th>
<th>( \varepsilon_{4}^{(4)} )</th>
<th>( \varepsilon_{4}^{(5)} )</th>
<th>( \varepsilon_{4}^{(6)} )</th>
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<td>( \frac{32}{9} )</td>
<td>( \frac{32}{9} )</td>
</tr>
</tbody>
</table>

however, we briefly discuss earlier methods of assuming equilibrated stress fields that are complete polynomials through the use of stress functions.

1.3. The stress-function approach

In the early stages of development of hybrid-stress elements it was common to enforce the constraint \( s \geq d - r \) (see (1.11) and the discussion following it) and increase the number \( 's' \)
Table 1(e)

\[ \mathbf{\sigma}(I_5) : \mathbf{\varepsilon}(I_5) \rightarrow \]

<table>
<thead>
<tr>
<th>( \mathbf{\epsilon}^{(1)} )</th>
<th>( \mathbf{\epsilon}^{(2)} )</th>
<th>( \mathbf{\epsilon}^{(3)} )</th>
<th>( \mathbf{\epsilon}^{(4)} )</th>
<th>( \mathbf{\epsilon}^{(5)} )</th>
<th>( \mathbf{\epsilon}^{(6)} )</th>
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</thead>
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<td></td>
<td></td>
</tr>
<tr>
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<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
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<td>( \mathbf{\sigma}^{(6)} )</td>
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<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
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</tr>
<tr>
<td>( \mathbf{\sigma}^{(7)} )</td>
<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
<td>[\frac{10}{9}]</td>
</tr>
<tr>
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<td>[\frac{259}{15}]</td>
<td>[\frac{259}{15}]</td>
<td>[\frac{259}{15}]</td>
<td>[\frac{259}{15}]</td>
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</tr>
<tr>
<td>( \mathbf{\sigma}^{(9)} )</td>
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<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
</tr>
<tr>
<td>( \mathbf{\sigma}^{(10)} )</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
</tr>
<tr>
<td>( \mathbf{\sigma}^{(11)} )</td>
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<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
</tr>
<tr>
<td>( \mathbf{\sigma}^{(12)} )</td>
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<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
<td>[\frac{32}{9}]</td>
</tr>
</tbody>
</table>
until the kinematic deformation modes were removed [9]. In fact an interesting recent study [10] suggests that the problem of kinematic modes can be resolved if complete polynomial expansions are used for stress assumptions and the resultant abnormally large number of stress parameters can be reduced partially through the use of equilibrium as well as compatibility conditions. Similar attempts were made earlier in [11]. In all these cases the major drawback is that the number of stress parameters may be impractically large. However, for purposes of completeness of comparison we briefly discuss the development based on the stress-function approach.

It is well known that an equilibrated stress field in three dimensions can be derived through three stress functions (in the so-called Maxwell representation) as

\[
\begin{align*}
\sigma_{11} &= \frac{\partial^2 \phi_3}{\partial x_3^2} + \frac{\partial^2 \phi_1}{\partial x_1^2}, \\
\sigma_{22} &= \frac{\partial^2 \phi_2}{\partial x_2^2} + \frac{\partial^2 \phi_1}{\partial x_1^2}, \\
\sigma_{33} &= \frac{\partial^2 \phi_1}{\partial x_1^2} + \frac{\partial^2 \phi_2}{\partial x_2^2}.
\end{align*}
\]

Although it is clear from (1.37) that the stress functions \( \phi_i \) \( (i = 1, 2, 3) \) are at least of second order their polynomial composition is primarily determined by the characteristics of the sibling stress field, a factor which complicates their interpretation. Trial and error approaches have been found necessary in the past to identify the polynomial selections, complete or incomplete, which produce stiffness matrices of the correct rank. Furthermore, since the stress-function approach does not necessarily generate natural irreducible representations, it contains a significant degree of redundancy and invariably requires more stress parameters than the currently discussed least-order stress-space formulations. However, it is possible, through a direct (computer) examination of the rank of \( G \) to start out with a broad range of polynomial options and to eliminate the redundant parameters such that the desired rank of \( G \) is still maintained. We discuss some results of this trial and error process also in the following.

2. Numerical results

Since the primary aim of this paper is the study of kinematic modes at the element level, rather than presenting results for subjective loading cases, we restrict our attention in this part of the paper to eigenvalue studies of stiffness matrices. We consider a unit cube of linear elastic material \( (E = 10^7 \text{ psi}, \nu = 0) \). A quantitative measure of comparison of stiffness matrices used here is the ratio \( T/E \) where \( T \) is the trace (sum) of the eigenvalues. As another measure not only the trace but the individual eigenvalues as well of the hybrid-stress stiffness matrices are compared with those of the stiffness matrices of identical elements generated by the standard displacement method.

2.1. Least-order stable invariant stress spaces

(a) 8-noded element. As mentioned earlier, there are 8 choices of an 18-parameter stress
Table 2
Least-order stress selections (8-node)

<table>
<thead>
<tr>
<th>Choice</th>
<th>Selection</th>
<th>Trace/E</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO8: 1</td>
<td>$\sigma_4^{(1)}$, $\sigma_8^{(1)}$, $\sigma_8^{(6)}$</td>
<td>4.0139</td>
</tr>
<tr>
<td>LO8: 2</td>
<td>$\sigma_4^{(1)}$, $\sigma_8^{(5)}$, $\sigma_8^{(6)}$</td>
<td>4.0416</td>
</tr>
<tr>
<td>LO8: 3</td>
<td>$\sigma_4^{(2)}$, $\sigma_5^{(2)}$, $\sigma_5^{(6)}$</td>
<td>4.2638</td>
</tr>
<tr>
<td>LO8: 4</td>
<td>$\sigma_4^{(2)}$, $\sigma_5^{(2)}$, $\sigma_5^{(6)}$</td>
<td>4.7916</td>
</tr>
<tr>
<td>LO8: 5</td>
<td>$\sigma_4^{(1)}$, $\sigma_8^{(3)}$, $\sigma_8^{(6)}$</td>
<td>4.3888</td>
</tr>
<tr>
<td>LO8: 6</td>
<td>$\sigma_4^{(1)}$, $\sigma_8^{(3)}$, $\sigma_8^{(6)}$</td>
<td>4.4166</td>
</tr>
<tr>
<td>LO8: 7</td>
<td>$\sigma_4^{(1)}$, $\sigma_8^{(3)}$, $\sigma_8^{(6)}$</td>
<td>4.6388</td>
</tr>
<tr>
<td>LO8: 8</td>
<td>$\sigma_4^{(1)}$, $\sigma_8^{(3)}$, $\sigma_8^{(6)}$</td>
<td>4.6666</td>
</tr>
</tbody>
</table>

*All choices contain $\sigma_1^{(1)}$, $\sigma_2^{(1)}$, $\sigma_3^{(1)}$ and $\sigma_8^{(6)}$.

field; these are shown in Table 2. The choices are labeled in the order of increasing stiffness from which it is apparent that LO8:1 generates the most flexible stiffness matrix. It is interesting to observe that one of the 8 choices presented in Table 2, viz. LO8:2, is identical to that presented recently in [12]. An examination of all the 8 choices in Table 2 would reveal that, while each incorporates the constant stress states essential to an 8-noded linear displacement block, none has complete linear or quadratic order terms.

(b) 20-noded block. A 54-parameter (least-order) stable invariant stress field can accrue from any one of 384 previously defined choices, 8 of which are detailed and labeled in Table 3. In interpreting these selections it is possible to apply an aesthetic but by no means essential mechanics criterion of complete quadratic variation in all stress components, thereby reducing the options to 160. Table 4 lists the 8 chosen selections in the order of increasing stiffness. It is noteworthy that the selection LO20: 1, which has incomplete quadratic stresses, is the most

Table 3
Least-order stress combinations (20-node)

<table>
<thead>
<tr>
<th>Selection</th>
<th>(ii)</th>
<th>(iv)</th>
<th>(v)</th>
<th>(vi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO20: 1</td>
<td>$\sigma_1^3$</td>
<td>$\sigma_1^3$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_8^3$</td>
</tr>
<tr>
<td>LO20: 2</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_8^2$</td>
<td>$\sigma_8^2$</td>
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<tr>
<td>LO20: 3</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
</tr>
<tr>
<td>LO20: 4</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
</tr>
<tr>
<td>LO20: 5</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
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<tr>
<td>LO20: 6</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
</tr>
<tr>
<td>LO20: 7</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
</tr>
<tr>
<td>LO20: 8</td>
<td>$\sigma_2^2$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
<td>$\sigma_5^3$</td>
</tr>
</tbody>
</table>

*Each with essential terms: $\sigma_1^1$, $\sigma_2^1$, $\sigma_3^1$, $\sigma_4^1$, $\sigma_8^1$, $\sigma_8^1$, $\sigma_8^2$, $\sigma_8^3$, $\sigma_8^4$, $\sigma_8^5$, $\sigma_8^6$.

Table 4
Least-order stress selections (20-node)

<table>
<thead>
<tr>
<th>Selection</th>
<th>Trace/E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LO20: 1</td>
</tr>
<tr>
<td>2</td>
<td>LO20: 2</td>
</tr>
<tr>
<td>3</td>
<td>LO20: 3</td>
</tr>
<tr>
<td>4</td>
<td>LO20: 4</td>
</tr>
<tr>
<td>5</td>
<td>LO20: 5</td>
</tr>
<tr>
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</tr>
<tr>
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<td>LO20: 7</td>
</tr>
<tr>
<td>8</td>
<td>LO20: 8</td>
</tr>
</tbody>
</table>
flexible of the entire 384 possibilities. Selection LO20: 2 is the most flexible of the subset with complete quadratic terms. None of the stress selections in Table 3 have complete cubic variations in any stress component, but all give the requisite constant and linear stress spaces in this 20-noded quadratic displacement block.

2.2. Stress function approach

(a) 8-noded block. In assessing the performance of the stress function algorithm with respect to that of least-order stress-spaces, two stress-function polynomial selections are considered. The first, denoted SF8-27, has a total of 27 stress parameters and reproduces the incomplete stress field of optimal least-order 18-parameter selection LO8: 1 of Table 2.

The second, SF8-48, produces a complete quadratic stress field by means of 48 stress-function parameters. Both SF8-27 and SF8-48 have sparse incomplete quadratic, cubic and quartic stress functions, the polynomials of which are presented in Table 5 (only $\phi_1$ is listed, from which $\phi_2$ and $\phi_3$ can be generated by cyclic permutation).

Computed stiffness factors $T/E$ for the elements LO8: 1, SF8-27, SF8-48, and the standard assumed-displacement 8-noded element (labeled here as DM8) are shown in Table 6 from which it is seen that all the hybrid-stress elements are more ‘flexible’ than the standard displacement element. Least-order LO8: 1 is optimal but is almost matched in performance by SF8-27. However, SF8-48, although containing ‘complete’ stress assumptions, possesses too many terms and is almost as ‘stiff’ as the standard displacement element.

These characteristics are further confirmed by a study of the eigenvalues and eigenvalues ratios (i.e., ratio of individual eigenvalues of the standard displacement element to the corresponding eigenvalue of the hybrid-stress element) shown in Table 7. Displacement element eigenvalues are markedly greater than those of LO8: 1 and SF8-27, indicating greater frequencies of vibration and stiffer behavior, but differ little from those of SF8-48!

(b) 20-noded element. The investigation of the 20-noded block is analogous to that of the 8-noded case and involves two polynomial selections. A 72-term selection, SF20-72, represents an attempt to duplicate the stress field of the least-order stress selection LO20: 2 while a 90-term choice delivers complete cubic stresses. The polynomial terms of both SF20-72 and SF20-90 are listed in Table 8, each consisting of incomplete quadratics, cubics, quartics and quintics.

As in the 8-noded case the excessive stiffness of the 20-noded standard displacement element (labelled here as DM20) with respect to the hybrid models LO20: 2, SF20-72 and

<table>
<thead>
<tr>
<th>Selection</th>
<th>Term #</th>
<th>Terms in $\phi_1^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SF8-27/48</td>
<td>Quadratic</td>
<td>1-6 y^2, yz</td>
</tr>
<tr>
<td>SF8-27/48</td>
<td>Cubic</td>
<td>7-15 y^2z, yz^2, xyz</td>
</tr>
<tr>
<td>SF8-27/48</td>
<td>Quartic</td>
<td>16-27 y^3z, yz^3, xy^2z, xyz^2</td>
</tr>
<tr>
<td>SF8-48</td>
<td>Cubic</td>
<td>28-33 xy^2, xz^2</td>
</tr>
<tr>
<td>SF8-48</td>
<td>Quadratic</td>
<td>34-48 x^2y^2, x^2yz, x^2z^2, y^3z^2, xy^3</td>
</tr>
</tbody>
</table>

Table 5
Stress function polynomials (SF8-27/48)

Table 6
Comparison of model flexibility (8-node)

<table>
<thead>
<tr>
<th>Model</th>
<th>Trace/E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LO8: 1</td>
</tr>
<tr>
<td>2</td>
<td>SF8-27</td>
</tr>
<tr>
<td>3</td>
<td>SF8-48</td>
</tr>
<tr>
<td>4</td>
<td>DM8</td>
</tr>
</tbody>
</table>

* $\phi_2$, $\phi_3$ by cyclic permutation.
Table 7

<table>
<thead>
<tr>
<th>Normalized eigenvalues*</th>
<th>Eigenvalue ratios</th>
</tr>
</thead>
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<tr>
<td></td>
<td>DM8</td>
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<tr>
<td>1 0 0 0 0 0</td>
<td>0 0 0 0 0 0</td>
</tr>
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</tr>
<tr>
<td>0 0 0 0 0 0</td>
<td>0 0 0 0 0 0</td>
</tr>
<tr>
<td>0.08333 0.04166 0.041666 0.08333</td>
<td>2.0 2.0 1.0</td>
</tr>
<tr>
<td>0.08333 0.04166 0.041666 0.08333</td>
<td>2.0 2.0 1.0</td>
</tr>
<tr>
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<td>2.666 2.666 1.0666</td>
</tr>
<tr>
<td>0.111111 0.046296 0.0486111 0.104166</td>
<td>2.4 2.2857 1.0666</td>
</tr>
<tr>
<td>0.111111 0.046296 0.0486111 0.104166</td>
<td>2.4 2.2857 1.0666</td>
</tr>
<tr>
<td>0.25 0.046296 0.0486111 0.20833</td>
<td>5.4 5.14285 1.2</td>
</tr>
<tr>
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<td>3.0 3.0 1.2</td>
</tr>
<tr>
<td>0.25 0.08333 0.08333 0.20833</td>
<td>3.0 3.0 1.2</td>
</tr>
<tr>
<td>0.25 0.08333 0.08333 0.20833</td>
<td>3.0 3.0 1.2</td>
</tr>
<tr>
<td>0.25 0.08333 0.08333 0.20833</td>
<td>3.0 3.0 1.2</td>
</tr>
<tr>
<td>0.3333 0.3333 0.3333 0.3333</td>
<td>1.0 1.0 1.0</td>
</tr>
<tr>
<td>0.5 0.5 0.5 0.5</td>
<td>1.0 1.0 1.0</td>
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<td>1.0 1.0 1.0</td>
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<td>1.0 1.0 1.0</td>
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</tr>
<tr>
<td>0.5 0.5 0.5 0.5</td>
<td>1.0 1.0 1.0</td>
</tr>
</tbody>
</table>

*All $\times (0.1 \times 10^8)$, i.e. $E$.

SF20-90 is confirmed by the calculated $T/E$ factors, shown in Table 9. Of the choices presented in Table 9, LO20: 2 is most flexible but even the case SF20-72 is substantially stiffer, a possible explanation lying in the fact that while SF20-72 at least replicates the stress field of LO20: 2 it incorporates additional stress terms as well.

The features of this stiffness factor comparison carry over to the examination of individual eigenvalues and eigenvalue ratios in Table 10. The least-order selection LO20: 2 is considerably more flexible than the displacement element at each corresponding eigenmode, while SF20-72 and SF20-90 are moderately so.

We note that while a stress-function approach was outlined earlier in [11], the stiffness matrix was not adequately investigated. An 8-noded block with 21 stress-function parameters as indicated in [11] was found by the present authors to have three kinematic modes. In likewise fashion a 20-noded block with 57 parameters had six kinematic modes. However, a relative eigenvalue analysis of an 8-noded block with an unspecified number of parameters was shown [11] to produce mildly stiff eigenvalue ratios identical to those of the SF8-48 case shown in the present Table 7.
### Table 8
Stress function polynomials (SF20-72/90)

<table>
<thead>
<tr>
<th>Selection</th>
<th>Term #</th>
<th>Terms in $\phi_1^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SF20-72/90</td>
<td>Quadratic</td>
<td>1–6 $y^2$, $yz$</td>
</tr>
<tr>
<td>SF20-72/90</td>
<td>Cubic</td>
<td>7–21 $x y^2$, $x^2 z$, $y^2 z$, $x y z$, $x y z$</td>
</tr>
<tr>
<td>SF20-72/90</td>
<td>Quartic</td>
<td>22–48 $x^2 y^2$, $x^2 z^2$, $x^2 y z$, $x y^3$, $x y^2 z$, $x y z^2$, $y^2 z^2$, $y^3 z^3$</td>
</tr>
<tr>
<td>SF20-72/90</td>
<td>Quintic</td>
<td>49–72 $x^4 z$, $x^3 y z$, $x^3 y^3$, $x^3 y^2 z$, $y^5$, $y^2 z^3$, $y z^4$, $z^5$</td>
</tr>
<tr>
<td>SF20-90</td>
<td>Quintic</td>
<td>73–90 $x^4 y$, $x^3 y z^2$, $x^2 y z^3$, $x y^3 z$, $y^4 z$, $y^3 z^2$</td>
</tr>
</tbody>
</table>

*$\phi_2$, $\phi_3$ by cyclic permutation.

### Table 9
Comparison of model flexibility (20-node)

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<tr>
<th>Model</th>
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<tbody>
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<tr>
<td>2</td>
<td>SF20-72</td>
</tr>
<tr>
<td>3</td>
<td>SF20-90</td>
</tr>
<tr>
<td>4</td>
<td>DM20</td>
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</table>

### Table 10
Normalized eigenvalues*  
Eigenvalue ratios

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<th></th>
<th>DM20</th>
<th>LO20: 2</th>
<th>SF20-72</th>
<th>SF20-90</th>
<th>DM20</th>
<th>LO20: 2</th>
<th>SF20-72</th>
<th>SF20-90</th>
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<td>0</td>
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<td>0.0339</td>
<td>0.036995</td>
<td>31.2598</td>
<td>1.1118</td>
<td>1.0188</td>
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<td>0.0012057</td>
<td>0.0339</td>
<td>0.036995</td>
<td>31.2598</td>
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<td>1.0188</td>
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<td>0.036995</td>
<td>31.2598</td>
<td>1.1118</td>
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<tr>
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<td>0.036995</td>
<td>0.049021</td>
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<td>0.04604</td>
<td>0.049021</td>
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<td>0.053037</td>
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<tr>
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<td>0.053037</td>
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<td>1.59296</td>
<td>1.03347</td>
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<tr>
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</table>
### Table 10 (continued)

<table>
<thead>
<tr>
<th>Normalized eigenvalues*</th>
<th>Eigenvalue ratios</th>
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<tbody>
<tr>
<td></td>
<td>DM20</td>
</tr>
<tr>
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</tbody>
</table>

*All \( (0.1 \times 10^n) \), i.e. E.

### 3. Concluding remarks

We believe that the presently described way of constructing stable invariant stress-spaces for cubes and squares is a good and essential first step in removing some of the curses that
have plagued the hybrid-stress method since the beginning. Further work is necessary and is currently underway in the authors' group, concerning the extension of the present concepts to isoparametric distorted elements as well as the planar $C^1$ elements of the plate bending type.

Appendix A. 2-dimensional 4-noded square element

The symmetry group $G$ of a square, consisting of rotations and reflections, has the following representations

\[
\begin{align*}
C_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; & C_2 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; & C_3 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \\
C_4 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; & C_5 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\end{align*}
\]

As argued in [6] the above group must have 5 irreducible representations. As before, representation $\Gamma_1$ can, in this case, be defined by the action of $G$ on $(x^2 + y^2)$, which leads to

\[
\Gamma_1: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\]

(A.2)

$\Gamma_2$ is defined through action of $G$ on $(x^2 - y^2)$

\[
\Gamma_2: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}
\]

(A.3)

and $\Gamma_3$ is defined through action of $G$ on $(xy)$

\[
\Gamma_3: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \end{bmatrix}
\]

(A.4)

$\Gamma_4$ can be defined through action of $G$ on $(xy^2 - yx^2)$

\[
\Gamma_4: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\]

(A.5)

$\Gamma_5$ can be defined through its action on $(x, y)$ and thus is similar to $G$.

Now, for a 4-noded square, the displacement representation is

\[
U = (1, x, y, xy)X + (1, x, y, xy)Y.
\]

(A.6)

Using procedures described earlier we obtain the strain decomposition into irreducible subspaces
We seek a 5-parameter stress field. A linear equilibrated stress field with 7 parameters has the form

\[ \sigma = [\mu_1 + \mu_2 x + \mu_3 y]X^2 + [\mu_4 + \mu_5 x + \mu_6 y]Y^2 + 2[\mu_7 - \mu_8 x - \mu_9 y]XY. \]  

(A.8)

The irreducible representation of the above stress field is

\[ \Gamma_1: \quad \mathbf{\sigma}_1 = \beta_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \Gamma_2: \quad \mathbf{\sigma}_2 = \beta_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \Gamma_3: \quad \mathbf{\sigma}_3 = \beta_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \Gamma_5: \quad \mathbf{\sigma}_5^{(1)} = \beta_4 \begin{bmatrix} 0 \\ x \\ -y \end{bmatrix} + \beta_5 \begin{bmatrix} -x \\ y \\ 0 \end{bmatrix}, \quad \mathbf{\sigma}_5^{(2)} = \beta_6 \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} + \beta_7 \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}. \]  

(A.9)

The matrix \( \mathbf{G} \) can be easily assembled as

\[
\begin{array}{cccc|c}
\sigma_1 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\
\hline
\sigma_1 & 0 & NZ & 0 & 0 & 0 \\
\sigma_2 & 0 & NZ & 0 & 0 \\
\sigma_3 & 0 & 0 & NZ & 0 \\
\sigma_4 & 0 & 0 & 0 & (2 \times 2) \\
\sigma_5 & 0 & 0 & 0 & (2 \times 2) \\
\end{array}
\]  

(A.10)

where NZ represent 'non-zero'. Each of the \( 2 \times 2 \) matrices in the last column of (A.10) is non-singular.

Thus a 5-parameter stable, invariant stress-space for the 4-noded square is: \( \mathbf{\sigma}_1, \mathbf{\sigma}_2, \mathbf{\sigma}_3, \) and \( \mathbf{\sigma}_5^{(1)} \) or \( \mathbf{\sigma}_5^{(2)} \). Thus there are two choices: the one involving \( \mathbf{\sigma}_5^{(1)} \) is identical to that presented in [12].

Acknowledgment

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