The meshless local Petrov-Galerkin method for the analysis of heat conduction due to a moving heat source, in welding

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\section{Introduction}

With recent advances in numerical simulation methods and increases in computer processing power, it has become increasingly possible to perform structural analysis of welding phenomena with a sufficiently high level of accuracy, by FEM thermal–elastic–plastic analysis. However, if the object to be analyzed is large (e.g., a hull structure) and it has a complex shape, much time is required in preprocessing, and in performing tasks such as mesh generation, which can be problematic.

Meshless methods such as the meshless local Petrov-Galerkin (MLPG) method \cite{1-6, 10} have been developed that have the potential to resolve these problems. The MLPG method is a completely meshless method, which never employs mesh divisions in analysis. Consequently, information on the connectivity between nodal points and elements is not required, and the preprocessing time is expected to be significantly reduced, and thus the operational efficiency increases. Accordingly, the MLPG method has attracted attention as a next-generation method for structural analysis to replace FEM.

The MLPG method has been applied to steady heat conduction problems \cite{11, 12, 18} and transient heat conduction/transfer problem \cite{13-16}. In this study, our objective is to investigate the analysis of transient deformation and stress under welding, using the MLPG method. Toward this end, the heat conduction analysis of the moving heat source problem, which is necessary in transient welding, is studied using the MLPG method. Many welding problems involve large local temperature gradients and the heat source moves. The material properties are temperature dependent. In this study, we focus on these aspects and investigate the applicability of the MLPG method to welding problems. The method is applied to bead-on-plate welds, which are produced using a moving heat source. The fundamental characteristics of the method are examined. The influence of the nodal point distribution density (number of nodal points) is investigated, and adaptive analysis methods involving the addition and removal of nodal points during analysis, which are expected to be applicable to welding problems are also studied.

\section{Theory of heat conduction analysis by the MLPG method}

In the MLPG method, the moving least-squares (MLS) approximation \cite{7} is often used to create an approximation function for the physical parameter to be solved. In this section, an overview of this
method is given and the formulation of heat conduction analysis
methods based on the concept of MLPG5 [7], which is reported to
have a relatively high accuracy, is described.

2.1. MLS approximation

The approximation function for temperature at an arbitrary
evaluation point \( T^q(x) \) is defined as:

\[
T^q(x) = \sum_{j=1}^{m} p_j(x)a_j(x) = \mathbf{p}^T(x)a(x)
\]  
(1)

In this case, in a two-dimensional problem, \( \mathbf{p}^T(x) \) is a basis
function vector defined by the following equations:

\[
\mathbf{p}^T(x) = [1, x, y] \quad \text{Linear basis : Components : 3}
\]  
(2)

\[
\mathbf{p}^T(x) = [1, x, y, x^2, xy, y^2] \quad \text{Quadratic basis : Components : 6}
\]  
(3)

\( a(x) \) in Eq. (1) is an undetermined coefficient vector, which is
decided by minimizing the evaluation function \( J \) as shown below:

\[
J = \sum_{k=1}^{N} w(x - x_k) \left[ \mathbf{p}^T(x_k)a(x) - \bar{T} \right]^2
\]  
(4)

In this case, \( N \) represents the number of nodal points near an
arbitrary evaluation point and \( \bar{T} \) indicates the nodal temperature.
\( x_k \) indicates the nodal coordinate vector located near an arbitrary
evaluation point \( I \) (see Fig. 1). Here, \( \Omega_{tr}^k \) in Fig. 1 is the region of
influence of each nodal point (local sub-domain), \( \rho_k \) gives the radius of
the area where it is assumed to be circular. \( r_k \) indicates the distance between the evaluation point and the surrounding node,
and \( w(x - x_k) = w(r_k) \) indicates the weight function when the
least-squares method is applied. In this study, the following quartic
spline function is used:

\[
w(r_k) = \begin{cases} 
1 - 6 \left( \frac{r_k}{\rho_k} \right)^2 + 8 \left( \frac{r_k}{\rho_k} \right)^3 - 3 \left( \frac{r_k}{\rho_k} \right)^4 & 0 \leq r_k \leq \rho_k \\
0 & r_k \geq \rho_k
\end{cases}
\]  
(5)

From the stationary condition for the undetermined coefficient
vector \( a(x) \) in Eq. (4), we obtain:

\[
\begin{pmatrix} A(x) \\ a(x) \end{pmatrix} = \begin{pmatrix} B(x) \\ T \end{pmatrix}
\]  
(6)

Here, \( A(x) \), \( B(x) \), and \( T \) are given by:

\[
A(x) = \sum_{k=1}^{N} w(x - x_k)p(x_k)p^T(x_k)
\]  
(7)

\[
B(x) = \left[ w_1(x)p(x_1), w_2(x)p(x_2), \ldots, w_N(x)p(x_N) \right]
\]  
(8)

\[
\bar{T} = \left[ \begin{array}{c} T_1 \\ \vdots \\ T_N \end{array} \right] = \left[ \begin{array}{c} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{array} \right]
\]  
(9)

By substituting \( a(x) \) obtained from Eq. (6) into Eq. (1), the
following equation, which includes the interpolation function \( \phi(x) \),
is derived.

\[
T^q(x) = \sum_{k=1}^{N} \phi^k(x)\bar{T}^k
\]  
(10)

Here, \( \phi^k(x) \) is given by:

\[
\phi^k(x) = \sum_{j=1}^{m} p_j(x) \left[ A^{-1}(x)B(x) \right]_{jk}
\]  
(11)

2.2. Heat conduction analysis by the MLPG method

In general, the governing equation for the two-dimensional
unsteady heat conduction problem in the domain \( \Omega \) bounded by \( \Gamma \)
(see Fig. 2) is described by:

\[
cp \frac{\partial T}{\partial t} = \lambda \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \dot{Q} \quad \text{in } \Omega
\]  
(12)

where \( T \) is the temperature, \( c \) is the specific heat, \( \rho \) is the density, \( \lambda \)
is the heat conductivity, and \( \dot{Q} \) is heat input per unit time per unit
volume. The parameters \( c, \rho, \) and \( \lambda \) are all temperature dependent
and they change significantly when the local temperature increases in
welding problems (see Fig. 6 in the next section).

It is possible to express Eq. (12) by the Eq. (13) when the
weighted residual method is applied to each integration region \( \Omega_{te} \).

\[
\int_{\Omega_{te}} \left\{ c \frac{\partial T}{\partial t} - \lambda \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) - \dot{Q} \right\} \, v \, d\Omega = 0
\]  
(13)

where \( v \) is the weight function. When we integrate Eq. (13) by parts
and use the divergence theorem, we obtain:

Fig. 1. Definition of domain of MLS approximation for trial function at point I.

Fig. 2. Schematic illustration of MLPG method.
\[
\begin{align*}
\int_{\Omega_r} \left( \frac{\partial \mathbf{T}}{\partial t} - \mathbf{Q} \right) \nu \, d\Omega + \int_{\Omega_r} \lambda \left( \frac{\partial \nu}{\partial x} + \frac{\partial \nu}{\partial y} \right) d\Omega \\
- \int_{\partial \Omega_r} \lambda \left( \frac{\partial \mathbf{T}}{\partial x} n_x + \frac{\partial \mathbf{T}}{\partial y} n_y \right) v \, d\Gamma = 0
\end{align*}
\]

where \( \partial \Omega_r \) represents the boundary of the integration region, and \( n = [n_x, n_y] \) is the outward normal vector at that boundary.

When the Heaviside function is applied as follows for the weight function \( v \) for the integration region \( \Omega_r \),

\[
v(x) = \begin{cases} 1 & \text{at } x \in \Omega \\ 0 & \text{at } x \notin \Omega \end{cases}
\]

it is possible to transform the equation (14) as:

\[
\int_{\Omega_r} \left( \frac{\partial \mathbf{T}}{\partial t} - \mathbf{Q} \right) \nu \, d\Omega + \int_{\partial \Omega_r} \lambda \left( \frac{\partial \nu}{\partial x} n_x + \frac{\partial \nu}{\partial y} n_y \right) v \, d\Gamma = 0
\]

If we consider the boundary conditions for Eq. (16) to be the temperature constraint by the penalty method and the heat transfer to the outside, then it can be expressed by the following equation, which includes the penalty coefficient \( \alpha \) and the heat transfer coefficient \( \beta \).

\[
\int_{\Omega_r} \left( \frac{\partial \mathbf{T}}{\partial t} - \mathbf{Q} \right) \nu \, d\Omega - \int_{\Gamma_{ss}} \alpha(T - T_0) d\Gamma + \int_{\Gamma_{sy}} \beta(T - T_0) d\Gamma = 0
\]

where \( T \) indicates the temperature on the temperature constraint boundary \( \Gamma_{ss} \) and \( T_0 \) indicates the external temperature of the heat transfer boundary \( \Gamma_{sy} \). \( \Gamma_{ss} \) and \( \Gamma_{sy} \) are parts of the boundary on the integration region \( \partial \Omega_r \), and sections positioned on the boundary \( \Gamma \) of the object to be analyzed. \( \Gamma_{ss} \) indicates the other sections in the object. Consequently, if \( \Omega_r \) is fully contained within the object to analyze, and there is no exchange of heat with the outside, then the left-hand side of Eq. (17) can be simplified to just the integral terms relating to \( \Omega_r \) and \( \Gamma_{ss} \). In such cases, Eq. (17) can be transformed as:

\[
\int_{\Omega_r} \left( \frac{\partial \mathbf{T}}{\partial t} - \mathbf{Q} \right) \nu \, d\Omega - \int_{\Gamma_{ss}} \alpha(T - T_0) d\Gamma + \int_{\Gamma_{sy}} \beta(T - T_0) d\Gamma = 0
\]

Moreover, by using the MLS approximation described in Eq. (10) to determine the temperature at the evaluation point by interpolating from the temperature of the surrounding node \( \mathcal{T} \), and rearranging for \( \mathcal{T} \), the following is obtained:

\[
C_i \sum_{j=1}^N \left( C_{ij} \mathcal{T}_j + K_{ij} \mathcal{T} \right) = f_i
\]

where \( C_i \), \( K_{ij} \), and \( f_i \) are given by:

3. Investigation of analysis of the moving heat source problem of welding by MLPG heat conduction analysis and MLPG parameters

To verify the appropriateness of the MLPG heat conduction analysis shown in Section 2, an analysis is conducted of the head-on-plate welding, which is a moving heat source problem in welding.

3.1. Analysis conditions

Fig. 3 shows the shape and size of a plate of the head-on-plate weld problem to be analyzed. The plate length \( l \) is set to 200 mm, the breadth \( b \) is set to 200 mm, and the height \( h \) is set to 2 mm. Analysis is conducted on only half of the modeled plate (the part indicated by diagonal lines in Fig. 3) by considering the symmetry of the problem. Fig. 4 shows the nodal point distribution when there are 1891 nodal points. The support size (SS), a parameter specific to the MLPG method shown in Fig. 5, indicates the radius of the region over which MLS approximation is conducted when determining the temperature at an arbitrary evaluation point, while the test function size (TS) indicates the region of the radius over which the numerical integration is performed to satisfy Eq. (19). In this study, a linear basis is used in the MLS. The welding conditions are taken to be a heat input of \( Q = 100 \) J/mm and a welding speed of \( v = 10 \) mm/s. Fig. 6 shows the assumed temperature dependency of the physical constants, including the material properties.

3.2. Influence of MLPG parameters

First, to evaluate the fundamental properties of this method, an analysis is conducted that does not account for the temperature dependency of the material or heat transfer on the plate surface, in order to compare with theoretical values. As an index of the characteristics of the temperature distribution, accuracy verification is investigated using the average temperature increase \( \Delta T_{av} \) which is
an index of the degree of preservation of the total heat input into the plate, and the molten length $L_m$ in the welding direction on the weld line, which is an index of the heat conductivity. The average temperature increase $\Delta T_{av}$ and the average temperature after complete cooling $T_{av}$ are described by:

$$\Delta T_{av} = \frac{Q_{net}}{c \rho L B h}$$  \hspace{1cm} (22)

$$T_{av} = T_{room} + \Delta T_{av}$$  \hspace{1cm} (23)

In this case, $Q_{net}$ indicates the total heat input into the plate. Specific heat $c$ and density $\rho$ indicate the values of physical properties at room temperature. $T_{room}$ indicates room temperature, which in this study is taken to be 15 °C. In addition, the molten length $L_m$ is defined as the maximum length in the direction of the welding line in the region that is over 1200 °C during welding.

To begin with, to investigate the influence of TS, on the average temperature of the plate after complete cooling (the average temperature increase $\Delta T_{av}$), a series analysis is conducted for cases when there are 861 and 1891 nodal points. Fig. 7 shows the results of this. The horizontal axis indicates TS and the vertical axis represents the estimation error $e_{av}$, calculated using the average temperature increase of the plate $\Delta T_{MLPG}$ after complete cooling and the theoretical average temperature increase $\Delta T_{Theoretical}$ is determined using Eq. (22).

$$e_{av} = \frac{\Delta T_{MLPG} - \Delta T_{Theoretical}}{\Delta T_{Theoretical}}$$  \hspace{1cm} (24)

Fig. 7(a) is when there is a relatively low number of nodal points (861) and (b) is the case when there are a large number of nodal points (1891). The two graphs reveal that the influence of SS and TS decrease as the number of nodal points increases and that when there are 1891 nodal points, the differences compared to the theoretical values in both cases are less than 4%. In particular, in the case from 0.3d to 0.8d in TS, the error is less than 1%.

Fig. 8 shows the effect of TS and SS on molten length $L_m$ at 15 s after the commencement of welding. Fig. 8(a) shows the case when the number of nodal points is relatively low at 861, whereas Fig. 8(b)
shows the case when the number is large at 1891. The dotted lines in the graphs is the analytical solution when the mesh division is sufficiently fine, which is obtained by FEM analysis. These graphs show that when TS is larger, \( L_m \) approaches the value obtained by FEM analysis. The best agreement is obtained when TS is between 0.6 and 0.8. In these cases, Fig. 7 confirms that the average temperature increase \( T_{av} \) also agrees with the theoretical value.

Fig. 9 (a) shows a temperature distribution for this case (nodal points: \( SS = 2.3d \), \( TS = 0.6d \)) obtained by analysis by the MLPG method. Fig. 9(b) shows the results obtained by conducting the same analysis using FEM. These two graphs show that when the two indexes investigated in Figs. 7 and 8 (the average temperature increase \( T_{av} \), which is an index of the degree of preservation of total heat input, and molten length \( L_m \), which is an index of the heat conductivity) approximately agree, the temperature distributions also agree. Moreover, the sparsity of the matrix of the simultaneous equations to be solved can be considered to be an objective index for the computation time. In relation to FEM, which uses the two-dimensional isoparametric elements shown in Fig. 8, when there are 1891 nodal points, the ratio of non-zero components to total components in the matrix to be solved is 0.26%. On the other hand, it is clear in relation to the related matrix of the MLPG method in Eq. (21) \( K_{ij} \) that if, for example, \( SS = 2.1d \), there is a 0.66% ratio, and if \( SS = 2.3d \) or 2.6d, there is a 1.06% ratio of non-zero components. Table 1 shows these results, compiled in a form that allows the accuracies to be compared. Here, only the cases \( TS = 0.3d \) and 0.6d are shown.

### 3.3. Influence of heat transfer and the temperature dependency of the material properties

In the previous section, to be compared with the analytical solution, the analysis is conducted that does not account for the temperature dependency of the material properties or for heat transfer. Hence, in this section, we perform the analysis using an analytical model of the same kind as the actual phenomena. Specifically, the analysis in this section differs from the analytical model in the previous section in that it considers heat transfer on the plate surface and the temperature dependencies of the physical constants, including the material properties (see Fig. 6).

Fig. 10 shows the results of molten length \( L_m \) at 15 s after commencement of welding. The horizontal axis indicates the number of nodal points, and the vertical axis represents the molten length \( L_m \). Fig. 11 shows the temperature distribution when \( SS = 2.3d \) and \( TS = 0.6d \). Figs. 10 and 11 show that it is possible to perform a very high accuracy analysis using \( TS = 0.6d \), even when the temperature dependency of the material and heat transfer coefficient are taken into account.

In addition, Fig. 12 shows the temperature history at points A–F that are indicated in the graph. Fig. 12(b) is an enlargement of Fig. 12(a). The broken line in the graph shows the FEM results, while the solid line indicates the MLPG analysis results when \( SS = 2.3d \) and \( TS = 0.6d \). The solid and broken lines roughly coincide with each other in these graphs. This indicates that a sufficiently accurate analysis can be performed using the MLPG method regardless of the heating process, until the cooling process, even when the

### Table 1

<table>
<thead>
<tr>
<th>Ratio of Non-zero elements</th>
<th>( SS = 2.1d )</th>
<th>( SS = 2.3d )</th>
<th>( SS = 2.6d )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NP: 861</strong></td>
<td>0.06 (%)</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td><strong>NP: 1841</strong></td>
<td>0.66 (%)</td>
<td>1.06</td>
<td>1.06</td>
</tr>
</tbody>
</table>

temperature dependency of the material constants and the heat transfer coefficient are taken into account.

3.4. Investigation of flexibly configurable nodal point distances

One of the characteristics of welding problems is local nonlinearity. In welding problems, the heating caused by the welding arc affects only a very narrow local melted area relative to the whole analysis region. Therefore, when we consider the nodal point distribution, we can assume that the computation time can be reduced by concentrating the nodal point distribution only in the vicinity of the welding line (i.e., the area in which the temperature change is larger) and employing a coarser distribution which is farther from the welding area.

Hence, in this study, we created an analytical model in which the intervals between nodal points become larger with increasing distance from the welding area. Fig. 13 shows this distribution of the nodal points. In this way, in analysis using the MLPG method, the nodal point distribution can be set relatively freely. However, it is difficult to generate a freely arranged nodal point distribution like that is shown in Fig. 13 is for a four-node quadrilateral element in FEM. In such cases, the analytical accuracy generally decreases since it is necessary to arbitrarily introduce a triangular element [19]. The MLPG parameters, SS and TS, are taken to be 2.3$d_1$ and 0.6$d_2$ respectively, the base radii are $d_1$ and $d_2$ which are determined by the process described below.

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For SS,
1. Take eight nodal points which are closest to the evaluation point.
2. Let the largest of the distances between the eight points and the evaluation point be the base radius \( d_1 \).

For TS,
1. Let the shortest distance to the evaluation point be the base radius \( d_2 \).

Next, using the analytical model of Fig. 13, the bead-on-plate weld described in the previous section is analyzed. Fig. 14 shows these results. The figure shows the temperature distribution along a longitudinal section of the welding line, that is, \( A-A' \) indicated in Fig. 14(b). The circles in Fig. 14(a) indicate the results of analysis by the MLPG method using the nodal point distribution (nodal points: 724) shown in Fig. 13, and the squares indicate the results of FEM, which has a larger nodal point density with 1891 nodal points. In Fig. 14(b) the dotted line in the graph indicates the temperature distribution along \( B-B' \), a central section, at 12 s after commencement of welding. The broken line is for a time of 15 s, and the solid line is for a time of 21 s. These results show that the temperature distribution by FEM and the temperature distribution obtained as an MLPG solution (of which the number of nodal points is less than half) coincide well. Next, the temperature distribution along a center section, (i.e., along \( B-B' \)) is shown for the same times in Fig. 14(b). These graphs show that the heat concentration in the heated area diffuses with the passage of time. They also show that there is good agreement between the MLPG solution and the FEM results.

From the above, we could confirm that a better solution is obtained by the MLPG method, even when the nodal point distribution is considered to have high applicability to welding problems with a high density of nodal points which are near the welding line.

3.5. Investigation of adaptively alterable nodal point distributions

An analysis with a high accuracy and low computational cost should be possible if we can make the nodal point distribution dense near the welding torch (which has a large temperature gradient) and coarse elsewhere. However, in the moving heat source problem in welding, a region, in which the temperature gradient is large, moves with the torch. Hence, the method described in the previous section (i.e., of making the dense nodal point distribution beforehand for the whole route over which the welding torch will move) is often used. Therefore, in this analysis it is possible that the dense nodal point distribution moves with the torch, it may be considered an effective method for reducing the computation time. However, in the moving heat source problem in welding, if this kind of analysis is conducted using FEM, special processing (see Ref. [9]) will be necessary. Consequently, in this section, we propose a method in which the number of nodal points can be reduced and high accuracy analysis can be achieved by using the MLPG method for the welding problem.

When adaptively adding and removing nodal points in analysis using FEM, if the nodal points are arranged by not considering the connectivity between nodes and elements, mesh division can not be made. However, it is not necessary to consider this point at all with the MLPG method. It is possible to generate and eliminate nodal points in the required places, if necessary. This is because meshless methods such as the MLPG method differ from FEM in that variables such as temperature and their differential are preserved as continuous values. This is a significant advantage of completely meshless method, MLPG. In this section, we conducted an analysis to highlight this point.

First, we assumed a coarse nodal point distribution with the intervals between nodal points at 10 mm in equal distances, as shown in Fig. 15(a). However, when analyzing the area around the welding torch with this nodal point distribution (number of nodal points: 231), Figs. 7, 8, and 10 predict that the analytical accuracy will decrease. Therefore, by using a nodal point distribution in which a dense nodal point interval of 3.3 mm (see Fig. 15(b)) is added only to areas near the torch, high accuracy analysis is possible, regardless of the low number of nodal points. The MLPG parameters, TS and SS, are defined in the same way as the method described in the previous section. The shape and size of the specimen to be analyzed are taken to be those in Fig. 3 (which are used in the previous section), and we used an analysis that took account of the temperature dependency of the material constants.

Fig. 16 shows the temperature distributions at 7, 15, and 23 s after the commencement of welding, obtained by the above-mentioned adaptive MLPG method. The region with a dense nodal point distribution moves together with the torch. The results agree with the analytical solution when the number of nodal points is sufficiently large (see Fig. 11) in whole region. This shows that this method of adaptively adding and removing nodal points during analysis can provide sufficiently accurate results in practical problem.
The analytical method using a freely alterable nodal point distribution shown in this section is considered to be compatible with adaptive methods [8,17] such as the $r$ and $h$ methods. By introducing the concept of error assessment, it should be possible to conduct analysis with higher efficiency and accuracy; thus the use of the MLPG method is expected to grow in the future in welding studies.

4. Conclusions

With the goal of developing a new method of structural analysis for welding by the MLPG method, we applied this computational technique to transient heat conduction involving a moving heat source. To investigate the applicability of this computational method to welding problems, it is applied to the bead-on-plate weld problem including a moving heat source, and the fundamental properties of the method are considered. In addition to investigating the influence of the nodal point density (number of nodal points), an adaptive computational method is proposed, which includes the addition and removal of nodal points, which is expected to be applicable to welding problems. The following findings are obtained:

1) It is clear that an analysis is possible to obtain a high level of accuracy, when the number of nodal points is sufficiently high, without depending on the MLPG parameters, SS and TS.
2) Even in cases when the temperature dependency of the material properties is accounted for, and heat transfer to the surrounding medium associated with welding problems is accounted for, it is clear that analysis with a high level of accuracy is possible.
3) We have demonstrated that this method can be used to perform computations using nodal points that are adaptively added and removed during analysis, and that highly accurate analysis is possible using the MLPG method.

Acknowledgment

We would like thank Dr. Koji Masaoka of MASAOKA Technical Development for useful discussion in the execution of this study.

References


