Path-independent integral in fracture mechanics of quasicrystals

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Abstract

Path-independent integrals are successfully utilized for accurate evaluation of fracture parameters in crystalline materials, where atomic arrangement is periodic. In quasicrystals (QC) the atomic arrangement is quasiperiodic in one-, two- or three-directions. The 2-d elastic problem for quasicrystal is described by coupled governing equations for phonon and phason displacements. Conservation laws for quasicrystals are utilized to derive path-independent integrals for cracks. The relation between the energy release and stress intensity factor for a crack under the mode I is given for decagonal QCs. The path-independent integral formulation is valid also for cracks in QCs with continuously varying material properties.

1. Introduction

Since the pioneering work of Eshelby [1], and independent discovery of Rice [2], innumerable number of papers have appeared in literature concerning path-independent integrals (familiy so-called J-integral) and their application to mechanics of fracture. Both papers initiated important studies, on conservation laws in finite and infinitesimal elasticity, by Knowles and Sternberg [3], and the interpretation of these in the context of the mechanics of cracks and notches in 2-dimensional bodies by Budiansky and Rice [4]. Above mentioned studies are restricted to elasto-statics, and the crack-extension considered, if any, is of quasi-static nature, i.e. inertia is considered negligible. Atluri [5] extended path-independent integrals to finite elasticity with general body forces, arbitrary crack-face tractions and nonstationary conditions. Recently, Han and Atluri [6,7] have published more general papers on conservation laws, to improve upon the work of Eshelby, Knowles, and Sternberg. The MLPG is applied together with these conservation laws there.

In many fracture problems it has been derived a unique relation between the stress intensity factors and path-independent J-integral value. Stress intensity factors characterize stress fields at the crack tip vicinity and determine fracture processes. Their evaluation from the asymptotic expansion expression of stresses is inaccurate due to numerical difficulties connected with modelling strong field gradients at the crack tip vicinity. However, the J-integral characterize fracture processes also, however, it is determined by fields sufficiently far from the crack tip. Therefore, the approach based on
the evaluation stress intensity factors on the J-integral is more accurate than direct approach based on asymptotic expansion of stresses. It is well known in conventional elasticity of crystals [8]. Now, it is a natural attempt to extend this idea into fracture of quasicrystals.

According to the review article [9] and monograph [10], which comprehensively give the state of the art of investigations on the mechanical analyses of QCs, the crack problems are presented in literature quite seldom. It is despite reality that quasicrystals are brittle. Therefore, to understand the effect of cracks on the mechanical behavior of a quasicrystal, the crack analysis of quasicrystals, including the determination of the stress intensity factors, the elastic field, the strain energy release rate and so on, is a prerequisite. Although a sufficiently large number of solutions associated with cracks have been obtained for various problems of theoretical and practical importance in conventional linear fracture mechanics for crystals, very few investigations are known for quasicrystals [11]. Due to the coupling of phonon and phason displacements in quasicrystals, the crack problems for these materials are more complicated than those in conventional crystals. Therefore, crack analyses in the QC are focused mainly on Griffith cracks in an infinite body, where analytical solutions are available for one and two-dimensional quasicrystals [9,12–14]. The governing equations for anti-plane crack problems are simpler than that for in-plane problems and analytical solutions are available also for the crack mode III too [15,16].

The goal of this paper is to derive path-independent integrals for crack problems in quasicrystals with nonstationary conditions. The first effort is made by Shi [17], where a special version of Noether’s theorem for the sake of absolute invariance on invariant variational principles is used to obtain conservation laws of a three-dimensional solid periodically stacked in a two-dimensional quasiperiodic structure with decagonal symmetry. Conservation laws in material space depend on the material coefficients [18,19] and, indeed, may not be easy to be observed by experiment. It is well known that some path-independent integrals valid in an isotropic elastic medium does not exist in orthotropic medium. For quasicrystal, there are many kinds of symmetries of the material coefficients [20]. These material symmetries of quasicrystals should be considered to prove the existence and determine the characteristic of their conservation laws. In the present paper conservation laws for quasicrystals are utilized to derive path-independent integrals for cracks. The relation between the energy release and stress intensity factor for a crack under the mode I is given for decagonal QCs. The path-independent integral formulation is valid also for cracks in QCs with continuously varying material properties.
2. Governing equations for quasicrystals

A quasicrystal material is deformable under applied forces, thermal load and certain internal effects. The Landau density wave theory [21–23] can be considered as a base of elasticity theory of quasicrystals. In the theory of elasticity of crystals the displacement field (phonon displacement field $u_i(x)$) represents the phenomenological field corresponding to translational motion of atoms in crystals. Due to quasi-periodic lattice structure in quasicrystals, additional degrees of freedom corresponding to atomic rearrangements are introduced in the phenomenological theory via phason displacements $w_i(x)$. In the classical theory of elasticity, the strains $\varepsilon_{ij}(x)$ of crystals is described by the gradients of (phonon) displacements $u_{ij}(x)$. The strain energy density has the following form [10]

$$W = \frac{1}{2} c_{ijkl} \varepsilon_{ik} \varepsilon_{jl} + \frac{1}{2} K_{ijkl} w_{il} w_{jl} + \frac{1}{2} R_{ijkl} w_{il} w_{jl} + \frac{1}{2} R_{ijk} w_{ij} \varepsilon_{kl},$$

(1)

where

$$R_{ijkl} = R_{jikl} \quad \text{and} \quad R_{ijkl} = R_{klij},$$

and the phonon and phason strains are defined as

$$\varepsilon_{ij}(x) = \frac{1}{2} (u_{ij} + u_{ji}),$$

(2)

$$w_{ij}(x) = w_{ij}(x),$$

(3)

in which the phason strains $w_{ij}(x)$ are not necessarily to be a symmetric tensor in contrast to the phonon strains $\varepsilon_{ij}(x)$.

The generalized Hooke’s law of quasicrystals:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = c_{ijkl} \varepsilon_{kl} + R_{ijkl} w_{kl},$$

$$H_{ij} = \frac{\partial W}{\partial w_{ij}} = K_{ijkl} w_{kl} + R_{klij} \varepsilon_{kl},$$

(4)

where $c_{ijkl}$, $K_{ijkl}$ and $R_{klij}$ denote phonon elastic tensor, phason elastic tensor and phonon-phason coupling elastic constant tensor, respectively.

The quasi-periodicity leads to two different elementary excitations in the material: phonons $u_i$ and phasons $w_i$. The phonon modes can be interpreted as vibrations of the quasicrystal lattice. For the phonon modes, there are different points of view. According to Bak [21] the phason structure disorders are realized by fluctuations in quasicrystals. The balance of momentum is valid for phonon deformation and similarly for phason oscillations with the same mass of density playing role in relaxation processes:

$$\sigma_{ij}(x, \tau) + X_i(x, \tau) = \rho \ddot{u}_i(x, \tau),$$

(5)

$$H_{ij}(x, \tau) + g_i(x, \tau) = \rho \ddot{w}_i(x, \tau),$$

(6)

where $\ddot{u}_i$, $\ddot{w}_i$, $\rho$, $X_i$ and $g_i$ denote the acceleration of the phonon and phason displacements, the mass density, and the body force vectors, respectively.

According Lubensky et al. [24] the phonon and phason fields play different roles in the hydrodynamics of quasicrystals, because phason displacements are insensitive to spatial translations. Furthermore, the relaxation of the phonon strain is diffusive and is much slower than rapid relaxation of conventional phonon strains. Recently, more advanced elasto-hydrodynamic model has been introduced by Fan et al. [25]. Rochal and Lorman [26] published the minimal model which is similar to Fan’s model. The following governing equations one can write

$$\sigma_{ij}(x, \tau) + X_i(x, \tau) = \rho \ddot{u}_i(x, \tau),$$

(7)

$$H_{ij}(x, \tau) + g_i(x, \tau) = D \ddot{w}_i(x, \tau),$$

(8)

where $D = 1/\Gamma_w$, and $\Gamma_w$ denotes the kinematic coefficient of phason field of the material defined by Lubensky et al. [24].

The aforementioned models investigate the phasons in a very different manner. Pure diffusion is very different from wave propagation without attenuation. Therefore, it is a motivation to formulate a model, which unifies the two approaches (the elastodynamic model of wave type and the elasto-hydrodynamic model) as asymptotic cases of it and moreover provides a theory for the phason dynamics valid in the whole range of possible wavelengths. Recently, Agiasofitou and Lazar [27] have introduced the elastodynamic model of wave-telegraph type for the description of dynamics of quasicrystals. Phonons are represented by waves, and phasons by waves damped in time and propagating with finite velocity; that means the equations
of motion for the phonons are partial differential equations of wave type, and for the phasons partial differential equations of telegraph type.

The decagonal quasicrystals frequently occur since almost one half of all observed quasicrystals belong to this category. So this kind of solid phases plays an important role. They have ten-fold rotation symmetries belong to the class of two-dimensional (2-d) quasicrystals, where the atomic arrangement is quasiperiodic in a plane, and periodic in the third direction. The problem can be decomposed into plane and anti-plane elasticity. Here, we consider only the plane elasticity, because the anti-plane elasticity is a classical one and independent of phason variables. An opposite case is valid for the one-dimensional quasicrystals, where the atom arrangement is quasi-periodic in one direction and periodic in the plane perpendicular to the quasi-periodic arrangement. The plane problem is there represented by the conventional classical elasticity equations. The generalized Hooke's law for plane elasticity of decagonal QC is given as [10]

\[
\sigma_{ij} = c_{ij} \varepsilon_{ij} + c_{ijkl} \varepsilon_{kl} + R(w_{ij} + w_{kl}),
\]

where \(c_{ij}, c_{ijkl}, R\) and \(K_i\) denote the classical phonon elastic coefficients, the phonon-phason coupling parameter and the phason elastic coefficients, respectively.

3. A crack in decagonal quasicrystals under mode I

Li et al. [11] found an analytical solution for a Griffith crack in decagonal quasicrystals. He showed that both phonon and phason stresses exhibit the singularity \(r^{-1/2}\), where \(r\) is the radial coordinate with origin at the crack-tip (Fig. 1). The asymptotic crack-tip field expressions are obtained under stationary conditions, therefore, they have to be the same for both the Bak's and the elasto-hydrodynamic models. The following asymptotic stress expressions can be written for the mode-I crack under a pure phonon load [28]:

\[
\sigma_{11}(r, \theta) = \frac{K_{1}^\parallel}{\sqrt{2\pi r}} \cos \frac{1}{2} \theta \left(1 - \sin \frac{1}{2} \theta \sin \frac{3}{2} \theta \right),
\]

\[
\sigma_{22}(r, \theta) = \frac{K_{1}^\parallel}{\sqrt{2\pi r}} \cos \frac{1}{2} \theta \left(1 + \sin \frac{1}{2} \theta \sin \frac{3}{2} \theta \right),
\]

\[
\sigma_{12}(r, \theta) = \sigma_{21}(r, \theta) = \frac{K_{1}^\parallel}{\sqrt{2\pi r}} \cos \frac{1}{2} \theta \cos \frac{3}{2} \theta,
\]

\[
H_{11}(r, \theta) = -\frac{d_{21}K_{1}^\parallel}{\sqrt{2\pi r}} \sin \theta \left(2 \sin \frac{3}{2} \theta + \frac{3}{2} \sin \theta \cos \frac{5}{2} \theta \right),
\]

\[
H_{22}(r, \theta) = \frac{d_{21}K_{1}^\parallel}{\sqrt{2\pi r}} \frac{3}{2} \sin^2 \theta \cos \frac{5}{2} \theta,
\]

\[
H_{12}(r, \theta) = -\frac{d_{21}K_{1}^\parallel}{\sqrt{2\pi r}} \frac{3}{2} \sin^2 \theta \sin \frac{5}{2} \theta,
\]

\[
H_{21}(r, \theta) = \frac{d_{21}K_{1}^\parallel}{\sqrt{2\pi r}} \sin \theta \left(2 \cos \frac{3}{2} \theta - \frac{3}{2} \sin \theta \sin \frac{5}{2} \theta \right),
\]

where

\[
d_{21} = \frac{R(K_1 - K_2)}{4(MK_1 - R^2)},
\]

\[
K_{1}^\parallel = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{22}(r, 0),
\]

in which \(M = (c_{11} - c_{12})/2\).
Non-vanishing phason stress field arises due to the coupling relationship between the phonon and the phason fields even at a pure phonon load. If the Griffith crack with the length $2a$ under a uniform phonon tension $p$ is considered, the stress ahead of the crack-tip is given as

$$\sigma_{22}(x_1, 0) = p \left( \frac{x_1}{x_1^2 - a^2} - 1 \right).$$

Then, the phonon stress intensity factor has the following form

$$K_I^p = p\sqrt{2\pi a}. \quad \text{(13)}$$

Up to now, we are not able to prescribe a finite value of the generalized stresses $H_{ij}$. Despite it, mathematically it has been shown by Fan [10] that the singularity of generalized stresses around the crack-tip also exhibits the square-root if the considered cracked body is under a pure phason load. Therefore, one can define the stress intensity factor for phason stresses as

$$K_I^\gamma = \lim_{r \to 0} \sqrt{2\pi r} H_{22}(r, 0). \quad \text{(14)}$$

It follows from Eq. (13) that the phonon stress intensity factor for the Griffith crack is identical with the stress intensity factor for crystals and it is independent on material parameters. Also the phonon stresses (10) are independent on material parameters. However, the phason stresses (11) are dependent on the material parameters expressed by $d_{21}$. Li et al. [11] derived the phonon displacement for the Griffith crack

$$u_2(x_1, 0) = \frac{p}{2} \left( \frac{K_1}{MK_1 - R^2} + \frac{1}{L + M} \right) \sqrt{a^2 - x_1^2}, \quad \text{(15)}$$

where $L = c_{12}$.

One can observe that the phonon displacement field at the crack-tip vicinity is dependent on material parameters, like in conventional linear elastic fracture mechanics for crystals.

Fan et al. [28] derived the energy release rate for the Griffith crack in quasicrystals and it has the following form

$$G = \frac{1}{4} \left( \frac{1}{L + M} + \frac{K_1}{MK_1 - K} \right) (K_I^p)^2. \quad \text{(16)}$$

Evaluation of the stress intensity factor from the asymptotic expression (12) is not convenient due to inaccurate stresses computed at the crack tip vicinity. If we able to compute accurately energy release the stress intensity factor can be computed accurately from (16). In the next paragraph the path-independent integral for energy release is given.

### 4. Path-independent integral

Consider the dynamic propagation of a crack in a self-similar fashion such that the crack length increases by $(da)$ in time $dt$. In the self-similar crack propagation it is assumed that the crack tip fields are self-similar at time $\tau$ and $\tau + dt$. However, their intensities can be different. The energy release to the crack tip per unit of crack extension is denoted by $G$ and it follows from the energy balance [8]

$$G = \frac{DW_{ex}}{Da} - \frac{DU}{Da} - \frac{DT}{Da}, \quad \text{(17)}$$

where $W_{ex}$, $U$ and $T$ represent the external work, internal energy and kinetic energy, respectively. If the crack is along $x_1$ coordinate, the total derivative in Eq. (17) follows from the self-similar crack propagation [8].
The traction and the generalized traction vectors are defined by
\[ t_i(x, \tau) = \sigma_i(x, \tau) \eta_i(x), \]
\[ h_i(x, \tau) = H_i(x, \tau) \eta_i(x), \]
where \( \eta_i(x) \) is the outward unit normal vector to the boundary \( \Gamma \) of the cracked body \( \Omega \).

If traction vector \( t_i \) is prescribed on the part of the boundary \( \Gamma_1 \) and the generalized traction \( h_i \) on \( \Gamma_h \), the energy release (17) can be rewritten into the following form
\[ G = \int_{\Gamma_1} t_i \frac{\partial u_i}{\partial a} d\Gamma + \int_{\Gamma_h} h_i \frac{\partial w_i}{\partial a} d\Gamma + \int_{\Omega} X_i \frac{\partial u_i}{\partial a} d\Omega + \int_{\Omega} g_i \frac{\partial w_i}{\partial a} d\Omega - \frac{D}{\partial a} \int_{\Omega} (W + T) d\Omega, \tag{21} \]
where \( W \) is the density of total stress work and the density for the kinetic energy is considered to admit phonon and phason contributions
\[ T = \frac{1}{2} \rho \dot{u}_i \dot{u}_i + \frac{1}{2} \rho \dot{w}_i \dot{w}_i. \]

We consider an arbitrarily small loop \( \Gamma_x \) surrounding the crack tip, such that the volume inside \( \Gamma_x \) is \( \Omega_x \) (Fig. 1). If we consider \( \Omega = \Omega_x \), which does not include the crack tip, the energy release has to be vanishing for considered domain. For vanishing traction and generalized traction vectors, \( t_i = 0, h_i = 0 \) on both crack surfaces \( \Gamma_x \) one can write
\[ 0 = \int_{\Gamma_x} t_i \frac{\partial u_i}{\partial a} d\Gamma + \int_{\Gamma_x} h_i \frac{\partial w_i}{\partial a} d\Gamma + \int_{\Omega_x} X_i \frac{\partial u_i}{\partial a} d\Omega + \int_{\Omega_x} g_i \frac{\partial w_i}{\partial a} d\Omega - \frac{D}{\partial a} \int_{\Omega_x} (W + T) d\Omega. \tag{22} \]

Subtracting (22) from (21) one gets
\[ G = \lim_{\epsilon \to 0} \left\{ \int_{\Gamma_1} t_i \frac{\partial u_i}{\partial a} d\Gamma + \int_{\Gamma_h} h_i \frac{\partial w_i}{\partial a} d\Gamma + \int_{\Omega} X_i \frac{\partial u_i}{\partial a} d\Omega + \int_{\Omega} g_i \frac{\partial w_i}{\partial a} d\Omega - \frac{D}{\partial a} \int_{\Omega} (W + T) d\Omega \right\}. \tag{23} \]

Applying the total derivative (18) to the last term in (23) and Gauss divergence theorem one can write
\[ \frac{D}{\partial a} \int_{\Omega} (W + T) d\Omega = \int_{\partial \Omega} \frac{\partial}{\partial a} (W + T) d\Omega - \int_{\Gamma_x} (W + T) n_1 d\Gamma. \tag{24} \]

If the total derivative (18) is applied also to other terms in (23) we get
\[ G = \lim_{\epsilon \to 0} \left\{ \int_{\Gamma_1} \left[ (W + T) n_1 - t_i \frac{\partial u_i}{\partial a} - h_i \frac{\partial w_i}{\partial a} \right] d\Gamma + \int_{\Gamma_h} \left[ t_i \frac{\partial u_i}{\partial a} + h_i \frac{\partial w_i}{\partial a} \right] d\Gamma \right\} \]
\[ - \lim_{\epsilon \to 0} \left\{ \int_{\Omega_x} \frac{\partial}{\partial a} (W + T) - X_i \frac{\partial u_i}{\partial a} - g_i \frac{\partial w_i}{\partial a} + X_i \frac{\partial u_i}{\partial a} + g_i \frac{\partial w_i}{\partial a} \right\} d\Omega \right\} \]
\[ = \lim_{\epsilon \to 0} \left\{ \int_{\Gamma_1} \left[ (W + T) n_1 - t_i \frac{\partial u_i}{\partial a} - h_i \frac{\partial w_i}{\partial a} \right] d\Gamma \right\}. \tag{25} \]

In the above equation we have utilized that terms in the second line are vanishing in the limit \( \epsilon \to 0 \), due to the fact that \( \partial W/\partial a \) and \( \partial T/\partial a \) are still order \( 1/r \) near the crack tip, \( t_i \) is of \( O(r^{-1/2}) \), while \( \partial u_i/\partial a \) is \( O(r^{1/2}) \).

Since on the boundary with prescribed displacements \( \Gamma_u \) it has to valid
\[ \frac{\partial u_i}{\partial a} = \frac{\partial u_i}{\partial x_1}, \]
the energy release (21) can be rewritten into the form
\[ G = \int_{\Gamma} \left[ (W + T) n_1 - t_i \frac{\partial u_i}{\partial x_1} - h_i \frac{\partial w_i}{\partial x_1} \right] d\Gamma + \int_{\Gamma_h} \left[ t_i \frac{\partial u_i}{\partial a} + h_i \frac{\partial w_i}{\partial a} \right] d\Gamma \]
\[ - \int_{\Omega} \frac{\partial}{\partial a} (W + T) - X_i \frac{\partial u_i}{\partial a} - g_i \frac{\partial w_i}{\partial a} + X_i \frac{\partial u_i}{\partial a} + g_i \frac{\partial w_i}{\partial a} \right\} d\Omega. \tag{26} \]

If we apply total derivative (18) to (26) we get
\[ G = \int_{\Gamma} \left[ (W + T) n_1 - t_i \frac{\partial u_i}{\partial x_1} - h_i \frac{\partial w_i}{\partial x_1} \right] d\Gamma + \int_{\Gamma_h} \left[ t_i \frac{\partial u_i}{\partial a} + h_i \frac{\partial w_i}{\partial a} \right] d\Gamma \]
\[ - \int_{\Omega} \frac{\partial}{\partial a} (W + T) - X_i \frac{\partial u_i}{\partial a} - g_i \frac{\partial w_i}{\partial a} + X_i \frac{\partial u_i}{\partial a} + g_i \frac{\partial w_i}{\partial a} \right\} d\Omega. \tag{27} \]
Note that the second term in Eq. (27) does not vanish in a finite domain. If we consider domain $\Omega - \Omega_c$ and corresponding boundary $\Gamma - \Gamma_c$ in Eq. (27) we get

$$G = \int_{\Gamma + \Gamma_c} \left[ (W + T)_{n_1} - t_i \frac{\partial u_i}{\partial x_1} - h_i \frac{\partial w_i}{\partial x_1} \right] \, d\Gamma + \int_{\Gamma - \Gamma_c} \left[ t_i \frac{\partial u_i}{\partial a} + h_i \frac{\partial w_i}{\partial a} \right] \, d\Gamma$$

$$- \int_{\Omega - \Omega_c} \left[ \frac{\partial}{\partial a} (W + T) - X_i \frac{\partial u_i}{\partial a} - g_i \frac{\partial w_i}{\partial a} + X_i \frac{\partial u_i}{\partial x_1} + g_i \frac{\partial w_i}{\partial x_1} \right] \, d\Omega. \quad (28)$$

For a quasi-static crack growth $Du/Da = 0$. Then, it is valid $\partial u_i/\partial a = \partial u_i/\partial x_1$. Keeping it in mind and applying Gauss divergence theorem to the second integral in (28) one gets

$$G = \int_{\Gamma + \Gamma_c} \left[ (W + T)_{n_1} - t_i \frac{\partial u_i}{\partial x_1} - h_i \frac{\partial w_i}{\partial x_1} \right] \, d\Gamma$$

$$- \int_{\Omega - \Omega_c} \left[ \frac{\partial}{\partial x_1} (W + T) - \sigma_{ij} \frac{\partial u_i}{\partial x_1} - \sigma_{ij} \frac{\partial^2 u_i}{\partial x_1 \partial x_j} - H_{ij} \frac{\partial u_j}{\partial x_1} - H_{ij} \frac{\partial^2 u_j}{\partial x_1 \partial x_j} \right] \, d\Omega. \quad (29)$$

The gradient of the strain energy density (1) for quasicrystals with continuously varying material properties is given by

$$W_m(\epsilon_{ij}, w_{ij}, x_i) = \sigma_{ij} u_{ij,m} + H_{ij} w_{ij,m} + \left( \frac{\partial W}{\partial x_m} \right)_{\text{exp}}, \quad (30)$$

where

$$\left( \frac{\partial W}{\partial x_m} \right)_{\text{exp}} = \frac{1}{2} c_{ijkl,m} \epsilon_{ij} \epsilon_{kl} + \frac{1}{2} k_{ijkl,m} \epsilon_{ij} w_{kl} + \frac{1}{2} r_{ijkl,m} \epsilon_{ij} w_{kl} + \frac{1}{2} r_{ijkl,m} w_{ij} \epsilon_{kl}. $$

If gradient of strain energy in (30) is considered in $x_1$ direction and substituted into (29) one gets

$$G = \int_{\Gamma + \Gamma_c} \left[ (W + T)_{n_1} - t_i \frac{\partial u_i}{\partial x_1} - h_i \frac{\partial w_i}{\partial x_1} \right] \, d\Gamma - \int_{\Omega - \Omega_c} \left[ \frac{\partial^2 T}{\partial x_1} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} - H_{ij} \frac{\partial w_i}{\partial x_1} \right] \, d\Omega \quad (31)$$

Divergences of stresses and generalized stresses can be expressed by governing Eqs. (5) and (6) in the Bakh’s model. If they are substituted into (31) and kinetic energy too, the energy release is given by

$$G = \int_{\Gamma + \Gamma_c} \left[ (W + T)_{n_1} - t_i \frac{\partial u_i}{\partial x_1} - h_i \frac{\partial w_i}{\partial x_1} \right] \, d\Gamma$$

$$- \int_{\Omega - \Omega_c} \left[ \frac{\partial W}{\partial x_1} \right]_{\text{exp}} \, d\Omega. \quad (32)$$

The energy release involves both path and volume integrals. The path $\Gamma$ can be arbitrary. A pure path integral formulation is obtained only for elastostatic case ($u_i = \bar{u}_i = 0, w_i = \bar{w}_i = 0$) with vanishing body forces and for a homogeneous material.

5. Numerical examples

Recently, we have analyzed a straight central crack in a finite quasicrystal strip under a pure phonon load is analyzed (Fig. 2) by the meshless local Petrov–Galerkin method (MLPG) [27], where stress intensity factor is evaluated directly from asymptotic expansion of stresses at the crack tip vicinity [30]. Both the FEM and MLPG have been proved as powerful computational tools to analyze crack problems. Each one has its own advantages and drawbacks. Meshless formulations are becoming popular due to their high adaptivity and low costs in preparation of input and output data for numerical analyses. The continuity of the MLS approximation is given by the minimum between the continuity of the basis functions and that of the weight function. So continuity of trial functions can be tuned to a desired degree in the MLPG method, resulting in better accuracies [29]. In FEM, increasing the order of continuity of the trial function is a nontrivial problem. On the other hand, the price which should be paid for the above mentioned advantages of meshless methods, in general, is the lower computational efficiency due to more complex shape functions in the approximation formula. Note that even this drawback can be overcome by using analytical integration instead of Gauss–Legendre or other numerical quadrature schemes. It brings significant reduction of CPU times needed for the creation of the system matrix. Sladek and Sladek [31] and later Soares et al. [32] have reduced the amount of evaluations of the shape functions and their derivatives to nodal points instead of the integration points by introducing the analytical integrations in LIEs implemented by meshless approximations for field variables. Furthermore, they have developed a modified differentiation scheme for approximation of higher order derivatives of displacements appearing in the discretized formulations.

The strip is subjected to a stationary or impact mechanical load with Heaviside time variation and the intensity $\sigma_0 = 1$ Pa on the top-side of the strip. The material coefficients of the strip correspond to Al-Ni-Co quasicrystal and they are given by
Homogeneous material properties are considered in this example to test accuracy of the present approach. The crack-length $2a = 1.0$ m, crack-length to strip-width ratio $a/b = 0.4$, and strip-height $h = 1.2b$ are considered. Due to the symmetry of the problem with respect to the crack-line as well as vertical central line, only a quarter of the strip is numerically analyzed. The boundary value problem and quantities for evaluation of the energy release in Eq. (32) are computed by the MLPG. Both phonon and phason displacements in the quarter of the strip are approximated by using 930 (31 nodes equidistantly distributed. We have considered two different integration paths $\Gamma$ in Eq. (32). The first one, more close to the crack tip, is created by nodes EFGD, where $EA = (b - a)/2$ and $GD = h/2$. The second integration path is given by ABCD. The discrepancies of the numerical values obtained for these fracture parameters from the two paths were less than 1%.

Numerical results for stress intensity factors (SIF) are given in Table 1. One can observe that the SIF increases with increasing value of the coupling parameter. Current decagonal QC have coupling parameter lower than it is maximum value in Table 1. The considered Al-Ni-Co material has the value $R = 1.1$ GPa, and it corresponds to $R/M = 0.012$ [8]. Results for larger values are presented only for better imagination to see influence of the coupling parameter on the SIF value.

The same cracked strip under an impact load with Heaviside time variation $\sigma_0 H(t - 0)$ is analyzed too. The normalized stress intensity factor corresponding to the Bak’s model is compared with the FEM results in Fig. 3 for conventional elasticity, i.e., $R/M = 0$. The time variable is normalized as $\alpha = c_1 \tau / h$, where $c_1 = \sqrt{\rho / \mu}$ is the velocity of longitudinal wave. The SIF is evaluated by the path-independent integral method on the integration path ABCD illustrated in Fig. 2. One can observe that the stress intensity factor for a finite value of the coupling parameter is only slightly larger than the corresponding factor in conventional elasticity. The peak value is shifted to larger time instants for cracks in quasicrystals. We have selected the time-step $\Delta \tau = 0.25 \cdot 10^{-4}$ s in our numerical analyses.

An edge crack in a finite strip is analyzed in the second example. The following geometrical parameters are considered: $a = 0.5$, $a/b = 0.4$ and $h/b = 1.2$. Due to the symmetry with respect to $x_1$ only a half of the strip is modeled. We have used 930 nodes equidistantly distributed for the MLS approximation of the physical fields. On the top of the strip a uniform impact tension $\sigma_0 = 1$ Pa is applied. In this example homogeneous and continuously varying material properties are considered. In the functionally graded quasicrystal an exponential variation for all material parameters (generally denoted by $f$) is used

$$ f(x) = f_0 \exp(\gamma x_1), $$

where $f_0$ corresponds to parameters used in the previous example.

The influence of the material gradation on the phonon crack displacement is illustrated in Fig. 4. Parameter gradation $\gamma = 2$ m$^{-1}$ is considered in numerical analyses. One can observe that the phonon displacement for a finite value of the coupling parameter is larger than the corresponding displacement in conventional elasticity. However, the material gradation reduces the crack displacements. The stress intensity factor normalized to that in conventional elasticity is equal to $K_1^f / \sigma_0 \sqrt{\pi a} = 2.108$ for a crack in corresponding homogeneous crystal material. The coupling parameter has vanishing influence on the stress intensity factor, since the normalized SIF is equal 2.106 for $R/M = 0.5$. However, the material gradation in

$$ c_{11} = L + 2M = 23.43 \cdot 10^{10} \text{Nm}^{-2}, \quad c_{12} = L = 5.74 \cdot 10^{10} \text{Nm}^{-2}, \quad K_1 = 12.2 \cdot 10^{10} \text{Nm}^{-2}, $$

$$ K_2 = 2.4 \cdot 10^{10} \text{Nm}^{-2}, \quad M = (c_{11} - c_{12})/2, $$

$$ \rho = 4180 \text{kg/m}^3, \quad \Gamma_w = 4.8 \cdot 10^{-19} \text{m}^3/\text{kg}. $$

Fig. 2. Central crack in a finite quasicrystal strip.
the FGM quasicrystal reduces the stress intensity factor. The normalized SIFs in crystal and quasicrystal material are almost the same and they are equal 1.95.

We have analyzed the same cracked strip also under an impact load with the Heaviside time variation $r_0H(t - 0)$. The coupling parameter for homogeneous or FGM quasicrystal is considered as $R/M = 0.5$. The SIF is evaluated by the path-independent integral method on the integration path ABCD.

The temporal variation of the SIF in the cracked strip under a pure mechanical load is presented in Fig. 5. For a phonon mechanical load, we have obtained $K_{I}^{stat} = 2.642 \text{ Pa m}^{1/2}$ in stationary case for the corresponding crystal with vanishing coupling parameter. The dynamic stress intensity factor is about two times larger than the corresponding static SIF. For a gradation of mechanical material properties with $x_1$ coordinate and a uniform mass density, the wave propagation is growing with $x_1$. Therefore, the peak value of the SIF is reached in a shorter time instant in FGM quasicrystal strip than in a homogeneous one. The maximum value of the SIF is only slightly reduced for the FGM cracked strip.

**Table 1**

<table>
<thead>
<tr>
<th>$R/M$</th>
<th>$K_I/\sigma_0\sqrt{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.14</td>
</tr>
<tr>
<td>0.05</td>
<td>1.142</td>
</tr>
<tr>
<td>0.5</td>
<td>1.152</td>
</tr>
</tbody>
</table>

Fig. 3. Temporal variation of the normalized SIF for the central crack in a strip under an impact load according to the Bak’s model.

Fig. 4. Variations of the phonon displacement on the edge crack with the normalized coordinate $x_1/2a$ for a pure phonon load $\sigma_0 = 1 \text{ Pa}$.
6. Conclusions

Path-independent integral is presented for accurate evaluation of fracture parameters in quasicrystals. The path-independent J-integral characterize fracture processes at the crack tip by fields sufficiently far from the crack tip. Therefore, the approach based on the evaluation stress intensity factors on the J-integral is more accurate than direct approach based on asymptotic expansion of stresses. General body forces, arbitrary crack-face tractions and nonstationary conditions can be considered for evaluation of the stress intensity factor. The present path-independent integral formulation is valid also for cracks in QCs with continuously varying material properties. Since the relation between the energy release and stress intensity factor for a crack under the mode I is available only for decagonal QCs, numerical results for the SIF are presented only for this case.

The present dynamic fracture study is based on the Bak's models. The influences of the coupling parameter on the temporal variation of the stress intensity factor have been shown for the central crack. The stress intensity factor for a finite value of the coupling parameter is only slightly larger than the corresponding factor in conventional elasticity. The peak value is shifted to larger time instants for cracks in quasicrystals. The influence of the material gradation in the FGM quasicrystals is analyzed for the edge crack problem. The crack opening displacement is significantly reduced with increasing material gradation. However, the SIF is only slightly reduced. The peak values of the SIF are reached in shorter time instants in FGM quasicrystal strip than in a homogeneous one. It is due to higher wave velocity in the FGM. The maximum value of the SIF is slightly reduced for the FGM cracked strip.

Acknowledgements

The authors gratefully acknowledge the supports by the Slovak Science and Technology Assistance Agency registered under number APVV-0014-10 and the Slovak Grant Agency VEGA-2/0011/1/3.

References