Direct and Inverse Multi-Scale Analyses of Functionally Graded Layered Hollow Cylinders (Discs), with Different Shaped Reinforcements, under Harmonic Pressure Loads

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Abstract

A multi-scale model is proposed to investigate functionally graded (exponential or power-law grading in the radial direction) hollow cylinders (discs), with fibers, particles, or disc-shaped reinforcements, subjected to harmonic loading conditions. The stress analyses are performed by dividing the cylinders (discs) into several layers each with homogeneous properties, which are functionally graded through the thickness of the structures, with varying microstructural details. Good agreement can be obtained by comparing the present stress distributions for laminated structures with each layer being homogeneous against analytical solutions used as inner and outer boundary conditions or obtained for homogeneous and continuously graded structures. Furthermore, the Mori-Tanaka model is used to generate effective properties of each layer reinforced with fibers, particles or disc-shaped inclusions. The stress distributions in the cylinders in the radial direction are effectively investigated with the influence of either the shape (decided by aspect ratio) or the volume fraction of reinforcements. Finally, the particle swarm optimization technique is combined with the present framework to provide inverse calculations for microstructural details, in the effort of finding proper inclusion volume fractions or minimizing the shear stress along radial direction, which are necessary for the design of functionally graded structures.

Keywords: Multi-scale analysis; functionally graded layered cylinders; harmonic loading; Mori-Tanaka homogenization; Particle Swarm Optimization
1 Introduction

Functionally graded materials (FGMs) have gained extensive attentions since the concept was first proposed by a group of material scientists in Japan in 1984 [1-2]. FGMs have various applications in aerospace industry, civil and marine structures, as well as bio-technologies. The properties of the FGMs are always architected through non-uniform distributions of reinforcements in heterogeneous materials. The reinforcements can be categorized into different shapes, properties and sizes, depending on the purposes of the applications. Thus, the macroscopic effective properties of FGMs depend directly on the microstructural details. FGMs have numerous existing forms in the world, ranging from man-made materials reinforced with particles using manufacturing systems applied with centrifugal forces [3] to natural-occurring bio-materials (bamboos) reinforced with fiber vascular bundles [4].

The mechanical and thermal properties of FGMs have been investigated for inhomogeneous cylinders, spheres and plates. However, the development of general solutions of boundary value problems for inhomogeneous materials and structures is always a challenging task. Thus, almost all existing methods involving inhomogeneous media are based on simplifying assumptions and functions to describe functional gradations. Several analytical expressions are derived for FGM cylinders [5-8] and spheres [6] under symmetric loading conditions. Batra and his co-authors have also published a sequence of their work regarding the elastic solutions of FG cylinders and spheres and material tailoring designs [9-11]. Finite element (FE) analyses are also conducted to study cylinders or spheres in more complicated situations [12]. In addition, some authors also focused on the investigations of FGMs as heat-shielding materials [13-17].

It should be noted that solving boundary value problems of inhomogeneous materials has two strategies – discretizing the space with local trial functions or using global analytical trial functions. One could use a trial function expansion for the elastic solutions since the material properties vary in the elastic domain and no simple solutions can represent the stresses or displacements [5-6,8]. The alternative is to divide the cylinders, spheres, plates into several layers with perfect contact conditions; and each layer is treated as a homogeneous body, leading to much simpler elastic solutions [7]. In addition, more general loading conditions can be considered with the easier equations.
Additionally, most of the work presented focused on the stress analyses by directly assuming macroscopic effective properties and ignoring the details of microscopic reinforcements. However, it should be noted that FGMs are usually manufactured by mixing several materials by powder metallurgy methods [6]. One example of the manufacturing process is that of applying the centrifugal force where particle or fiber distributions are formed [18]. Although FGMs are mainly achieved through tailoring the microstructural reinforcements, there is still not much work to provide a unified multi-scale analytical or numerical framework to conduct stress analysis while also considering the microstructural details. Kukui and Yamanaka [18] provided an expression for the gradation of particle distribution and then applied it to a functionally graded (FG) thick-walled tube, but not much microstructural simulation was provided. Salzar [19] proposed an optimization algorithm to reduce the effective stress by changing the fiber volume fraction in FG metal matrix composite tubes. Reddy and Cheng [20] provided three-dimensional solutions for FGM plates using the transfer matrix method, while the overall properties are calculated by Mori-Tanaka (M-T) method. Nie et al. [10] provided a technique to tailor FG materials and used rule-of-mixtures (ROM) and the Mori-Tanaka method for finding the effective properties.

Although some work has been conducted for the material designs of FG structures [10], there is still not a systematic procedure for the inverse calculations of both structural and microstructural details based on practical engineering requirements. Combined with the fact of rapid development of computational technology, it is necessary to introduce a simple and stable optimization technique to fulfill this purpose.

In order to present a sophisticated multi-scale framework in this contribution, composite $N$-layered cylinders and discs are investigated with each layer being considered as homogeneous, and the properties are graded through the thickness. The effective properties of each layer reinforced with fibers, particles, or discs of different aspect ratios are calculated using the Mori-Tanaka (M-T) model, firstly proposed by Mori and Tanaka [21], then illustrated by Benveniste [22], and popularized by Weng and his co-authors [23-24] by giving the explicit expressions. Finally, the particle swarm optimization technique [25] is introduced to solve the inverse problem of providing a design procedure for thick cylinders with the consideration of microstructural details.
The present contribution is organized as follows: Section 2 introduces the theoretical framework by conducting the relevant derivations for hollow composite cylinders (discs) under harmonic internal and external loading conditions. Section 3 provides validation for the equations in Section 2 and illustrates that the analysis of \(N\)-layered composites can be effectively used to generate the analytical solutions for continuously functionally graded structures. Section 4 briefly explains the Mori-Tanaka model which is employed to generate the effective properties of fiber, particle, or disc reinforced layers. Section 5 presents the particle swarm optimization method for both the inverse calculation of volume fraction to generate the necessary properties in each layer as well as the efforts of minimizing shear stress in the FG structures. Section 6 presents some conclusions.

2 Theoretical Framework

To provide a simple procedure to analyze functionally graded thick cylinders (wherein the gradation along radial direction is either of the exponential or power-law type) under harmonic pressure loadings, we first present analytical solutions for an \(N\)-layered cylinder wherein each layer has different homogeneous elastic properties. Thus, first, \(N\)-layered composite cylinders (discs), wherein each layer has homogeneous properties, are investigated under harmonic internal and external pressures with different magnitudes. The inner and outer radii of the structures are \(r_i = a\) and \(r_{N+1} = b\), respectively. Each layer is considered to be reinforced with secondary fibers, spheres, or discs with different aspect ratios \(\alpha\), while \(\alpha > 1\) for fibers, \(\alpha = 1\) for spheres, and \(\alpha < 1\) for discs. The cross-section of a circular cylinder is shown in Fig. 1. In the stress analysis, each layer is treated as being homogeneous, with the effective material properties having been calculated using the Mori-Tanaka homogenization model, which is introduced in Section 4. In this section, the basic governing equations are first provided for the generic \(i\)-th layer of the cylinders, and the interlayer traction reciprocity and displacement continuity conditions, as well as external boundary conditions are then employed to establish the relationships among the layers. Thus, the quantities in this section are considered within the \(N\)-layered structural level.
First of all, the harmonic boundary loading conditions at the inner and outer peripheries of the layered cylinders can be expressed as

\[
p_{in} = P_{in}^0 + \sum_{m=1}^{M} [P_{in}^1(m) \sin m\theta + P_{in}^2(m) \cos m\theta]
\]

\[
p_{out} = P_{out}^0 + \sum_{m=1}^{M} [P_{out}^1(m) \sin m\theta + P_{out}^2(m) \cos m\theta]
\]  

(1)

in which \( P_{in}^0, P_{in}^1, P_{in}^2 \) and \( P_{out}^0, P_{out}^1, P_{out}^2 \) are the amplitudes of the harmonic terms, and \( m \) stands for the \( m \)-th order. The problem can be solved by obtaining Navier’s equations of elasticity for the \( i \)-th layer (\( i = 1, \ldots, N \)), which can be derived by combining equilibrium equations, stress-strain relationship and strain-displacement relationship:

\[
\begin{align*}
C_{11}^{(i)} \left( \frac{\partial^2 u_r^{(i)}}{\partial r^2} + \frac{1}{r} \frac{\partial u_r^{(i)}}{\partial r} - \frac{u_r^{(i)}}{r^2} \right) + C_{12}^{(i)} \frac{\partial^2 u_r^{(i)}}{\partial \theta^2} + C_{11}^{(i)} + C_{12}^{(i)} \frac{\partial^2 u_\theta^{(i)}}{\partial r \partial \theta} - \frac{3C_{11}^{(i)} - C_{12}^{(i)}}{2} \frac{\partial u_\theta^{(i)}}{\partial \theta} &= 0 \\
C_{11}^{(i)} - C_{12}^{(i)} \left( \frac{\partial^2 u_\theta^{(i)}}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta^{(i)}}{\partial r} - \frac{u_\theta^{(i)}}{r^2} \right) + C_{11}^{(i)} + C_{12}^{(i)} \frac{\partial^2 u_r^{(i)}}{\partial r \partial \theta} + \frac{3C_{11}^{(i)} - C_{12}^{(i)}}{2} \frac{\partial u_r^{(i)}}{\partial \theta} &= 0
\end{align*}
\]  

(2)

in which
\[ C_{11}^{(i)} = \frac{E^{(i)}(1-v^{(i)})}{(1-2v^{(i)})(1+v^{(i)})}, \quad C_{12}^{(i)} = \frac{E^{(i)}v^{(i)}}{(1-2v^{(i)})(1+v^{(i)})} \]

for plane strain assumption;

\[ C_{11}^{(i)} = \frac{E^{(i)}}{1-(v^{(i)})^2}, \quad C_{12}^{(i)} = \frac{E^{(i)}v^{(i)}}{1-(v^{(i)})^2} \]

for plane stress assumption.

where \( E^{(i)} \) and \( v^{(i)} \) are the homogenized Young’s modulus and Poisson’s ratio for the \( i \)-th layer, respectively. Eq. (2) can be solved by assuming Fourier series expansions:

\[ u_r^{(i)}(r,\theta) = \sum_{m=1}^{\infty} \left[ f_m^{(i)}(r) \cos m\theta + g_m^{(i)}(r) \sin m\theta \right] \]

\[ u_\theta^{(i)}(r,\theta) = \sum_{m=1}^{\infty} \left[ f_m^{(i)}(r) \sin m\theta + g_m^{(i)}(r) \cos m\theta \right] \]

Substituting Eq. (3) into Eq. (2) leads to the expressions for displacement field with different harmonic terms:

\[ u_r^{(i)}(r,\theta) = F_{01}^{(i)} a_{\xi} + F_{02}^{(i)} a_{\xi}^{-1} + \sum_{j=1}^{3} a_{\xi}^{p_{ij}} (F_{ij}^{(i)} \cos \theta + G_{ij}^{(i)} \sin \theta) \]

\[ + \sum_{m=2}^{\infty} \sum_{j=1}^{4} a_{\xi}^{p_{mj}} (F_{mj}^{(i)} \cos m\theta + G_{mj}^{(i)} \sin m\theta) \]

\[ u_\theta^{(i)}(r,\theta) = \sum_{j=1}^{3} a_{\beta}^{p_{ij}} (F_{ij}^{(i)} \sin \theta - G_{ij}^{(i)} \cos \theta) + \sum_{m=2}^{\infty} \sum_{j=1}^{4} a_{\beta}^{p_{mj}} \xi^{p_{mj}} (F_{mj}^{(i)} \sin m\theta - G_{mj}^{(i)} \cos m\theta) \]

Substituting Eq. (3) into Eq. (2) leads to the expressions for displacement field with different harmonic terms:

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\[ + \sum_{m=2}^{\infty} \sum_{j=1}^{4} a_{\xi}^{p_{mj}} (F_{mj}^{(i)} \cos m\theta + G_{mj}^{(i)} \sin m\theta) \]

\[ u_\theta^{(i)}(r,\theta) = \sum_{j=1}^{3} a_{\beta}^{p_{ij}} (F_{ij}^{(i)} \sin \theta - G_{ij}^{(i)} \cos \theta) + \sum_{m=2}^{\infty} \sum_{j=1}^{4} a_{\beta}^{p_{mj}} \xi^{p_{mj}} (F_{mj}^{(i)} \sin m\theta - G_{mj}^{(i)} \cos m\theta) \]

\[ \xi = r/a \]

where \( \xi \) is introduced as a non-dimensionalized parameter. The introduction of \( \xi \) is very important for structures with large dimensions and helps mitigating the ill-conditioned equations that affect the accuracy of solutions.

The corresponding eigenvalues \( p_{mj} \) in Eq. (4) are given as

\[ p_{m1} = m + 1, \quad p_{m2} = m - 1, \quad p_{m3} = -(m + 1), \quad p_{m4} = -(m - 1) \]

and eigenvectors are expressed in the form of

\[ \beta_{mj}^{(i)} = \frac{2C_{22}^{(i)}(1-p_{mj}^2) + m^2(C_{22}^{(i)} - C_{23}^{(i)})}{m[(C_{22}^{(i)} + C_{23}^{(i)})p_{mj} - 3C_{22}^{(i)} + C_{23}^{(i)}]} \]

The radial, shear and tangential stress distributions of \( i \)-th layer are then obtained through strain-displacement relationship and stress-strain relationship:
\[
\sigma^{(i)}_{rr}(r, \theta) = (C^{(i)}_{11} + C^{(i)}_{12})F_{01}^{(i)} - (C^{(i)}_{11} - C^{(i)}_{12})F_{02}^{(i)} \xi^{-2} + \sum_{j=1}^{3} F^{(i)}_{1j} \xi^{P_{ij} - 1} (F^{(i)}_{1j} \cos \theta + G^{(i)}_{1j} \sin \theta) \\
+ \sum_{m=2}^{n} \sum_{j=1}^{4} P^{(i)}_{mj} \xi^{P_{mj} - 1} (F^{(i)}_{mj} \cos m\theta + G^{(i)}_{mj} \sin m\theta)
\]

\[
\sigma^{(i)}_{r\theta}(r, \theta) = \sum_{j=1}^{3} R^{(i)}_{1j} \xi^{P_{1j} - 1} (F^{(i)}_{1j} \sin \theta - G^{(i)}_{1j} \cos \theta) + \sum_{m=2}^{n} \sum_{j=1}^{4} R^{(i)}_{mj} \xi^{P_{mj} - 1} (F^{(i)}_{mj} \sin m\theta - G^{(i)}_{mj} \cos m\theta)
\]

\[
\sigma^{(i)}_{\theta\theta}(r, \theta) = (C^{(i)}_{11} + C^{(i)}_{12})F_{01}^{(i)} + (C^{(i)}_{11} - C^{(i)}_{12})F_{02}^{(i)} \xi^{-2} + \sum_{j=1}^{3} S^{(i)}_{1j} \xi^{P_{1j} - 1} (F^{(i)}_{1j} \cos \theta + G^{(i)}_{1j} \sin \theta) \\
+ \sum_{m=2}^{n} \sum_{j=1}^{4} S^{(i)}_{mj} \xi^{P_{mj} - 1} (F^{(i)}_{mj} \cos m\theta + G^{(i)}_{mj} \sin m\theta)
\]

where \( P^{(i)}_{mj} = C^{(i)}_{11} p^{(i)}_{mj} + C^{(i)}_{12} (1 + m \beta^{(i)}_{mj}) \), \( R^{(i)}_{mj} = (C^{(i)}_{11} - C^{(i)}_{12})/2 \cdot [(p^{(i)}_{mj} - 1) \beta^{(i)}_{mj} - m] \), and \( S^{(i)}_{mj} = C^{(i)}_{11} (1 + m \beta^{(i)}_{mj} ) + C^{(i)}_{12} p^{(i)}_{mj} \).

In order to solve the problem of the \( N \)-layered thick cylinders, the stress and displacement continuity conditions between \( i \)-th and \((i+1)\)-th layers are applied at \( r = r_{i+1} \):

\[
\begin{align*}
&u^{(i)}_{r}(r_{i+1}, \theta) = u^{(i+1)}_{r}(r_{i+1}, \theta), \quad u^{(i)}_{\theta}(r_{i+1}, \theta) = u^{(i+1)}_{\theta}(r_{i+1}, \theta) \\
&\sigma^{(i)}_{rr}(r_{i+1}, \theta) = \sigma^{(i+1)}_{rr}(r_{i+1}, \theta), \quad \sigma^{(i)}_{r\theta}(r_{i+1}, \theta) = \sigma^{(i+1)}_{r\theta}(r_{i+1}, \theta)
\end{align*}
\]

Through matching harmonic terms with same orders and applying orthogonality, the relationship of unknown coefficients in Eqs. (4,7) is established for

(1) \( n = 0 \) terms:

\[
A_{0}^{i}F_{0}^{i} = B_{0}^{i+1}F_{0}^{i+1}
\]

in which \( F_{0}^{i} = \left[ F_{01}^{i}, F_{02}^{i} \right]^T \), and

\[
A_{0}^{i} = \begin{bmatrix}
1 & (r_{i+1}/a)^{-2} \\
C^{(i)}_{11} + C^{(i)}_{12} & -(C^{(i)}_{11} - C^{(i)}_{12})(r_{i+1}/a)^{-2}
\end{bmatrix}, \quad B_{0}^{i+1} = \begin{bmatrix}
1 & (r_{i+1}/a)^{-2} \\
C^{(i+1)}_{11} + C^{(i+1)}_{12} & -(C^{(i+1)}_{11} - C^{(i+1)}_{12})(r_{i+1}/a)^{-2}
\end{bmatrix}
\]

(10)

(2) \( n = 1 \) terms:

\[
A_{1}^{i}F_{1}^{i} = B_{1}^{i+1}F_{1}^{i+1}
\]

where \( F_{1}^{i} = \left[ F_{11}^{i}, F_{12}^{i}, F_{13}^{i} \right]^T \), and

7
Finally, the boundary loading conditions are applied at the inner periphery \((r_i = a)\) of the innermost layer #1:

\[
\sigma_{rr}^{(i)}(r, \theta) = (C_{11}^{(i)} + C_{12}^{(i)}) F_{01}^{(i)} - (C_{11}^{(i)} - C_{12}^{(i)}) F_{02}^{(i)} + \sum_{m=1}^{4} \sum_{j=1}^{4} P_{mj}^{(i)} \xi P_{p_i}^{-1} (F_{mj}^{(i)} \cos m\theta + G_{mj}^{(i)} \sin m\theta) = p_{in}
\]

\[
\sigma_{r\theta}^{(i)}(r, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{4} R_{mj}^{(i)} \xi P_{p_i}^{-1} (F_{mj}^{(i)} \sin m\theta - G_{mj}^{(i)} \cos m\theta) = 0
\]

and at the outer periphery \((r_{N+1} = b)\) of the outermost layer #N

\[
\sigma_{rr}^{(N)}(r, \theta) = (C_{11}^{(N)} + C_{12}^{(N)}) F_{01}^{(N)} - (C_{11}^{(N)} - C_{12}^{(N)}) F_{02}^{(N)} + \sum_{m=1}^{4} \sum_{j=1}^{4} P_{mj}^{(N)} \xi P_{p_i}^{-1} (F_{mj}^{(N)} \cos m\theta + G_{mj}^{(N)} \sin m\theta) = p_{out}
\]

\[
\sigma_{r\theta}^{(N)}(r, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{4} P_{mj}^{(N)} \xi P_{p_i}^{-1} (F_{mj}^{(N)} \sin m\theta - G_{mj}^{(N)} \cos m\theta) = 0
\]
By combining Eqs. (11,13,15-16), a system of equations is established by matching the \( m \)-th \((m \geq 1)\) order harmonic terms for the \( N \)-layer composite cylinders:

\[
\begin{bmatrix}
P^{(1)}_m & 0 & \ldots \\
0 & P^{(1)}_m & \ldots \\
R^{(1)}_m & 0 & \ldots \\
\vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots \\
0 & P^{(N)}_m & 0 \\
\vdots & \vdots & \vdots \\
R^{(N)}_m & 0 & \ldots \\
0 & R^{(N)}_m & \ldots \\
\end{bmatrix}
\begin{bmatrix}
P^1_m(m) \\
p^2_m(m) \\
p^1_{out}(m) \\
p^2_{out}(m)
\end{bmatrix}
\begin{bmatrix}
F^1_m \\
G^1_m \\
F^N_m \\
G^N_m
\end{bmatrix}
\]

where \( P^{(i)}_m = P^{(i)}_{m_j}(r_{i+1}/a)^{p_{m_j}} \) and \( R^{(i)}_m = R^{(i)}_{m_j}(r_{i+1}/a)^{p_{m_j}} \).

Similar equations can also be set up for \( m = 0 \) terms, which are not repeated here. After obtaining the unknown coefficients, the displacement and stress fields in Eqs. (4,7) can be easily generated.

The boundary value problem is solved for the \( N \)-layered structures. It is found that the procedure of obtaining the solutions after dividing the continuously functionally graded thick cylinders into several layers are much simpler and more understandable than the work done by Horgan and Chan [5], Tutuncu [6], Xiang et al. [7], Tutuncu and Ozturk [8] for continuously graded structures. In addition, the solutions for the continuously functionally graded thick cylinders under harmonic pressures cannot be easily obtained using analytical methods. Similar ideas has also been applied in solving the bending of laminates [26]. Several examples in the next section will validate the accuracy of the present derivations.

3 Validation

3.1 A homogeneous thick cylinder

The present derivation is firstly validated through generating the stress distributions in a homogeneous thick cylinder, modeled as 10 layers under constant internal pressure
\( p_{in} = -100 \text{MPa} \). The analytical stress expressions are already given for a single homogeneous cylinder [8]:

\[
\begin{align*}
\sigma_{rr}^{HC}(r) &= p_{in} a^2 (r^2 - b^2) / r^2 (a^2 - b^2) \\
\sigma_{\theta\theta}^{HC}(r) &= p_{in} a^2 (r^2 + b^2) / r^2 (a^2 - b^2)
\end{align*}
\]  

(18)

The inner and outer radii of the cylinder are \( a = 50 \text{mm} \) and \( b = 100 \text{mm} \), respectively. It should be noted that the stress distributions for homogeneous cylinder are not directly related with the material properties, Eq. (18). The existing stress distributions \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) are compared between the present solutions and analytical expressions in Fig. 2, where good agreement is obtained.
Figure 2 Comparisons of radial and tangential stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ between 10-layer composite cylinders and homogeneous cylinder under constant inner pressure.

Then an extended study is conducted for the same 10 layered cylinders by applying internal and external pressures expressed using harmonic terms, which has not been used before:

\[
\begin{align*}
  p_{\text{in}} &= p^0 + p^0 \sin 4\theta + p^0 \cos 4\theta \\
  p_{\text{out}} &= p^0 \sin 2\theta + p^0 \cos 2\theta
\end{align*}
\]

which are also treated as the analytical solutions at boundaries for comparisons, and $p^0 = -100\text{MPa}$.

First of all, the stress distributions $\sigma_{rr}$ are compared at the inner and outer perimeters of the homogeneous cylinder in Fig. 3, where open circles are generated from Eq. (19). It can be easily observed that good agreement is obtained between the present analysis and analytical solutions, where 4th order and 2nd order harmonic terms are applied at inner and outer peripheries, respectively. The offset magnitude at the inner periphery is caused by the constant pressure applied, Eq. (19). In addition, the shear stress distributions $\sigma_{r\theta}$ are also generated (not shown), and practical zeros are obtained along the tangential directions.
Figure 3 Comparisons of radial stress $\sigma_{rr}$ at inner and outer peripheries of the 10-layer cylinder, to mimick a thick homogeneous cylinder.
Figure 4 Distributions of stress components $\sigma_{rr}$, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ along radial direction at $\theta = \pi/2$ for homogeneous cylinder under the pressure as described in Eq. (19).

Fig. 4 illustrates the distributions of stress components $\sigma_{rr}$, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ along the radial direction at $\theta = \pi/2$. $\sigma_{rr}$ arrives at -200MPa and 100MPa at inner and outer peripheries, respectively, which coincides with Fig. 2, while $\sigma_{r\theta}$ touches zero at both sides. In addition, greater magnitude of $\sigma_{r\theta}$ is generated along the radius, which cannot be easily captured in constant loading case [8].

3.2 Functionally-graded layered thick hollow cylinders

The deformations and stresses of a hollow cylinder with continuously exponentially graded material properties in radial direction was derived by Xiang et al. [7]. Same parameters are adopted in this scenario for consistency, where the stiffness coefficients of the FG cylinder are assumed along the radial direction as

$$C_{11}(r) = C_{11}^0 e^{\lambda r/b}, \quad C_{12}(r) = C_{12}^0 e^{\lambda r/b}$$

in which $C_{11}^0$ and $C_{12}^0$ are initial stiffness at the inner radius $r_i = a$, with $C_{11}^0 = 81.707\,\text{GPa}$ and $C_{12}^0 = 31.775\,\text{GPa}$. $\lambda$ is the constant describing the material gradient and $\lambda = 2.379\,\text{m}^{-1}$. 

\[C_6 = 612.0 \times 792.0 \times 14\]

\[\text{Figure 4} \quad \text{Distributions of stress components } \sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta} \text{ along radial direction at } \theta = \pi/2 \text{ for homogeneous cylinder under the pressure as described in Eq. (19).}

\[\text{Fig. 4 illustrates the distributions of stress components } \sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta} \text{ along the radial direction at } \theta = \pi/2. \quad \sigma_{rr} \text{ arrives at -200MPa and 100MPa at inner and outer peripheries, respectively, which coincides with Fig. 2, while } \sigma_{r\theta} \text{ touches zero at both sides. In addition, greater magnitude of } \sigma_{r\theta} \text{ is generated along the radius, which cannot be easily captured in constant loading case [8].}

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The analytical solutions [7] are compared with the present solutions wherein the continuously graded [as per Eq. (20)] cylinder is modeled as $N$-layered cylinders, with each layer having constant material properties. The material properties of each layer are obtained by picking the values at the mid-point of the local layers, following the gradation in Eq. (20). Another type of loading is applied at the inner radius and the outer pressure is zero:

\[
p_{in} = p^0 + p^0 \cos 2\theta \\
p_{out} = 0
\]

where $p^0 = -100\text{MPa}$.

First of all, the effect of the number of layers on the radial, shear and tangential stresses is tested in Fig. 5 along the radial direction at $\theta = \pi/4$, where not much difference is observed for $\sigma_{rr}$ and $\sigma_{r\theta}$ (except for $N=2$), but significantly reduced discontinuities are obtained for $\sigma_{\theta\theta}$ with the increase of the number of layers. It can be easily assumed that a smooth curve will be generated for infinite number of layers along the radial direction. Furthermore, by degenerating constant loading pressure $p_{in} = p^0, p_{out} = 0$, the midpoint interpolated stress components are investigated for the 40 layer cylinder and compared with analytical solutions of continuously functionally graded structure [7] in Fig. 6. The radial stress $\sigma_{rr}$ is still continuous along the radial direction and well matched with the analytical result. The tangential stress $\sigma_{\theta\theta}$ should be discontinuous in this situation between layers (Fig. 5), but using stress interpolation at middle points of local layers (dash line) still recovers the analytical results for a continuously functionally graded structure. Based on the comparison results, it can be safely concluded that the stress interpolation among the midpoints of layered composites generates exactly the same stress distributions as the analytical solutions of continuously FG structures. Thus, the interpolated stress distributions can be employed to illustrate further numerical characteristics in this presentation. What should also be mentioned is that the shear stress components $\sigma_{r\theta}$ becomes significant under harmonic loading conditions, making itself an important factor in the consideration of delaminations of the composite structures. Thus, a careful examination of shear stress is necessary.
Figure 5 Stress components $\sigma_{rr}$, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ along the radius of the cylinders at $\theta = \pi/4$ with increasing number of layers, wherein each layer has constant properties.
Figure 6 Comparisons of interpolated radial and tangential stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ between $N$-layer cylinders wherein each layer has constant properties and a continuously functionally graded cylinder [7].

3.3 Exponential function vs Power law function in a FG cylinder

Several descriptive functions have been used in the past, to describe the FG material properties of cylinders (discs) or spheres in the radial direction. Exponentially-varying functions were adopted by Tutuncu [8], leading to the complicated expressions in the form of Frobenius method; Power law functions were employed by a larger group of researchers [5-6,27-28] mainly because a simpler Navier’s equation can be obtained. Linear functions are sometimes applied but with much less popularity. It has been reported that the effect of the variation of Poisson’s ratio on stress distributions can be neglected and a constant Poisson’s ratio is thus usually assumed along the radial direction as in the literature [5,16,29].
Figure 7 Effective property distributions $C_{11}$ by employing (a) exponential and (b) power law functions.
Figure 8 Radial stress distributions $\sigma_{rr}$ using (a) exponential and (b) power law functions.
In order to show the effects of the functions used to describe the material gradation, Fig. 7 generates the varying modulus $C_{11}$ of a 20-layer FG cylinder using both exponential and power functions with parameters of different values. Same dimensions and loading conditions are employed here as the in last section, while different values of parameter $\lambda$ are chosen. It could be easily noted that the exponential functions change the modulus with much larger magnitude, which may not be easily achieved in real practice through reinforcements. However, the power law model provides a much more reasonable interpretation of modulus in the radial direction, along with the simpler Navier’s equation, making it a more popular choice.

Also, radial stresses $\sigma_{rr}$ of composite cylinders with modulus expressed in exponential and power gradient functions are generated in Fig. 8 with the constant loading conditions $p_{in} = -100\text{MPa}, p_{out} = 0$. Greater differences are noticed for power law function with different parameters, facilitating its applications in the design process.

4 Mori-Tanaka model for homogenization of properties

4.1 M-T model for effective material parameters

Mori and Tanaka [21-22] proposed the exact elastic homogenization technique in accordance with the Eshelby-type embedding approach [30]. M-T model is generally considered as the classical method that provides accurate homogenized moduli and easily-to-implement expressions [31]. The advantages and shortcomings of M-T model can be referred to [32].

A brief description is provided below of the micromechanical model. Based on the average matrix stress assumption, the fibrous, spherical or disc inclusions are embedded into the matrix phase, and then far-field macroscopic strains were applied to generate strain concentration matrix $A'_{mve}$, which is used to relate the averaged fiber strain with averaged matrix strain

$$\bar{\varepsilon}_f = A'_{mve}\bar{\varepsilon}_m$$

(22)

For any two-phase composite materials,

$$\bar{\varepsilon} = v_f\bar{\varepsilon}_f + v_m\bar{\varepsilon}_m \Rightarrow \bar{\varepsilon} = [v_fA'_{mve} + v_mI]\bar{\varepsilon}_m$$

(23)

from which the strain concentration matrix relating the average fiber strain with the macroscopic average strain becomes

$$A' = A'_{mve}[v_fA'_{mve} + v_mI]^{-1}$$

(24)
Then the final elastic stiffness matrix can be expressed as

\[ C^{M,T} = C^m + v_f (C^f - C^m) A^f_m \left[ v_f A^f_m + v_m I \right]^{-1} \]  

(25)

from which Young’s modulus and Poisson’s ratio are obtained. The explicit expressions were offered by Tandon and Weng [23] for heterogeneous media reinforced with different shaped inclusions.

### 4.2 A real application for composite cylinders with spherical inclusions

Thick-walled corundum particulate reinforced FGM plaster tubes, manufactured by a method of centrifugal casting, are considered by Fukui and Yamanaka [18]. The material properties of individual constituents are given in Table 1. The inner and outer radii of the tubes are \( a = 30 \text{mm} \), \( b = 45 \text{mm} \), respectively. Controlled parameters for the graded distributions of particles are the mean volume fraction \( \overline{v}_f \) (which are 15%, 25%, and 35% in this case) and multiples of gravity \( G \), which is the ratio of the centrifugal force to the gravity and chosen as 2.1. Fukui and Yamanaka [18] provided an equation for the distributions of the volume fraction along the radius of cylinders:

\[ v_f(r) = A(r/b)^3 + B(r/b)^2 + C \]  

(26)

in which \( A, B \) and \( C \) are functions of the mean volume fraction \( \overline{v}_f \) and multiples of gravity \( G \).

<table>
<thead>
<tr>
<th>Materials</th>
<th>Young’s modulus ( E ) (GPa)</th>
<th>Poisson’s ratio ( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corundum</td>
<td>360</td>
<td>0.333</td>
</tr>
<tr>
<td>plaster</td>
<td>35</td>
<td>0.333</td>
</tr>
<tr>
<td>Sic</td>
<td>420</td>
<td>0.16</td>
</tr>
<tr>
<td>Aluminum</td>
<td>73</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Table 1 Material properties used in the present calculations
Fig. 9 presents the recovered distributed volume fractions from Eq. (26) in the radial direction, with three mean volume fractions: 15%, 25%, and 35%. The volume fractions behave in a systematic manner while mean values happen at around $r/b=0.833$. The corresponding stress components are generated by relating the structural analysis with micromechanical calculations by M-T model under loading conditions:

$$
\begin{align*}
\sigma_{in} &= p^0 + p^0 \sin 2\theta + p^0 \cos 2\theta \\
\sigma_{out} &= p^0 /2 + p^0 /2 \sin 2\theta + p^0 /2 \cos 2\theta 
\end{align*}
$$

(27)

where $p^0 = -100$MPa.

Fig. 10 shows the distributions of $\sigma_{rr}$, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ along the radial direction at $\theta = \pi/4$ for three different volume fraction distributions dictated by Fig. 0. Surprisingly, nearly invisible difference appears in the three stress distributions, which means stress distributions are not significantly affected by the magnitudes of volume fraction distributions of similar patterns under the present loading condition. One of the main reasons for the similar stress distributions is due to the similar modulus functions generated from the same volume fraction distribution in Eq. (25) with different unknown coefficients. The stress distributions generated in Fig. 10 show
engineers and designers of minimum effect of fiber volume fraction for future guidance. However, more general case needs to be investigated for further guidance.
Figure 10 Radial and tangential stresses $\sigma_{rr}$, $\sigma_{r\theta}$, and $\sigma_{\theta\theta}$ along the radial direction at $\theta = \pi/4$ using the three different volume fraction functions.

### 4.3 Effect of reinforcements on the overall structures

The micromechanical details may affect the stress distributions at the structural level. Thus, 40 layered aluminum composite cylinders with inner and outer radii $a = 30\text{mm}$, $b = 100\text{mm}$ are investigated when subjected to same loading conditions as last section. Each layer of the cylinders is reinforced using SiC aligned fibrous ($\alpha = 10$), spherical ($\alpha = 1$), or disc ($\alpha = 0.1$) inclusions, respectively. The following simple linear function is used to describe the volume fraction along the radial direction

$$v_f = (r-a)/(b-a)$$

(28)

where the volume fraction each layer is fixed by picking the value at mid-point. The effective properties of each layer are generated through M-T model [23]. Then the homogenized properties are employed as in Section 2 to analyze the stress and displacement fields at the structural level. The stiffness element $C_{11}$ is firstly shown in Fig. 11 along the radius of the cylinders, where $C_{11}$ converges to around 100GPa at inner radius ($v_f = 0$) and 460GPa at outer radius ($v_f = 1$), which are the essentially the modulus of matrix and fiber, respectively. It can be
easily observed that $C_{11}$ is the smallest for particle reinforcement while progresses interchangeably between two other reinforcements. Fig. 12 investigates the radial, shear and tangential stresses $\sigma_{rr}$, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$, as well as radial displacement $u_r$ along the thickness of the cylinders at $\theta = \pi/4$. Nearly same radial stress component $\sigma_{rr}$ is generated for composite cylinders with different aspect ratios, while shear stress $\sigma_{r\theta}$ shows minor distinctions: Fibrous reinforcement generates higher shear stress at locations where radius is smaller while disc reinforcement produces higher magnitude at larger-radius location, Fig. 12b. Less systematical patterns happen for tangential stress $\sigma_{\theta\theta}$: the three types of reinforcements lead to curves with different trends, Fig. 12c. The reinforcements also play roles in affecting the radial displacement distributions. Fibers and discs generate larger and smaller magnitude of displacements, respectively, while spherical reinforcement produces displacement between them, Fig. 12d. The effects of reinforcements with different aspect ratios on stress and displacement components naturally promote the idea of tailoring the parameters for better manipulations of structures. Next section we introduce particle swarm optimization technique to fulfill this purpose.

![Figure 11. Stiffness coefficient $C_{11}$ along the thickness of the 40 layered aluminum cylinders with $SiC$ reinforcements.](image)
Figure 12 Comparisons of (a) radial stress $\sigma_{rr}$, (b) shear stress $\sigma_{r\theta}$, and (c) tangential stress $\sigma_{\theta\theta}$, as well as (d) radial displacement $u_r$ for aluminum cylinders with SiC reinforcements.
5 Design of composite cylinders using Particle Swarm Optimization

Optimization techniques could be extraordinarily useful in the design and applications of FG composite materials, providing the inverse calculations of structural properties as well as structural designs of FG materials. Particle swarm optimization has received more and more acceptance in the realm of FG materials since its proposition in 1995 [25]. Fereidoon et al. [33] used particle swarm-based algorithms to search for the optimal volume fractions of FGMs, trying to minimize the peak residual stresses and maximize the safety factor. Kou et al. [34] use PSO to optimize the generic gradation patterns. The applications in FG materials have proved that the PSO is a stable and efficient technique that is recommended for further calculations. Wang and Pindera [35] combined the PSO with elasticity-based homogenization technique to search for the optimal parameters to minimize the local microstructural stresses in stress analyses.

The technique combines the concept of social interaction with solving boundary value problems, and constitutes a swarm of particles keeping searching the target (best solution) by continuously updating the previous experience. First, a swarm of particles with different dimensions of variables are initiated. The position of \( i \)-th particle is denoted as \( X_i = (x_1, x_2, \ldots, x_D) \), where \( D \) stands for the dimension of variables. The velocity of the same particle is \( V_i = (v_1, v_2, \ldots, v_D) \). \( pBest \) and \( gBest \) denote the best experience of \( i \)-th particle and other particles, respectively, and they are used to update the positions continuously. The updating algorithms are expressed as \([25,34]\)

\[
\begin{align*}
\alpha_{i}^{k+1} &= \chi[\omega_{i}^{k+1} + a_1 \text{rand}(pBest_{i}^{k} - x_{i}^{k}) + a_2 \text{rand}(gBest_{i}^{k} - x_{i}^{k})] \\
x_{i}^{k+1} &= x_{i}^{k} + \alpha_{i}^{k+1}
\end{align*}
\]

in which superscript \( k \) and \( k+1 \) are the iteration numbers. \( \text{rand}(\cdot) \) are random numbers with uniform distributions in the interval \([0,1]\), and \( a_1, a_2 \) are acceleration constants. The parameter \( \omega \) is the inertia weight parameter defined in terms of its initial and final values, \( \omega_{\text{max}} \) and \( \omega_{\text{min}} \), and the current and maximum iteration numbers \( k \) and \( k_{\text{max}} \):

\[
\omega = \omega_{\text{max}} - k(\omega_{\text{max}} - \omega_{\text{min}})/k_{\text{max}}
\]

The constriction factor \( \chi \) is introduced to ensure convergence:

\[
\chi = \frac{2}{2 - \varphi - \sqrt{\varphi^2 - 4\varphi}}
\]
where $\varphi = a_1 + a_2$ such that $\varphi > 4$. The optimized results are iterated until the desired accuracy or maximum iteration is reached.

5.1 Inverse calculation of particle volume fraction

PSO is first used in this paper, in conjunction with the Mori-Tanaka model to generate the optimal volume fractions that produce the desired properties of composite materials. Aluminum composite spheres (15 layers) reinforced with SiC particles (aspect ratio $\alpha = 1$) in each layer are considered in this scenario. Typically employed as the radial variation function of Young’s modulus in the literature, power law model is used here for the calculation. The inner and outer radii of the composite spheres are $a = 50\text{mm}$, $b = 100\text{mm}$, respectively. The function of the Young’s modulus can be expressed as

$$E(r) = E^0 (r/b)^\lambda$$

(32)

in which $\lambda$ is the parameter interpolating the Young’s modulus, which is picked as $\lambda = 3$ here. Initial Young’s modulus is $E(r = 0) = E^0 = 420\text{GPa}$. The target function of PSO is expressed as

$$F_i = \text{minimize} \left[ \frac{E^{M-T}(v_f) - E(r)}{E(r)} \right]$$

(33)

where $E^{M-T}(v_f)$ is the Young’s modulus generated using M-T model by searching for the best $v_f$ that is matched with $E(r)$ in Eq. (32) at the mid-point of each layer. 20 particles and 20 maximum iterations are designed. Total 3.03 seconds are used to generate the converged results at 15 locations along the radius, due to the efficient PSO execution and M-T model. Fig. 13 (a) shows the convergence of the PSO after each iteration step, and practically zero is obtained after the 9th iteration, indicating well-matched results between optimized and target modulus. The optimized volume fractions are plotted and the corresponding Young’s modulus is recovered in Fig 13 (b).
Figure 13 Errors of PSO at each iteration (a), and the optimized volume fractions and corresponding Young’s modulus (b).
5.2 Design of composite cylinders by tailoring parameters

As is already mentioned in previous sections, the shear stresses in the transverse directions of composite cylinders are important in determining the delamination of structures. In addition, several factors play roles in determining the stress and displacement distributions, such as thickness of the cylinders and radial varying functions. Thus, two corresponding parameters are designed for the structural optimization of layered hollow cylinders with spherical reinforcement, which are inner and outer radius ratio \( a/b \) and the parameter \( \lambda \) in the function of varying volume fraction \( \nu_j = (r/b)^\lambda \). The designated upper and lower bounds of the parameters are set as \( a/b \in [0.1;0.9] \) and \( \lambda \in [0;10] \) while the target function is to designed to minimize the shear stresses along the thickness of the cylinders:

\[
F_2 = \text{minimize} \left( \left| \sigma_{r\theta}^{\max}(r) \right| \right)
\] (34)

40 particles and 40 maximum iterations are designed at the first step. The final optimized results are obtained after about 20.47 seconds using HP Intel(R) Core(TM) i7 CPU. The converged parameters are \( a/b = 0.501, \lambda = 5.140 \), and the final result is \( \left| \sigma_{r\theta}^{\max} \right| = 0.1401\text{MPa} \).

Fig.14 illustrates the initial distributions and final converged distributions of particles in the optimization process. A group of initial particles (open circles) are spared within the designated range, after about 23th iteration, the particles converge to a certain point (solid particles) by keeping updating their previous experiences. A closer observation of the stress distributions along the radial direction is investigated by changing one of the parameters \( a/b \) with several values in Fig. 15, it can be easily found that the optimized parameters (0.501) give the minimum absolute values of shear stress \( \left| \sigma_{r\theta}^{\max} \right| \), which are almost zeros along the radius of the cylinders. In addition, the tangential stress \( \sigma_{\theta\theta} \) also converges to minimum values along the thickness with the optimized parameter, while the radial stresses \( \sigma_{rr} \) are more systematic and smaller (absolute) stresses happen for thicker layered structures. Interpretation of other parameters generates similar situations.
Figure 14 Initial and final particle distributions before and after the optimization
Figure 15 Radial, shear and tangential stress distributions using different geometrical parameters $a/b$ compared with the optimized results.
6 Conclusions

Most of the published research work in prior literature is focused on the stress and deformation investigations of continuously functionally graded hollow cylinders (discs) or spheres under constant loading conditions. Herein a unified framework is established in the structural and microstructural analyses and designs on similar topics, by combining structural derivations, micromechanical homogenization and optimization techniques. The composite $N$-layered hollow cylinders (discs) with each layer being homogeneous are studied by applying harmonic loading conditions, the solutions of which are validated against the analytical solutions of boundary expressions and continuously graded elastic heterogeneous structures. It is proved that solutions in $N$-layered structures with each layer being homogeneous can generate the same stress distributions as in continuously graded structures, except with much simplicity. The significant shear stress distributions are also investigated under harmonic loading case, which are not present in the constant loading conditions. The Mori-Tanaka model is briefly introduced and employed in generating homogenized moduli for $N$ composite layers reinforced with different shaped inclusions with designated volume fractions. The accuracy of M-T model has been validated with easily implementable expressions. Several numerical examples also study the effects of the shape and volume fraction of inclusions on the stress distributions. Finally, the well known particle swarm optimization is employed in the present work to not only describe inverse calculations from homogenized moduli by altering volume fractions, but also more importantly to provide design procedures for many potential possibilities for structures by material property gradation or geometrical parameters. Both the quick executions and accurate predictions make the present procedures to be very simple for the analysis and design of continuously functionally graded thick cylinders. In the future work, the multiscale framework can be combined with more sophisticated computational micromechanics techniques that provide not only effective properties but also accurate localized stress concentrations, helping indentifying the possible damage initiations and propagations [36-37]. In addition, the interlayer shear stresses play important roles in the design of the functionally graded structures, especially when considering the delamination phenomenons. Thus, an investigation of shear stress components in other structures, such as (composite spheres), will be necessary and will be presented in our future work.
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References


