CONSTITUTIVE MODELING OF CYCLIC PLASTICITY AND CREEP, USING AN INTERNAL TIME CONCEPT

O. WATANABE and S. N. ATLURI
Georgia Institute of Technology

Abstract—Using the concept of an internal time as related to plastic strains, a differential stress-strain relation for elastoplasticity is rederived, such that (i) the concept of a yield-surface is retained; (ii) the definitions of elastic and plastic processes are analogous to those in classical plasticity theory; and (iii) its computational implementation, via a “tangent-stiffness” finite element method and a “generalized-midpoint-radial-return” stress-integration algorithm, is simple and efficient. Also, using the concept of an internal time, as related to both the inelastic strains as well as the Newtonian time, a constitutive model for creep-plasticity interaction, is discussed. The problem of modeling experimental data for plasticity and creep, by the present analytical relations, as accurately as desired, is discussed. Numerical examples which illustrate the validity of the present relations are presented for the cases of cyclic plasticity and creep.

I. INTRODUCTION

The characterization of material behavior at elevated temperatures plays an important role in the design of structures such as in hot sections of modern jet engines and other power plants. The ASME Code [1974] defines acceptable levels of stress and strain in critical components of power plants operating at elevated temperatures. The severe mechanical environment may often cause these structures to operate near or beyond the yield limit of the material. Consequently, a unified theory of creep and plasticity, applicable to cyclic loading, is often desirable.

Typical constitutive relations for creep reported and used in literature include the modified strain hardening rule developed by researchers at the Oak Ridge National Lab (PUGH et al. [1972], CORUM et al. [1974]), dislocation models (LAGEBOE [1971], GITTUS [1976]) based on metal physics, nonlinear viscoelasticity theory (BESSELING [1958]); and the kinematic hardening model (MALININ & KHAJINSKY [1972]) using an analogy to plasticity. However, recent efforts in material-constitutive-model development reveal a trend toward unifying creep and plasticity. Some experimental results (CORUM et al. [1974], JASKE et al. [1975]) have been reported concerning the interaction between creep and plasticity. These unifying theories may be roughly divided into the three categories of (i) potential theories, (ii) microphenomenological theories and (iii) nonlinear viscoplastic theories. Most studies employ these theories either individually or in combination. In the first category, one may cite the theories using time-dependent parameters (KRAJTKIVL & DILLON [1970]), the concept of kinematic hardening (KUJAWSKI & MROZ [1980]), micromechanical considerations (PONTIN & LECKIE [1976]), and a combination of viscoplastic theory (CHABOCHE [1977]). The phenomenological theories (BODNER & PARTOM [1975], HART [1976], HART et al. [1976], MILLER [1976a, 1976b], KRIEG et al. [1978], LEE & ZAVRE [1978], ROBINSON [1978], BODNER et al. [1979], STOFFER & BODNER [1979]) employ certain internal variables to reflect the micromechanics of deformation, such as involving dislocations. Most of these theories assume that the plastic strains are also time-dependent, as are creep strains, and
that the creep surface will translate and expand in the stress-space in a manner similar to that of isotropic and kinematic hardening used in classical plasticity theory. The nonlinear viscoplasticity theories have the variations, in which: the coefficients of the linear viscoelastic theory (CERNOCKY & KREMPLE [1980]) are expressed as a function of stresses and strains, the inelastic strains are divided into viscous and viscoelastic components (FINDLEY & LAI [1978]), and the internal time is measured by the (total) strain history (VALANIS [1971a, 1971b, 1980], BAZANT & BHAT [1976], BAZANT [1978], WU & YIP [1980, 1981], VALANIS & FAN [1983]). Of course, the fundamental aspects of inelastic deformation are also studied (RICE [1970, 1975], HILL & RICE [1972]) based on micromechanical considerations.

The intrinsic time theory, labeled "the endochronic theory" was presented by VALANIS [1971a, 1971b]. This theory held out the prospect of explaining the experimental phenomena of cross-hardening, cyclic hardening, and initial strain problems—the situations that classical plasticity theory could not cope with. BAZANT & BHAT [1976] also showed that the "endochronic" theory is effective in dealing with problems of inelasticity and failure in concrete, and that the Maxwell chain model can describe the creep behavior.

VALANIS [1980] later presented a slightly modified intrinsic time model, wherein the internal time is related to the inelastic strain. Recently, VALANIS & FAN [1983] presented an incremental or differential form of the integral relation of stress and strain (VALANIS [1980]) for plasticity. This differential relation (VALANIS & FAN [1983]) is of a fundamentally different form as compared to that of the classical plasticity theory and does not employ the notion of a yield surface nor the attendant concepts of "elastic" and "plastic" processes. Based on such a differential relation, VALANIS & FAN [1983] developed an "initial strain" type iterative finite element approach. In this approach, the determination of stress history (or the stress rate) from a given strain history (or the strain rate) is also highly iterative in nature.

While using the concept of an intrinsic time, which depends on plastic strains, and the integral relations of stress and strain (VALANIS [1980]), we rederive here a differential stress-strain relation, such that (i) the concept of a yield surface is retained; (ii) the definitions of "elastic" and "plastic" processes are analogous to those in classical plasticity theories; and (iii) it can be implemented in a computationally simple and efficient manner, via a "tangent-stiffness" finite element method, and a "generalized-midpoint-radial-return" algorithm for determining the stress history (or the stress rate) for a given strain history (or strain rate). The details of analytically modeling the test data, for monotonic or cyclic plasticity, as accurately as desired, through these differential relations, are discussed.

This paper also presents a simple theory for creep, using the concept of intrinsic time which is measured by the inelastic strain as well as Newtonian time, both of which are irreversible. Further, the present theory makes it possible to incorporate the effect of interaction between creep and plasticity in a simple fashion.

Numerical results are presented for cyclic plasticity and creep, in order to verify the validity of the present theories. It is shown that the present constitutive relations are simple in form, and the material constants involved are few in number. Thus they may be useful in practical analyses of inelastic behavior.

In Section II, we present the nomenclature; Section III contains theoretical developments for plasticity based on an intrinsic time measure; Section IV contains discussion of the issues related to the determination of material constants, characterization of monotonic and cyclic hardening plasticity, and certain pertinent numerical results; and Section V contains a unified theory for creep and plasticity, and pertinent numerical results.
II. NOMENCLATURE

Considerations in the present work are restricted to small strains and infinitesimal deformations. For simplicity, we use a Cartesian system of coordinates $x_i$, with basis $e_i$. The stress and strain tensors are represented by $\sigma = \sigma_{ij} e_i e_j$ and $\epsilon = \epsilon_{ij} e_i e_j$, respectively. The stress and strain deviators are represented by $s = s_{ij} e_i e_j$ and $e = e_{ij} e_i e_j$, respectively. If $A(A_{ij} e_i e_j)$ and $B(B_{mn} e_m e_n)$ are two second-order tensors, the notation: $A \cdot B = A_{im} B_{mn} e_i e_n$ and $A : B = A_{ij} B_{ij}$ is employed.

III. PLASTICITY: THEORETICAL DEVELOPMENT

Let $d\epsilon$ be the strain rate and $d\epsilon$ its deviator; $d\epsilon_m$ the mean strain; $d\epsilon = d\epsilon^p + d\epsilon^e$; the plastic strain rate $d\epsilon^p$ is purely deviatoric, i.e. $d\epsilon^p = d\epsilon^p$; and thus, $d\epsilon_m$ is purely elastic, i.e. $d\epsilon_m = d\epsilon^e_m$. We consider the solid to be elastically isotropic. Thus we have

$$d\epsilon^p = d\epsilon - ds/2\mu_0 .$$

Following VALANIS [1980], we define an endochronic (internal) time $\zeta$ (which is a Newtonian time-like parameter), such that

$$d\zeta = (d\epsilon^p : d\epsilon^p)^{1/2} , \quad dz = d\zeta f(\zeta) , \quad f(0) = 1 , \quad d\zeta \geq 0 ,$$

where $f(\zeta)$ is monotonically increasing.

As in VALANIS [1980], the stress in the elastic plastic solid is represented through the integral

$$s = 2\mu_0 \int_0^z \rho(z - z') \frac{d\epsilon^p}{dz'} dz' ,$$

where $\mu_0$ is the initial (elastic) shear modulus, and $\rho(z)$ is a material-specific kernel. Equation (3) thus appears to circumvent the need for a yield surface as well as for the flow rules of classical plasticity theory. Differentiation of (3) leads to

$$ds = 2\mu_0 \left[ d\epsilon + \frac{h(z)}{\rho(0) f(\zeta)} \left( \frac{d\epsilon - ds}{2\mu} \right) \left( \frac{d\epsilon - ds}{2\mu} \right)^{1/2} \right] ,$$

$$\rho(0) = \rho \text{ at } z = 0 , \quad \mu_0 = \mu_0 [1 + \rho(0)]^{-1} , \quad h(z) = \int_0^z \frac{\partial \rho}{\partial z'} (z - z') \frac{d\epsilon^p}{dz'} dz'.$$

While the classical loading/unloading criteria (or criteria for elastic or plastic processes) are apparently bypassed in eqn (4), there are, nevertheless, prices extracted for this seeming simplicity. Some of these counterbalancing difficulties of the above endochronic approach, as compared to a classical plasticity theory, are as follows: (i) The determination of stress history (and $d\sigma$) for a given strain history (or $d\epsilon$) at each material point becomes highly iterative in nature, as seen from (4). (ii) In a finite element/boundary-element/or other weak solution of the boundary value problem, the trial solution $d\epsilon$ is derived by differentiation of trial displacements $du$. To determine the trial stresses $ds$ and yet retain a piecewise-linear-equation solution strategy, there is no recourse other than to approximate eqn (4) as $ds = 2\mu_0 d\epsilon$. Thus the stiffness matrix at any stage of loading is essentially the linear-elastic stiffness matrix; and the
The elastic-plastic solution method becomes the so-called "initial-strain" method. (iii) To model the uniaxial stress-strain curve of a material that does exhibit a sharp "knee" near the elastic limit, the kernel $\rho(z)$ has to be weakly singular at $z = 0$. These drawbacks notwithstanding, Valanis & Fan [1983] have recently presented a series of papers dealing with a direct computational implementation of an iterative, initial strain method based on eqn (4) and using exponential functions for the kernel $\rho(z)$ in eqn (3). Details of computational times for achieving convergence of plasticity iterations of the global finite element equations, or of the iterations for stress integration, are not readily available in the work of Valanis & Fan [1983].

Here we present a rederivation of rate-type elastic-plastic constitutive relations using the essential concepts of an endochronic theory, but with the following features: (i) The notion of a yield-surface, and the demarcation in the definitions of the elastic processes and plastic processes, are retained. (ii) The stress history (or $d\sigma$), for a given strain history (or $d\varepsilon$), can be determined quite easily, as in a classical plasticity theory, by using a "generalized-midpoint-radial-return" algorithm. (iii) The finite-element formulation can be based on a piecewise linear "tangent-stiffness" approach, wherein the material constitutive law at each point can be chosen differently depending on whether an elastic process or a plastic process is postulated at each point during the current "load" increment. The starting point here is the representation of the kernel $\rho(z)$ in eqn (3) in the form (as also suggested by Valanis [1980])

$$\rho(z) = \rho_0\delta(z) + \rho_1(z) \tag{6}$$

where $\delta(z)$ is a Dirac function and $\rho_1(z)$ is a nonsingular function. It is seen in the sequel that the term $\rho_0\delta(z)$ in eqn (6) leads to the notion of a yield surface; the function $f(\xi)$ in (2) leads to the notion of yield-surface-expansion (isotropic hardening); and the function $\rho_1(z)$ in (6) leads to the notion of yield-surface translation (kinematic hardening). Use of (6) in (3) leads to

$$s = 2\mu_0\rho_0 \frac{d\varepsilon}{dz} + 2\mu_0 \int_0^z \rho_1(z - z') \frac{d\varepsilon}{dz} dz' \tag{7a}$$

$$= \tau_0^0 \frac{d\varepsilon}{dz} + r(z) \tag{7b}$$

wherein the definitions of $\tau_0^0$ and $r(z)$ (the "back stress") are apparent. Equation (7b) can be written as

$$d\varepsilon = \frac{(s - r)}{\tau_0^0 f} \cdot d\xi \quad , \quad d\xi \equiv 0 \tag{8}$$

Of course, eqn (8) is entirely reminiscent of the classical flow-rule and normality relation for plastic strain-rate using a Mises' yield criterion. However, at this point, this similarity is purely formal.

From the very definition of $d\xi$ as in (2), it follows that, during plastic flow,

$$\frac{d\varepsilon}{d\xi} : \frac{d\varepsilon}{d\xi} = 1 \quad , \quad \text{i.e.} \quad (s - r) : (s - r) = [\tau_0^0 f(\xi)]^2 \tag{9a,b}$$
Equation (9b) clearly indicates that during plastic flow, the stress point, in the deviatoric stress space, remains on a Mises-cylinder of radius $\tau_0 f(\xi)$, with the center of the surface at $r$.

By differentiating (8) with respect to $\xi$, one obtains the following relation which holds during plastic flow:

$$\left(\frac{d^2 e^p}{ds^2} + \frac{de^p}{df}\right) = \left(\frac{ds}{df} - \frac{dr}{df}\right) \frac{1}{\tau_0^2}. \quad (10)$$

From (1), and the definition of $r$ as in (7), respectively, we see that:

$$\frac{ds}{df} = 2\mu_0 \left(\frac{de}{df} - \frac{de^p}{df}\right). \quad (11)$$

and

$$\frac{dr}{df} = 2\mu_0 \left[\rho_1(0) \frac{de^p}{df} + \frac{h^*}{f}\right], \quad (12a)$$

where

$$h^* = \int_0^z \frac{d\rho_1}{dz'} (z - z') \frac{de^p}{df} dz'. \quad (12b)$$

Use of (11) and (12a) in (10) results in

$$de = \left[1 + \rho_1(0) + \frac{\tau_0^2 (df/d\xi)}{2\mu_0}\right] de^p + \frac{h^*}{f} \frac{d\xi}{df} + \frac{\tau_0^2}{2\mu_0} \left(\frac{d^2 e^p}{df^2}\right) d\xi. \quad (13)$$

Also, during plastic flow, it follows from (9a) and (8) that

$$\frac{d^2 e^p}{d\xi^2} \frac{de^p}{d\xi} = 0 = \left(\frac{d^2 e^p}{d\xi^2}\right) \left(\frac{s - r}{\tau_0^2 f}\right). \quad (14)$$

Taking the trace of both sides of (13) with $[(s - r)/(\tau_0^2 f)]$ [or which is also equal to $(de^p/d\xi)$] and using (14), one obtains

$$de : \frac{(s - r)}{\tau_0 f} = \left[1 + \rho_1(0) + \frac{\tau_0^2 (df/d\xi)}{2\mu_0} + \frac{h^*}{\tau_0 f} (s - r)\right] d\xi = C d\xi, \quad (15)$$

wherein the definition of $C$ is apparent. Equation (15) can be rewritten as

$$d\xi = \frac{1}{C} \left[\frac{(de): (s - r)}{\tau_0 f}\right] = \frac{1}{C} de : N, \quad (16)$$

where $N = (s - r)/\tau_0 f$ is a unit “Normal.” Now, by definition, during a “Plastic Process,” i.e. when $de^p \neq 0$, we have $d\xi > 0$. Thus (16) clearly indicates:
(A) **Definition of a plastic process** \( (P) \): \( d\gamma > 0 \)

\[ (P) \text{ if } (i) \ (s - r) : (s - r) = (\tau_0^f)^2 \text{ and } \mathbf{d}:\mathbf{N} > 0 . \]  

Equation (16) also indicates that a "plastic process" is not possible if \( \mathbf{N}:\mathbf{d}\mathbf{e} \leq 0 \). In conformity with this, we define an "elastic process" as follows:

(B) **Definition of an elastic process** \( (E) \): \( \gamma = 0 \)

\[ (E) \text{ (i) if } (s - r) : (s - r) < (\tau_0^f)^2 \]  

or

\[ (E) \text{ (ii) if } (s - r) : (s - r) = (\tau_0^f)^2 \text{ and } \mathbf{N}:\mathbf{d}\mathbf{e} \leq 0 . \]

It is interesting to observe that the (Elastic) and (Plastic) processes defined above, for the present endochronic theory, depend directly on whether \( \mathbf{N}:\mathbf{d}\mathbf{e} \geq 0 \); while in the classical plasticity theory these processes depend, *ab initio*, on whether \( \mathbf{N}:\mathbf{d}\mathbf{\sigma} \geq 0 \). In computational mechanics, the central problem of plasticity is to determine \( \mathbf{d}\mathbf{\sigma} \), for a given \( \mathbf{d}\mathbf{e} \). In this context, the (E) and (P) criteria of (17) and (18) are more direct and more meaningful. Using (16) in (8), we obtain:

During (P):

\[ \mathbf{d}\mathbf{\epsilon}_p = \frac{1}{C} \mathbf{N}(\mathbf{N}:\mathbf{d}\mathbf{e}) \equiv \frac{1}{C} \mathbf{N}(\mathbf{N}:\mathbf{d}\mathbf{\epsilon}) , \]  

since \( \mathbf{N} \) is deviatoric. Recall that \( \mathbf{r} \) [see (7)] and \( \mathbf{h}^* \) [see (12b)], and through them, the coefficient \( C \) [see (15)] depend on the kernel \( \rho_1(z) \).

A convenient choice for the kernel \( \rho_1(z) \) is

\[ \rho_1(z) = \sum_i \rho_{1i} \exp(-\beta_i z) , \]  

such that, from (7) it follows that

\[ \mathbf{r} = \sum_i 2\mu_0 \int_0^z \left\{ \rho_{1i} \exp[-\beta_i(z-z')] \frac{\mathbf{d}\mathbf{\epsilon}_p}{dz'} dz' \right\} \equiv \sum_i \mathbf{r}^{(i)} \]  

and

\[ \mathbf{h}^* = \sum_i \left( \frac{-\beta_i}{2\mu_0} \right) \mathbf{r}^{(i)} , \]

with \( \mathbf{r}^{(i)} \) being defined in an apparent fashion. From (21) it follows that

\[ d\mathbf{r} = 2\mu_0\rho_1(0) \mathbf{d}\mathbf{\epsilon}_p - \left\{ \sum_i \frac{\beta_i \mathbf{r}^{(i)}(0)}{f} \right\}(\mathbf{d}\mathbf{\epsilon}_p : \mathbf{d}\mathbf{\epsilon}_p)^{1/2} . \]
Thus the evolution equation for $r$ is *nonlinear* in $de^P$ and thus is similar to a non-linear-kinematic-hardening relation. It has been discussed in detail by Watanabe & Atluri [1986] that the present theory, with the translation of the yield surface as in (21), and the expansion of the yield surface as specified by

$$f = (1 + \gamma \xi) \quad \text{[linear]} \quad (23a)$$

or

$$f = a + (1 - a) \exp(-\psi \xi) \quad \text{[saturated]} \quad (23b)$$

where $\gamma$ and $\psi$ are constants, and $\xi = \int \mathrm{d}z$, includes the multiple-yield-surface theories of Mroz [1969], Krieg [1975] and Dafalias & Popov [1976] as special cases.

Based on (19), the stress-strain relation in the present theory may be written as

$$ds = 2\mu_0 [de - (1/C)N(N:de)] \quad (24a)$$

$$(d\sigma : I) = (2\mu + 3\lambda)(de : I) \quad (24b)$$

where $\Gamma = 1$ in (P) and $\Gamma = 0$ in (E).

It is worth noting that Valanis [1980] presents an entirely different derivation for $dr$ and obtains an equation similar to the present eqn (16). However, his result for the coefficient $C$ given in his eqn (3.34) (Valanis [1980]), which contains certain algebraic errors, differs from the value of $C$ given in the present eqn (15), even after the algebraic errors of Valanis' work do not contain the term $[\gamma^0(\mathrm{df}/\mathrm{d}z)]/2\mu_0$ as in the present eqn (15). It will be shown later in this paper that the present eqn (15) is in fact correct.

It is interesting to compare the present stress–strain relations with the familiar classical plasticity theory relations for isotropic and kinematic hardening (Atluri [1984]).

### III.1. Classical isotropic hardening

$$ds = 2\mu_0 \left[ de - 3\mu_0 \frac{N(N:de)}{1 + (1/3\mu_0)H} \right] \quad (25a)$$

$$N = \sigma_0(\epsilon^P) \quad (25b)$$

where $\sigma_0(\epsilon^P)$ is the uniaxial equivalent stress as a function of the equivalent plastic strain, and $H$ is the slope of the true stress vs the logarithmic strain relation.

### III.2. Classical linear kinematic hardening

$$ds = 2\mu_0 \left[ de - \frac{3\mu_0}{(c + 2\mu_0)} N(N:de) \right] \quad (26a)$$

where

$$N = (s - r)/\sigma_0^0 \quad , \quad \text{dr} = \dot{c} \, d\epsilon^P \quad (26b)$$
Thus the present endochronic relations (24a, b) are entirely analogous to those of classical plasticity theory (25a, b) and (26a, b).

By assuming \( I = 1 \) or 0 appropriately, one may proceed to develop a tangent-stiffness finite-element method in the usual fashion. If the stress \( \sigma_n \) at state \( C_n \), in an incremental solution, is known, the incremental stresses \( \Delta \sigma \) corresponding to the trial-solutions \( \Delta \epsilon \) for incremental strains are determined in the usual fashion. We assume that \( \sigma_n \) is on the yield surface and further assume that the process had been plastic\(^+\); i.e., \( \left[ \sigma'_n + 2 \mu \Delta \epsilon \right] - r_n \geq (\tau^0 f)_n \). Then, for any \( \theta \) such that \( 0 < \theta < 1 \), the algorithm for determining the actual stress-increment \( \Delta \sigma \) in the plastic process proceeds as follows:

\[
N_n = \frac{(s_n + 2 \mu \theta \Delta \epsilon) - r_n}{(s_n + 2 \mu \theta \Delta \epsilon) - r_n} ,
\]

\[
\Delta \epsilon^p = (1/C_n) (N_n : \Delta \epsilon) N_n \quad (27a)
\]

\[
\left\{ \begin{array}{l}
\text{where } C_n = \left[ 1 + \rho_1(0) + \frac{\tau^0_f (d f/d \xi)}{2 \mu_0} + \frac{h^* : (s - r)}{\tau^0_f} \right] \end{array} \right. \quad (27b)
\]

\[
\Delta S = 2 \mu [\Delta \epsilon - (1/C_n) (N_n : \Delta \epsilon) N_n] ,
\]

\[
\Delta \sigma : I = (2 \mu + 3 \lambda) (\Delta \epsilon : I) ,
\]

\[
\Delta \zeta = (\Delta \epsilon^p : \Delta \epsilon^p)^{1/2} , \quad f_{n+1} = f_n + \Delta f ,
\]

\[
r_{n+1} = r_n + \left\{ 2 \mu_0 \rho_1(0) \Delta \epsilon^p - \left( \sum_i \frac{\beta_i \sigma^{(i)}_n}{f_n} \right) \Delta \zeta \right\} .
\]

Of course, several variants of the above algorithm, such as subincremental ones, are possible. The above tangent-stiffness finite-element, and generalized midpoint-radial-return-stress integration, algorithm has been used by WATANABE & ATLURI [1984b] to solve several problems of cyclic plasticity and nonproportional biaxial loading. It has been found that the present models capture the experimentally observed phenomena of cyclic hardening, cross-hardening, ratcheting, etc.

Because of the superior predictive capabilities of the present model and the fact that it is no more difficult to implement than the usual (classical) plasticity models, it may be a candidate for further exploitation in general purpose computational programs.

IV. DETERMINATION OF MATERIAL CONSTANTS AND REPRESENTATION OF MONOTONIC AND CYCLIC HARDENING PLASTICITY

IV.1. General considerations

We develop here the stress-strain relations for uniaxial tension so as to be consistent with the presently developed alternative three-dimensional relations given in eqns (7) and (24) in integral and differential forms, respectively. We define the uniaxial tension

\footnote{For a discussion of a general plasticity algorithm, covering both elastic and plastic processes, see ATLURI [1985].}
response through the relations: $\sigma_{ij} \neq 0$ otherwise $\sigma_{ij} = 0$; $\epsilon_{ij} = \epsilon_{ij}^p = -\frac{1}{2} \epsilon_{ii}^p$, $\epsilon_{ij}^p = 0$ ($i \neq j$). Using (2), we obtain

$$d\xi^2 = \frac{1}{2} (d\epsilon_i^p)^2 = \frac{1}{2} (d\epsilon_i^p)^2 ,$$

or

$$d\xi = \sqrt{\frac{1}{2}} |d\epsilon_i^p|$$  \hspace{1cm} (28a)

and

$$\xi = \sqrt{\frac{1}{2}} \int |d\epsilon_i^p| .$$  \hspace{1cm} (28b)

We also introduce

$$dz = d\xi / f(\xi) , \quad f(\xi) = 1 + \beta \xi ,$$  \hspace{1cm} (29)

$$E_0 \rho(z) = E_0 \rho_0 \delta(z) + E_1 \epsilon^{-\alpha_1 z} + E_2 ,$$  \hspace{1cm} (30a)

and

$$\mu_0 \rho(z) = \mu_0 \rho_0 \delta(z) + \left( \frac{\mu_0}{E_0} \right) E_1 \epsilon^{-\alpha_1 z} + \left( \frac{\mu_0}{E_0} \right) E_2 .$$  \hspace{1cm} (30b)

The stress-strain relation under uniaxial tension that was used by Wu & Yip [1980,1981] is

$$\sigma_{11} = E_0 \int_0^z \rho(z-z') \frac{d\epsilon_i^p}{dz} \, dz' , \quad \text{for } \xi \geq 0^+ .$$  \hspace{1cm} (31)

Using (28) to (30a,b) in (31), we obtain, for monotonic uniaxial tension ($|d\epsilon_i^p| = d\epsilon_i^p$),

$$\sigma_{11} = E_0 \rho_0 \sqrt{\frac{2}{3}} \left( 1 + \beta \sqrt{\frac{3}{2}} \epsilon_i^p \right) + \sqrt{\frac{2}{3}} \left( \frac{E_1}{n_1 \beta} \right) \left( 1 + \beta \sqrt{\frac{3}{2}} \epsilon_i^p \right)$$

$$\times \left\{ 1 - \left( 1 + \beta \sqrt{\frac{3}{2}} \epsilon_i^p \right)^{-n_1} \right\} + E_2 \epsilon_i^p , \quad \text{for } \epsilon_i^p \geq 0 ,$$  \hspace{1cm} (32a)

where

$$n_1 = 1 + (\alpha_1 / \beta) .$$  \hspace{1cm} (32b)

It may be worth mentioning that Wu & Yip [1980,1981] use the definition that $d\xi = |d\epsilon_i^p|$ instead of the one in (28b). Hence the result in Wu & Yip [1980,1981] would agree with the present (32) if the constants $\rho_0$, $E_0$, $E_1$, $E_2$, $\beta$, $\alpha_1$ and $\alpha_1$ as used in Wu & Yip [1980,1981] are identified, instead, to be ($\rho_0 \sqrt{2/3}$), $E_0$, $E_1$, $E_2$, ($\beta \sqrt{3/2}$) and ($\alpha_1 \sqrt{3/2}$), respectively.
On the other hand, the present three-dimensional integral relation, eqn (7), reduces, for the uniaxial tension case, to

\[ \sigma_{11} = 3\mu_0 \rho_0 \frac{d\epsilon_{\tilde{r}11}}{dz} + 3\mu_0 \int_0^z \rho_1(z - z') \frac{d\epsilon_{\tilde{r}11}}{dz'} \, dz' \]

\[ = \sqrt{6}\mu_0 \rho_0 \left( 1 + \beta \frac{3}{\sqrt{2}} \epsilon_{\tilde{r}11} \right) + \frac{2}{\sqrt{3}} \left( \frac{3\mu_0}{E_0} \right) \frac{E_1}{\beta n_1} \left( 1 + \beta \frac{3}{\sqrt{2}} \epsilon_{\tilde{r}11} \right) \times \left\{ 1 - \left( 1 + \beta \frac{3}{\sqrt{2}} \epsilon_{\tilde{r}11} \right)^{-n_1} \right\} + \left( \frac{3\mu_0}{E_0} \right) E_2 \epsilon_{\tilde{r}11}, \text{ for } \epsilon_{\tilde{r}11} \geq 0. \]

(33)

In writing (34), monotonic loading has been assumed. By comparing (32) and (34), it may be seen that the Wu-YIP [1980,1981] relation agrees with the present, provided \( E_0 = 3\mu_0 \), i.e. the Poisson's ratio \( \nu_0 = 1/2 \), even in the elastic region. This is due to the fact that the integral relation used in Wu & YIP [1980,1981], which results in (31) and (32), is based on the assumption that the Poisson's ratio is constant through deformation.

We shall henceforth use (34) to evaluate material properties, for use in conjunction with the three-dimensional relations (7) and (24). We will assume that the elastic properties are related as \( \alpha_0 = E_0/2(1 + \nu_0) \), \( \lambda_0 = 2\mu_0\nu_0/(1 - 2\nu_0) \), \( 3\kappa_0 = (3\lambda_0 + 2\mu_0) \).

For large values of \( \epsilon_{\tilde{r}11} \), the asymptotic value of stress, denoted as \( \sigma_{11}^\infty \), may be obtained from (33) to be

\[ \sigma_{11}^\infty = \sqrt{6}\mu_0 \rho_0 \left( 1 + \beta \frac{3}{\sqrt{2}} \epsilon_{\tilde{r}11} \right) + \frac{2}{\sqrt{3}} \left( \frac{3\mu_0}{E_0} \right) \frac{E_1}{\beta n_1} \left( 1 + \beta \frac{3}{\sqrt{2}} \epsilon_{\tilde{r}11} \right) \left( \frac{3\mu_0}{E_0} \right) E_2 \epsilon_{\tilde{r}11}. \]

(35)

Assuming that the elastic constants \( (E_0, \mu_0) \) are known for the material, it is seen from (34) that the stress-plastic strain response of the material, given in (34), is governed by the five parameters: \( \rho_0 \), \( \beta \), \( \alpha_1 \), \( E_1 \) and \( E_2 \). We now discuss the determination of these five parameters from given test data for the material under monotonic uniaxial tension. To this end, first note that

\[ \frac{d\sigma_{11}}{d\epsilon_{\tilde{r}11}} = 3\mu_0 \rho_0 \beta + \left( \frac{3\mu_0}{E_0} \right) \frac{E_1}{n_1} \left[ 1 + (n_1 - 1) \left( 1 + \beta \frac{3}{\sqrt{2}} \epsilon_{\tilde{r}11} \right)^{-n_1} \right] + \left( \frac{3\mu_0}{E_0} \right) E_2 \]

(36)

and

\[ \frac{d\sigma_{11}^\infty}{d\epsilon_{\tilde{r}11}} = 3\mu_0 \rho_0 \beta + \left( \frac{3\mu_0}{E_0} \right) \frac{E_1}{n_1} + \left( \frac{3\mu_0}{E_0} \right) E_2. \]

(37)

We now define parameters \( \sigma_0^0 \), \( E_\rho \), \( \sigma_0^\infty \) and \( E_\iota \) as may be determined† from the test data as shown in Fig. 1(a), to be

†Here it is to be noted that the "knee" portion (near the "elastic" limit point) of the stress-strain test data is approximated by a straight line \( \sigma_1 = \sigma_0^0 + E_\rho \epsilon_1 \) for \( \epsilon_1 \ll 1 \); and for large values of \( \epsilon_1 \), the stress-strain test data is approximated by a straight line \( \sigma_1 = \sigma_0^\infty + E_\iota \epsilon_1 \) as shown in Fig. 1(a). Thus the parameters of \( \sigma_0^0 \), \( E_\rho \), \( \sigma_0^\infty \) and \( E_\iota \) are "read-off" from the test data for uniaxial tension.
Constitutive modeling of cyclic plasticity and creep

Fig. 1. Nomenclature used in analytical modeling of test data.

\[ \sigma^0 = \sigma_{11} \big|_{\varepsilon_{11}^p = 0} = \sqrt{6} \mu_0 \rho_0, \quad (38a) \]

\[ E_p = \frac{d\sigma_{11}}{d\varepsilon_{11}^p} \big|_{\varepsilon_{11}^p = 0} = 3 \mu_0 \rho_0 \beta + \frac{3 \mu_0}{E_0} E_1 + \frac{3 \mu_0}{E_0} E_2, \quad (38b) \]

\[ a_{11}^\infty = a_{11} \big|_{\varepsilon_{11}^p = 0} = \sqrt{6} \mu_0 \rho_0 + \sqrt{\frac{2}{3}} \frac{3 \mu_0}{E_0} \frac{E_1}{\beta n_1}, \quad (38c) \]

and

\[ E_t = \frac{d\sigma_{11}^\infty}{d\varepsilon_{11}^p} = 3 \mu_0 \rho_0 \beta + \left( \frac{3 \mu_0}{E_0} \right) \frac{E_1}{n_1} + \left( \frac{3 \mu_0}{E_0} \right) E_2. \quad (38d) \]

The four equations (38a)–(38d) are obviously not sufficient to determine the five constants \( \rho_0, \beta, \alpha_1, E_1 \) and \( E_2 \). To uniquely determine the five constants, the missing fifth relation may be arrived at by first noting that \( \rho_1(z) \) (involving \( E_1 \) and \( E_2 \)) describes the translation of the yield surface, and \( f(\xi) \) (involving \( \beta \)) describes the enlargement (or contraction, as the case may be) of the yield surface. Specifically, by integrating (33) for a loading-unloading-reloading case, assuming that \( f(\xi) = 1 + \beta \xi \), it may be shown that the stress-drop \( \Delta \sigma \) during the elastic part of the first unloading is \( 2 \sigma^0(1 + \beta \sqrt{\frac{3}{2}} |\varepsilon_{11}^p|) \) (see Fig. 3) where \( |\varepsilon_{11}^p| \) is the plastic strain at the beginning of elastic
unloading. Thus material constant $\beta$ may be determined. Now eqns (38a)-(38d) may be solved for the remaining four unknowns, as

$$n_1 = (1 + \alpha_1 / \beta) = 1 + \frac{\sqrt{3}}{3} \left( \frac{F_o - F_i}{\sigma_0^0 - \sigma_0^0} \right) \frac{1}{\beta}.$$  \hspace{1cm} (39a)

$$\rho_0 = (\sigma_0^0 / \sqrt{6} \mu_0).$$  \hspace{1cm} (39b)

$$E_1 = \frac{E_0}{3 \mu_0} \left[ (E_o - E_i) + \frac{\sqrt{3}}{2} (\sigma_0^\infty - \sigma_0^0) \beta \right].$$  \hspace{1cm} (39c)

and

$$E_2 = \frac{E_0}{3 \mu_0} \left( E_i - \frac{\sqrt{3}}{2} \sigma_0^\infty \beta \right).$$  \hspace{1cm} (39d)

If a more accurate approximation near the "knee" of the stress-strain curve (at $\epsilon_1^P = 0$) is needed, one may use, instead of (30b), the assumption

$$\mu_0 \rho(z) = \mu_0 \rho_0 \delta(z) + \left( \frac{\mu_0}{E_0} \right) E_1 \epsilon^{P_1} + \left( \frac{\mu_0}{E_0} \right) E_2 + \left( \frac{\mu_0}{E_0} \right) E_3 \epsilon^{P_3}.$$  \hspace{1cm} (40)

The corresponding solution becomes

$$\sigma_{11} = \sqrt{6} \mu_0 \rho_0 \left( 1 + \beta \frac{\sqrt{3}}{2} \epsilon_1^P \right) + \frac{2}{3} \left( \frac{3 \mu_0}{E_0} \right) \frac{E_1}{\beta n_1} \left( 1 + \beta \frac{\sqrt{3}}{2} \epsilon_1^P \right) \times \left( 1 - \left( 1 + \beta \frac{\sqrt{3}}{2} \epsilon_1^P \right)^{-n_1} \right) + \frac{2}{3} \left( \frac{3 \mu_0}{E_0} \right) \frac{E_3}{\beta n_3} \left( 1 + \beta \frac{\sqrt{3}}{2} \epsilon_1^P \right) \times \left( 1 - \left( 1 + \beta \frac{\sqrt{3}}{2} \epsilon_1^P \right)^{-n_3} \right) + \left( \frac{3 \mu_0}{E_0} \right) E_2 \epsilon_1^P,$$

where

$$n_3 = 1 + (\alpha_3 / \beta).$$  \hspace{1cm} (41)

The parameters from the test data,† as shown in Fig. 1(b), are now related as

$$E_i = 3 \mu_0 \rho_0 \beta + \left( \frac{3 \mu_0}{E_0} \right) \frac{E_1}{n_1} + \left( \frac{3 \mu_0}{E_0} \right) \frac{E_3}{n_3} + \left( \frac{3 \mu_0}{E_0} \right) E_2.$$  \hspace{1cm} (42a)

†In this improved representation, the "knee" portion of the test data is approximated by a straight line $\sigma_{11} = (\sigma_0^\infty)' + (E_o)' \epsilon_1^P$ for $\sigma_1^P \ll 1$, as shown in Fig. 1(b). However, for large values of $\epsilon_1^P$, the test data is approximated by the same straight line as in the earlier representation [shown in Fig. 1(a)], i.e. $\sigma_{11} = \sigma_0^\infty + E_i \epsilon_1^P$. Note also that the definitions of $(\sigma_0^0)$ and $(E_o)$, shown in Fig. 1(b), remain the same as before, i.e. $\sigma_0^0 = \sqrt{6} \mu_0 \rho_0$, and $E_o = 3 \mu_0 \rho_0 \beta + (3 \mu_0 / E_0) E_i$. 


Constitutive modeling of cyclic plasticity and creep

\[
\sigma_0^\infty = \sqrt{6} \mu_0 \rho_0 + \frac{2}{3} \left( \frac{3 \mu_0}{E_0} \right) E_1 + \frac{2}{3} \left( \frac{3 \mu_0}{E_0} \right) E_3 , \tag{42b}
\]

\[
(\sigma_0^0)' = \sqrt{6} \mu_0 \rho_0 , \tag{42c}
\]

\[
(E_p)' = 3 \mu_0 \rho_0 \beta + \left( \frac{3 \mu_0}{E_0} \right) E_1 + \left( \frac{3 \mu_0}{E_0} \right) E_3 + \left( \frac{3 \mu_0}{E_0} \right) E_2 . \tag{42d}
\]

With \( \beta, E_1 \) and \( E_3 \) being as before, the additional constants in the improved approximation are determined as

\[
n_3 = \frac{2}{\sqrt{3}} \frac{1}{\beta} \left( \frac{E_p' - E_p}{\sigma_0^0 - (\sigma_0^0)'} \right) + 1 \tag{43a}
\]

and

\[
E_3 = \left( \frac{E_0}{3 \mu_0} \right) \left[ (E_p' - E_p) + \frac{3}{\sqrt{2}} (\sigma_0^0 - \sigma_0^0') \beta \right] . \tag{43b}
\]

Similar procedures may be employed when an arbitrary number of terms are used in the expansion

\[
E_0 \rho(z) = E_0 \rho_0 \delta(z) + \sum_i E_i e^{-a_i z} , \tag{44}
\]

which makes it possible to represent the knee portion more and more accurately.

We now consider the presently derived differential form of the stress-strain relation, (24). For the uniaxial tension problem, eqn (24) becomes

\[
dS_{11} = 2 \mu_0 \left\{ \text{det} \begin{bmatrix} 1 & \frac{S_{11} - r_{11}}{Cf^2 S_p^{02}} \de_{11} - \frac{(S_{11} - r_{11})(S_{22} - r_{22})}{Cf^2 S_p^{02}} \de_{22} \\
- \frac{(S_{11} - r_{11})(S_{33} - r_{33})}{Cf^2 S_p^{02}} \de_{33} \end{bmatrix} \right\} . \tag{45}
\]

For uniaxial tension,

\[
S_{11} = \frac{1}{2} \sigma_{11} , \quad S_{22} = S_{33} = -\frac{1}{2} \sigma_{11} , \quad r_{22} = r_{33} = -\frac{1}{2} r_{11} ,
\]

\[
d\epsilon_{11} = d\epsilon_{11} - \frac{d \sigma_{11}}{9 K_0} , \quad d\epsilon_{22} = -\frac{1}{2} d\epsilon_{11} + \frac{d \sigma_{11}}{18 K_0} = d\epsilon_{33} ,
\]

\[
r_{11} = 2 \mu_0 \int_0^z \rho_1(z - z') \frac{d \rho}{dz'} \, dz' . \tag{46}
\]

Use of (46) in (45) results, for the monotonic uniaxial tension problem, in
\[ d\sigma_{11} = \frac{2\mu_0[1 - (1/C)]}{(2/3) + (2\mu_0/9K_0)[1 - (1/C)]} d\varepsilon_{11}, \]  
(47)

where

\[ C = 1 + \rho_1(0) + \frac{S_f f'}{2\mu_0} + \sqrt{\frac{3}{2}} \frac{h_{11}^*}{f}. \]  
(48)

For the presently assumed \( \rho_1(z) \) and \( f \), as in (30a) and (29), respectively, we have

\[ \rho_1(0) = \left( \frac{E_0}{E_1} \right) + \left( \frac{E_2}{E_0} \right), \]  
(49a)

\[ \frac{h_{11}^*}{f} = \frac{1}{f} \int_0^z \frac{\partial \rho_1}{\partial z'} (z - z') \frac{d\varepsilon_{11}^p}{dz'} dz' = -\sqrt{\frac{2}{3}} \left( \frac{E_1}{E_0} \right) \left( \frac{n_1 - 1}{n_1} \right) \{1 - f^{-n_1} \} \]  
(49b)

and

\[ C = 1 + \left( \frac{E_2}{E_0} \right) + \left( \frac{E_1}{E_0} \right) \frac{1}{n_1} \left[ 1 + (n_1 - 1)f^{-n_1} \right] + \rho_0 \beta. \]  
(49c)

Since

\[ d\varepsilon_{11} = d\varepsilon_{11}^p + \frac{d\sigma_{11}}{E_0}, \]  
(50)

it may be easily shown, by using (50) in (47), that

\[ \frac{d\sigma_{11}}{d\varepsilon_{11}^p} = 3\mu_0(C - 1) \]
\[ = 3\mu_0\rho_0\beta + \left( \frac{3\mu_0}{E_0} \right) \frac{E_1}{n_1} \left[ 1 + (n_1 - 1)f^{-n_1} \right] + \left( \frac{3\mu_0}{E_0} \right) E_2, \]  
(51)

which agrees with (36) derived from the integral relation. This agreement of (51) with (36) is then a confirmation of the validity of the differential stress–strain relations (24a) and (24b), as well as the correctness of the presently derived expression for \( C \) as in (15).

IV.2. Numerical results

IV.2.1. Plasticity: monotonic loading. The applications that we address here pertain to inelastic deformation at elevated temperatures. In this subsection, we will consider plasticity; and later in this paper, we deal with the problem of plasticity-creep interaction. Here we make use of experimental data for type-304 stainless steel at high temperatures, produced by the Power Reactor and Nuclear Fuel Development Corporation (hereafter denoted as PNC) in Japan (JSME [1981]). First we study the monotonic stress–strain curve at 550°C, for which the PNC data (JSME [1981]) is shown in
Fig. 2. The constitutive equation adopted by PNC (JSME [1981]) may be considered to be a modified version of BLACKBURN's [1972] equation, in which an increment of plastic strain is expressed by a power law in terms of stress. This makes the "knee" portion of the stress vs plastic strain curve to have a very steep initial slope, as shown in Fig. 2.

We now approximate the above PNC data by three different types of the present "intrinsic-time-plasticity" models, eqns (33) or (45). These three modes are designated as Cases A, B and C, respectively, in Tables 1 and 2. In Case A, a two-term approxi-
mation for \( \rho_1(z) \) as in (36) is used, and the yield-stress (in this case, \( \sigma_0^0 = \sigma_0^0 \)) is the highest among all the cases. In Cases B and C, a three-term approximation for \( \rho_1(z) \) as in (40) is used. The yield stress \( \sigma_0^0 \) is now equal to \( (\sigma_0^0)' \) in Cases B and C. Note that \( (\sigma_0^0)' \) in Case B is chosen to be higher than that in Case C. Material constants \( \sigma_0^0, \sigma_0^0, E_i, E_p, \) and \( E' \), as inferred from the test data are given in Table 1. Modeling parameters \( E_1, E_2, E_3, \alpha_1, \alpha_2, \) and \( \beta \) as calculated from eqns (39a)-(39d) and (43a,b) are given in Table 2. The value \( \beta = 5 \) is taken from Wu & Yip [1981] who base it on their analysis of experimental data for type-304 stainless steel at room temperature.

Figure 2 shows the comparison between the PNC data and the present modeling through eqns (33) or (45). The present results for each of the Cases A, B and C may be seen to agree excellently with experimental data, with the only differences between the three cases being in the knee region, as anticipated.

IV.2.2. Plasticity: cyclic loading. A typical cyclic loading, under tension-compression straining, is sketched in Fig. 3. The stress under this cyclic history of loading is calculated by using the differential stress-strain relation, (45). Note that now, \( d\xi = \sqrt{3} |d\epsilon_1^0|, \xi = \sqrt{3} \int |d\epsilon_1^0|, d\zeta = d\xi/f(\xi). \) Note also that the procedure for the present calculation using (45) or (47) is operationally very similar to that in classical plasticity theory, in that (17) and (18) apply. Referring to Fig. 3, the stress and strain are increased elastically from the “free state” point 0 to the yield point denoted by 1. From point 1 the material is subject to plastic deformation, and until the strain of magnitude \( \epsilon_{11} \) (see Fig. 3) the yield surface translates and expands. At point 2, the material is unloaded. During unloading the material behaves elastically, and the stress-state reaches a point on the opposite side of the yield surface. The stress drop, \( \Delta \sigma \) in Fig. 3, from point 2 to the elastic limit point 3 is given by \( 2\sigma_0^0 f_2 \) where \( f_2 \) is the value of \( f \) at point 2. Note that the increments of \( \Delta\epsilon_{11} \) between various points in the strain-path, defined, for instance, as \( \Delta\epsilon_{11}^{P24} = f_2|\Delta\epsilon_{11}^P|, \) are as shown in Fig. 3. We now discuss the quantitative features of the hysteresis loops under cyclic loading for two different types of functions \( f(\xi). \)

![Fig. 3. Schematic of cyclic hardening of elastic-plastic materials.](image-url)
(i) $f(\xi) = (1 + \beta \xi), \xi = \sqrt{\frac{2}{3}} \int \left| \text{de}_{11}^p \right|$

The analysis of cyclic loading is carried out for material data designated as Case B in Tables 1 and 2, along with the linear function $f(\xi) = (1 + \beta \xi)$. Figure 4 shows the calculated results for the $\sigma-\epsilon$ relation for the strain range of $\varepsilon = \pm 0.5\%$. Peak stresses at the loading-unloading points are denoted by $\sigma_{t,N}$ and $\sigma_{c,N}$, where the superscripts $t$ and $c$ imply tension and compression, respectively, while the subscript $N$ implies the $N$th cycle of loading. It is observed from Fig. 4 that these peak stresses increase monotonically with $N$, and do not reach a stable value as is normally observed in experiments. An examination of Fig. 3 clearly shows that the reason for this monotonic increase of the peak stress is the linearity of $f$ with respect to $\xi$.

(ii) $f(\xi) = \{a + (1 - a) \exp(-\gamma \xi)\}, a$ and $\gamma$ constants

Instead of a linear function, the above saturated function $f$ will be employed, wherein $a$ and $\gamma$ are appropriate material parameters. Note that such an $f$ has also been used by Wu & Yip [1981] so as to obtain certain analytical solutions in explicit form. In the

![Fig. 4. Analytical modeling of cyclic plasticity using a linear yield function $f$ (Case B).](image)
above, the parameter $a$ represents a saturated magnitude of the yield surface. If the initial slope at $\zeta = 0$ is equated for both the linear and saturated functions, the following relation is obtained:

$$(a - 1)\nu = \beta . \tag{52}$$

Even if a saturated function $f$ is used, we may determine the other material parameter $\rho_1(z)$ as if using a linear $f$, because a saturated function $f(\zeta)$ can be expected to have an influence only for large values of $\zeta(= \sqrt{\zeta} f |d\epsilon_p^0|)$ such as in cyclic loading. Appropriate experimental data for plasticity for cyclic loading of type-304 stainless steel at elevated temperatures does not appear to be readily available. However, as far as room temperature cyclic loading is concerned, data exists (JSME [1981]) for saturated peak stress $\sigma^p_0$ and initial yield stress $\sigma_0^0$ for type-304 stainless steel, as

$$\sigma_0^0 = 196 \text{ MPa} \quad \sigma^p_0 = 265 \text{ MPa} .$$

Ignoring the kinematic hardening for the time being, one may obtain the following rough overestimation, $a_0$ for $a$

$$a_0 = \sigma^p_0 / \sigma_0^0 = 1.35 .$$

We will henceforth assume that $a = 1.2$. We estimate $\gamma$ from (52) to be

$$\nu = 5/(0.2) = 25 .$$

Of course, if one can easily identify the point of departure from unloading to reloading, such as point 3 in Fig. 3, in the experimental data, one may estimate the values of $a$ and $\gamma$ more accurately. However, in general, point 3 cannot be so unambiguously identified from experimental data.

Figure 5 shows the presently computed results using the saturated function $f$. As may be seen, the hysteresis loops saturate after a few cycles of loading. The peak stresses $\sigma_0^p$ and $\sigma^p_0$ converge to stable values as shown in Fig. 6. Figure 7 shows the enlargement of the yield surface, i.e. $f(\xi)$, as a function of $\xi = \sqrt{\xi} f |d\epsilon_p^0|$. The corresponding values at each peak of the tension-compression loading-cycles are also depicted in Fig. 7. The increment between the peak of tension and the peak of compression points can be seen, from Fig. 7, to be almost the same regardless of the number of cycles of loading.

V. CREEP AND PLASTICITY

V.1. Theoretical development

In Section III, we employed an intrinsic time measure related, in a differential sense, to the norm of the differential plastic strain, to describe rate-independent plasticity. To characterize the creep and plasticity-creep interaction behavior, we now employ an intrinsic time measure as well as Newtonian time, both of which are nonnegative and irreversible quantities, as was initially suggested by Valanis [1975]. Specifically, the internal time increment, $dz$, is expressed as

$$(dz)^2 \equiv \frac{(d\xi)^2}{f^2(\xi)} + \frac{(dt)^2}{g^2} . \tag{53}$$
where

\[ d\gamma^2 = d\eta : d\eta , \]  

\[ d\eta = \text{inelastic strain differential} \]  

\[ (\text{plastic as well as creep, i.e. } d\eta = d\eta^c + d\eta^p) \]  

and

\[ g = \text{a scaling function} . \]

As before, assuming elastic isotropy, we have

\[ d\eta = \left( de - \frac{ds}{2\mu_0} \right) . \]

We will henceforth consider \( \alpha = 1 \). We assume that the governing equation for creep (and plasticity) is the same as (7), i.e.
Fig. 6. Convergence of peak stress with the number of cycles (Case B).

Fig. 7. The function $f(\tau)$. 
Constitutive modeling of cyclic plasticity and creep

\[ s = \tau^0_y \frac{d\eta}{dz} + r(z) \quad , \quad r(z) = \int_0^z \rho_1(z - z') \frac{d\eta}{dz'} \, dz' . \] (56a, b)

From (56a) it follows that

\[ \frac{|s - r|^2}{(\tau^0_y)^2} \, dz^2 = d\eta \cdot d\eta = d\xi^2 , \] (57)

i.e.

\[ \frac{|s - r|^2}{(\tau^0_y)^2 f^2(\xi)} = \frac{1}{f^2(\xi)} \frac{(d\xi)^2}{(dz)^2} + \frac{1}{g^2} \frac{(dt)^2}{(dz)^2} . \] (58)

Using (57) in (53), we have

\[ dz = \frac{1}{\sqrt{1 - \{(|s - r|)/\tau^0_y f(\xi)\}^2}} \left( \frac{dt}{g} \right) . \] (59)

The total inelastic strain-increment, d\eta, is given from (56a) and (59) as

\[ d\eta = \frac{(s - r)}{\tau^0_y} \frac{1}{\sqrt{1 - \{(|s - r|)/\tau^0_y f(\xi)\}^2}} \left( \frac{dt}{g} \right) . \] (60)

Finally we obtain the differential stress-strain relation in the presence of creep, by using (60) in (55) as

\[ ds = 2\mu_0 \left\{ d\varepsilon - \frac{(s - r)}{\tau^0_y} \frac{1}{\sqrt{1 - \{(|s - r|)/\tau^0_y f(\xi)\}^2}} \left( \frac{dt}{g} \right) \right\} . \] (61)

We will postulate, as did BAZANT [1978] and SCHAPERY [1968] in different contexts, that the scaling function g is a function of stress \( \sigma \) and the intrinsic time variable \( \zeta \), i.e. \( g = g(\sigma, \zeta) \). Specifically, we assume here that

\[ g = \frac{f(\zeta)}{B} \left( \frac{|s - r|}{\tau^0_y f(\zeta)} \right)^{1-m} , \] (62)

where B and m are constants, so that (61) becomes

\[ ds = 2\mu_0 \left\{ d\varepsilon - B \left( \frac{|s - r|}{\tau^0_y f(\zeta)} \right)^m \frac{(s - r)}{|s - r|} \frac{dt}{\sqrt{1 - \{(|s - r|)/\tau^0_y f(\xi)\}^2}} \right\} . \] (63)

When the magnitude of stress is small compared to the yield stress, we have

\[ \sqrt{1 - \left( \frac{|s - r|}{\tau^0_y f(\zeta)} \right)^2} \approx 1 . \] (64)
Then eqn (63) becomes

\[ ds = 2\mu_0 \left\{ \frac{de - B \left( \frac{|s - r|}{r_0 f(r)} \right)^m (s - r)}{|s - r|} dt \right\}. \] (65)

**V.2. Determination of material constants and numerical results**

Following the details in Section IV, for uniaxial tension, eqn (63) becomes

\[ \frac{d\sigma_{11}}{E_0} = \frac{2}{3} B \left( \frac{\sigma_{11} - \frac{1}{3}r_{11}}{\sigma_r^0 / f} \right)^m \frac{dt}{\sqrt{1 - \left[ \frac{\sigma_{11} - \frac{1}{3}r_{11}}{\sigma_r^0 / f} \right]^2}}. \] (66)

Under constant external load, i.e. \( d\sigma_{11} = 0 \), the creep strain rate is thus given by

\[ \frac{d\epsilon_{11}}{dt} = B \frac{2}{3} \left( \frac{\sigma_{11} - \frac{1}{3}r_{11}}{\sigma_r^0 / f} \right)^m \frac{1}{\sqrt{1 - \left[ \frac{\sigma_{11} - \frac{1}{3}r_{11}}{\sigma_r^0 / f} \right]^2}}. \] (67)

For small values of \( \sigma_{11} \) as compared to \( \sigma_r^0 \), (67) may be approximated as

\[ \frac{d\epsilon_{11}}{dt} \approx B \frac{2}{3} \left( \frac{\sigma_{11} - \frac{1}{3}r_{11}}{\sigma_r^0 / f} \right)^m, \] (68)

which is similar to the well-known Norton's power law for steady-state creep.

We assume that material parameters \( E_0, E_1, E_2, \rho_0 \) (or \( \sigma_r^0 \)) and \( \beta \) are determined as discussed in Section IV. We now discuss ways of determining material constants \( B \) and \( m \). When \( \sigma_{11} < \sigma_r^0 \), the yield surface retains its initial shape at \( t = 0 \), which implies that \( f = 1 \) and \( r_{11} = 0 \). More specifically, if (67) is evaluated at \( t = 0 \), we obtain

\[ \varepsilon_{11}^0 = \left. \frac{d\epsilon_{11}}{dt} \right|_{t=0} = \frac{2}{3} B \left( \frac{\sigma_{11}}{\sigma_r^0 / f} \right)^m \frac{1}{\sqrt{1 - \left( \sigma_{11} / \sigma_r^0 \right)^2}}. \] (69)

Thus we obtain

\[ \log \left[ \varepsilon_{11}^0 \sqrt{1 - \left( \sigma_{11} / \sigma_r^0 \right)^2} \right] = \log \left[ \frac{2}{3} B + m \log \left( \frac{\sigma_{11}}{\sigma_r^0 / f} \right) \right]. \] (70)

Experimental data for \( \varepsilon_{11}^0 [1 - \left( \sigma_{11} / \sigma_r^0 \right)^2]^{1/2} vs \ (\sigma_{11} / \sigma_r^0) \) may be plotted in a logarithmic scale. A straight line may be used to represent the test data in a least-square sense; and from this, \( B \) and \( m \) may be determined.

We will henceforth refer to the equation of PNC (JSME [1981]) for the elevated temperature (550°C) behavior of type-304 stainless steel, as a basis for comparison of the present creep relation (67) under constant load. As mentioned earlier, the PNC equation (JSME [1981]) is based on BLACKBURN'S equation [1972] and is capable of representing test data over a wide range of stress; however, the equations of JSME [1981] and BLACKBURN [1972] do not consider the interaction between creep and plasticity.

We first choose material parameters \( B \) and \( m \) at 550°C, according to (70). Figure 8 shows the \( \varepsilon_{11}^0 vs \sigma_{11} \) data in logarithmic scale, where the initial rate of creep strain, \( \epsilon_{11}^{0} \).
Constitutive modeling of cyclic plasticity and creep is taken according to the PNC equation (JSME [1981]). The results obtained, by a straightline curve fitting, for $B$ and $m$ for the cases A, B and C (with $\sigma_0^0 = 112.8$, 103.0, and 92.1 MPa, respectively, as shown in Table 1) are recorded here in Table 3.

We now present results for creep behavior under constant uniaxial loading. When the prescribed stress is less than the yield stress $\sigma_0^0$, the creep analysis is carried out after first raising the stress elastically to the given value. A linear function for $f$, viz., $f(\dot{\varepsilon}) = 1 + \beta \dot{\varepsilon}$, is employed, and the increment of internal time, $\Delta \tau$, is assumed to be $0.5 \times 10^{-4}$. Figure 9 shows the presently computed results for creep strain for values of $\sigma_{11}$, in each of the cases A, B and C, respectively, along with PNC's results (JSME [1981]). A good agreement may be noted between the present and PNC results for $\sigma_{11} = 58.8$ MPa. However, for larger magnitudes of $\sigma_{11}$ and for longer times, the results for case A overestimate and those for C underestimate the creep strain as compared to the PNC equation.

When the prescribed stress level in uniaxial tension is higher than the yield stress, a plasticity analysis is first performed prior to a creep analysis. The steady state as observed in creep experiments may then be regarded as the case when the yield-surface ceases to translate and enlarge.

Figure 10 shows the presently computed creep strain variation with time, along with

<table>
<thead>
<tr>
<th>Case</th>
<th>$B$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$4.87 \times 10^{-6}$</td>
<td>5.2</td>
</tr>
<tr>
<td>B</td>
<td>$2.74 \times 10^{-6}$</td>
<td>5.16</td>
</tr>
<tr>
<td>C</td>
<td>$1.23 \times 10^{-6}$</td>
<td>4.9</td>
</tr>
</tbody>
</table>

Table 3. Material constants $B$ and $m$ derived for present analytical modeling of creep.
Fig. 9. Prediction of creep strains, for lower magnitudes of stress, using a linear yield function $f$.

Fig. 10. Prediction of creep strains, for higher magnitudes of stress, using a linear yield function $f$. 
PNC [38] data, for two values of $\sigma_{11} > \sigma_{y}^0$, for cases A, B and C, respectively. In this set of results, a linear function $f$ is used. The present results are lower than the PNC data, and the discrepancy becomes larger as $\sigma_{11}$ increases. Next, we consider the effect of using a saturated function $f$ as discussed in subsection IV.1.2.(ii). Figure 11 shows the presently computed results for Case A, when the parameter $a$ in the saturated function $f$ is assigned two different values, $a = 1.2$ and 1.1, respectively. It is observed that the smaller is the value of $a$, the larger is the creep strain as $\sigma_{11}$ becomes higher. This is due to the fact that even a small difference between the linear and saturated yield functions $f$ will be magnified due to the power law as in Norton's equation, or eqn (67). It may also be noted that a saturated function $f$ has little or no effect on creep strain for lower values of $\sigma_{11}$, since the saturated function $f$ is almost identical to the linear $f_1$ for small values of $\xi$.

Figure 12 shows the calculated results for creep strain for Case A, with the value of $a$ being assigned 1.1, for various levels of $\sigma_{11}$ (from 117.7-176.5 MPa). The experimental results as well as those from the PNC equation (JSME [1981]) are also shown in Fig. 12. Reasonably good agreement is noted between the three sets of data for all stress levels. It is seen that the discrepancies in analytical modeling are somewhat pronounced in the primary stages of creep, while the discrepancies tend to vanish in the steady state. It may be seen that the yield surface in the present theory tends to translate and expand from the initial state to the steady state more rapidly than it should, but the yield surface in the steady state is modeled rather accurately.

![Fig. 11. Effects of using a saturated yield function $f$, at higher magnitudes of stress.](image-url)
VI. CONCLUSIONS

This paper presents a differential stress-strain relation for plasticity, based on an intrinsic-time theory, which is analogous to the classical plasticity relation. Therefore, the present relation may be incorporated readily into existing numerical algorithms. The presently derived equation can approximate the test data for stress-strain curve as accurately as desired.

This paper also presents a simple theory for creep based also on an "internal time" concept, wherein the "internal time" is characterized by both the inelastic strain and Newtonian time. The thus-derived equation employs the concept of a yield surface for plasticity, and makes it possible to incorporate the effects of interaction between creep and plasticity. The presently obtained numerical results may be considered to be reasonable, if the scarcity of available experimental data is kept in mind.

The unified theory presented herein results in constitutive relations that are simple in form; the material constants are few in number and can be easily determined as shown in the paper. It is, therefore, hoped that the present theory may be useful for a practical estimation of inelastic material behavior.

Acknowledgements—This work was supported by the National Aeronautics & Space Administration, Lewis Research Center, under a grant, No. NAG-346 to Georgia Institute of Technology. The authors acknowledge this support as well as the encouragement of Drs. L. Berke and C. Chamis. It is a pleasure to record here our thanks to Ms. J. Webb for her careful assistance in the preparation of this paper.

REFERENCES

Constitutive modeling of cyclic plasticity and creep


1974 ASME Boiler and Pressure Vessel Code, Sec. III, Case Interpretations, Code Case N-47-17, ASME.


Center for the Advancement of Computational Mechanics
School of Civil Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

(Received 5 October 1984; in final revised form 4 August 1985)