INTRODUCTION

Constitutive modeling of experimentally observed behavior of materials, involving plastic and creep deformations under monotonic and cyclic loading, continues to be a subject of intense research interest. Inelastic constitutive relations presented so far in literature may be classified into two categories: those that require a memory of the entire history of deformation and those that do not. Of the first type, one may cite the theory of "simple" materials (TruebDell & Noll [1965]), a "hereditary material" using analogies to spring-and-dashpot models (Flugge [1967]), and 'internal time' (endochronic) theories (Valanis [1980]; Watanabe & Atluri [1984]). Constitutive equations of this type involve integrals of strain history, while the memory theory of Lee & Zaveri [1979] results in a differential form of evolution laws by using limiting values of internal variables; and in the theory of Mroz [1983], the maximal prestress is assumed to essentially affect the material response.

The general theory of internal variables has also played a key role in the development of more and more realistic constitutive models to characterize inelastic behavior. Such internal variables that are widely employed are, for instance: (i) the tensor locating the center of the yield surface in the stress-space (the so-called "back-stress"), (ii) the parameters that characterize the expansion of the yield surface, (iii) the parameters that characterize the "bounding-surface" in multi-yield-surface theories (Mroz [1967, 1969]; Dafalias & Popov [1975, 1976]; Krieg [1975]) of plasticity, and (iv) the "back-stress" and "drag-stress" used to characterize the creep surface, etc., etc.

The multitude of constitutive relations proposed in literature for inelasticity appear, on the surface, to be unrelated to each other and to be based on totally diverse concepts. The main objective of this paper is to show that such is not the case, and that the "internal-time" (endochronic) theory (Valanis [1980]; Watanabe & Atluri [1984]), the multi-yield-surface theories (Mroz [1967, 1969]; Dafalias & Popov [1975, 1976]; Krieg [1975]) and the internal variable theories (Onat [1966, 1968, 1981]; Fardsisheh & Onat [1973]; Onat & Fardsisheh [1973]; Chaboche [1977]; Chaboche & Rou-
selier [1982]), are essentially one and the same, with only minor variations. It is also shown that the differential forms of the stress-strain relations for plasticity and creep as given in WATANABE & ATLURI [1984] are general enough to encompass all other relations reported in literature as special cases. Thus, the relations in WATANABE & ATLURI [1984], along with the differential relations for evolution of the pertinent internal variables as given in this paper, may prove to be very useful in the practical analysis of cyclic plasticity and creep. Towards this end, it is pointed out in this paper that the implementation of the present differential relations of the “internal-time” theory is no more difficult numerically than that of the classical Prager-Ziegler kinematic hardening theory; and, at the same time, it leads to (WATANABE & ATLURI [1984]) a realistic description of non-linear kinematic hardening, cyclic hardening, cross-hardening, and a unified treatment of plasticity and creep. It is also noted that CHABOCHE & ROUSSELIER [1982] presented an interesting similarity of their “non-linear-kinematic-hardening-rule” (CHABOCHE [1977]) for plasticity to the concept of multi-yield-surface of MROZ [1967, 1969].

The contents of the paper are as follows. In Section 2 we present the nomenclature. In Section 3 we present a concise summary of a general internal variable theory in the context of finite deformations. We postulate growth-laws for the Cauchy stress and the internal variables. In this set of laws, (arbitrary) objective rates of Cauchy stress and of the internal variables are expressed as isotropic tensor functions of the Cauchy stress and the internal variables. In this representation, the strain-induced anisotropy is basically characterized by the symmetry group of the set of internal variables (Onat [1981]). The possibility exists (DAFALIAS [1983]), however, of including some material descriptors in the aforementioned evolution equations. The “finite deformation” relations of Section 3 are then reduced to the case of infinitesimal deformations and strains to which the rest of the paper is restricted. Section 4 provides a summary of the differential form of stress-strain relations based on an internal-time (endochronic) concept as given earlier by the authors (WATANABE & ATLURI [1984]). It is shown that these differential forms of the evolution equations for stress and other internal variables fit into the general framework of internal variable theory sketched in Section 3. Further, it is shown that the multi-yield-surface theories of MROZ [1967, 1969], KRIEG [1975], and DAFALIAS & POPOV [1975, 1976], the nonlinear-kinematic-hardening rules of CHABOCHE [1977] and CHABOCHE & ROUSSELIER [1982], and the classical Prager-Melen rule, etc. can be deduced as special cases of the presently given differential relations of an internal-time theory, through an appropriate choice of the kernels. Section 5 is concerned with “geometrical” representations of various theories. It is shown that the presently described internal-time theory of plasticity is entirely analogous to a multi-yield-surface theory. Depending on the way the kernels are selected, the present internal-time theory may be viewed as an infinite-numbered yield-surface theory. However, apart from this instructive geometrical representation, it is shown that from a functional and algorithmic viewpoint, the internal-time theory is much more convenient to implement than a multi-yield-surface theory. It is also shown in this section that by an appropriate choice of kernels, the tangent modulus of the uniaxial stress versus plastic strain curve may be set to be as arbitrary a function of strain as desired, just as in the theory of DAFALIAS & POPOV [1975, 1976]. Finally, in Section 6, we discuss further the rate form of the unified plasticity-creep theory, as given earlier by WATANABE & ATLURI [1984] and compare it with other well-known theories, such as due to BAILEY & OROWAN [1946] and others. Finally, a geometrical representation of the present unified creep-plasticity theory is also given. The paper ends with a set of concluding comments.
II. PRELIMINARIES AND NOTATION

Even though the central theme of the paper is restricted to infinitesimal strains and deformations, we start off by considering finite deformations. For the most part, we employ a Cartesian coordinate system with basis \( e_i \). A tensor \( A \), for instance of second order, is written in component form as \( A = A_{ij} e_i e_j \). If \( A \) and \( B \) are two second-order tensors, the notations \( A \cdot B = A_{im} B_{mn} e_i e_n \), \( A : B = A_{ij} B_{ij} \), \( \text{tr} A = A_{ii} \), and \( \text{tr}(A \cdot B) = A : B' \) are employed, where the superscript \( t \) denotes a transpose. Let \( X_i \) be the initial and \( x_i \) the final coordinates of a material particle. We denote the deformation gradient by \( F = \frac{\partial X_i}{\partial x_j} \) and consider its polar-decomposition to be given by \( F = R \cdot U = V \cdot R \), where \( R \) is a rigid rotation and \( V \) and \( U \) are stretches. We use \( (\cdot)' \) to denote the derivative of \((\cdot)\) with respect to Newtonian time (or, in an appropriate apparent context, with respect to a Newtonian-time-like external parameter such as the external load). We use \( \mathcal{D} \) to denote the skew-symmetric tensor \( \mathcal{R} \cdot R' \) (note \( \mathcal{R} \cdot R' = I \), the identity tensor). When two different observers are employed to study the motion of the solid, the relative rigid rotation of the observer frames is denoted by \( Q (Q \cdot Q' = I) \). Let the velocity of a material particle in the current configuration (with material particles being identified by \( x_i \)) be \( v \). The velocity gradient is denoted by \( L(L_{ij} = \partial v_i / \partial x_j) \) and is decomposed in the form \( L = D + W \), where \( D \) is the symmetric rate-of-deformation tensor and \( W \) is the skew-symmetric "spin" tensor. Further, \( L = \dot{F} \cdot F^{-1} \). We denote the Cauchy stress tensor in the current configuration by \( \sigma \). We denote the deviator of \( \sigma \) by \( S \). When the deformations and strains are infinitesimal, we denote the displacement by \( u(u_i e_i) \) and the strains by

\[
\varepsilon = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) .
\]

The deviator of \( \varepsilon \) is denoted by \( e \). When deformations are infinitesimal, it is seen that one obtains the result \( D = d\varepsilon / dt = \dot{\varepsilon} \).

III. SOME GENERAL CONSIDERATIONS OF INTERNAL VARIABLE THEORIES

Let the Cauchy stress \( \sigma \) be expressed as a function of a set of internal variables \( A^{(i)} \) and the rate of deformation, \( D \). Following ONAT [1981] we assume that the internal variables \( A^{(i)} \) are irreducible even-order tensors. Further, we assume that \( A^{(i)} \) are objective quantities, i.e., under observer transformations, they transform as \( A^{(i)} = Q \cdot A^{(i)} \cdot Q' \) where \( Q \) is the rigid rotation between the two observer frames. Note that \( \sigma \) and \( D \) are objective. Let \( (\cdot)^* \) be an objective rate of \((\cdot)\), i.e., under observer transformations, \((\cdot)^* = Q \cdot (\cdot)^* \cdot Q' \). We now postulate† the following evolution equations for the objective rates of \( \sigma \) and \( A^{(i)} \):

\[
\sigma^* = f(A^{(i)}, \sigma, D) \quad i = 1 \ldots N
\]

\[
A^{(i)*} = g(A^{(i)}, \sigma, D) \quad i = 1 \ldots N .
\]

†We realize that, in the case of finite deformations, it is convenient (ATLURI [1984]) to postulate the constitutive laws in terms of objective rates of Kirchhoff stress, \( \tau = f \sigma \). However, no generality is lost in the ensuing discussions in the present paper.
We note that in eqns (1) and (2) $\sigma^*$ (and similarly $A^{(i)*}$) may represent any one of the infinitely many objective rates (see Atluri [1984], and Reed & Atluri [1985] for a detailed discussion of finite strain constitutive modeling based on [1] and [2]) such as, for instance,

$$\sigma^* = \dot{\sigma} - W \cdot \sigma + \sigma \cdot W$$  \hspace{1cm} (Jaumann-Zaremba-Noll) \hspace{1cm} (3a)

or $$\dot{\sigma} - L \cdot \sigma - \sigma \cdot L'$$  \hspace{1cm} (Truesdell) \hspace{1cm} (3b)

or $$\dot{\sigma} + L' \cdot \sigma + \sigma \cdot L'$$  \hspace{1cm} (Cotter-Rivlin) \hspace{1cm} (3c)

or $$\sigma - \Omega \cdot \sigma + \sigma \cdot \Omega$$  \hspace{1cm} (Green-McGinnis) \hspace{1cm} (3d)

etc.

where $\dot{\sigma}$ is the material rate of $\sigma$. Likewise, $A^{(i)}$ will be used to denote the material rate of $A^{(i)}$. Note that one may, in general, have unsymmetric objective rates (Atluri [1984]); however, since the choice of one stress rate over the other is largely irrelevant for purposes of postulating a constitutive relation as shown in Atluri [1984] and Reed & Atluri [1985], we shall restrict ourselves to symmetric objective rates henceforth.

The principle of objectivity necessitates that $f$ and $g$ be isotropic tensor functions, i.e., $f(A^{(i)}, \sigma', D') = Q \cdot f(A^{(i)}, \sigma, D) \cdot Q'$. The representation theorem of Rivlin and Ericksen (see Truesdell & Noll [1965], for instance) then leads to the results:

$$\sigma^* = \sum_{m} \Psi_{m}(P + P')$$  \hspace{1cm} (4a)

$$A^{(i)*} = \sum_{m} \theta_{m}(S + S')$$  \hspace{1cm} (4b)

where $P$ and $S$ consist of terms of the type:

$$I; A^{(i)}; \sigma; D; A^{(i)} \cdot \sigma; A^{(i)} \cdot D; \sigma \cdot D; A^{(i)} \cdot A^{(j)}; A^{(i)} \cdot A^{(j)} \cdot A^{(k)}; \ldots A^{(i)2}; \sigma^2; \ldots$$  \hspace{1cm} (5)

and each of the $\Psi_{m}$ and $\theta_{m}$ are functions of invariants of the type:

$$\text{trace } A^{(i)}; \text{tr } \sigma; \text{tr } D; \text{tr } (A^{(i)} \cdot \sigma); \text{tr } (A^{(i)} \cdot D); \text{tr } (\sigma \cdot D);$$

$$\text{tr } (A^{(i)} \cdot A^{(j)}); \text{tr } (A^{(i)} \cdot A^{(j)} \cdot A^{(k)}); \ldots$$  \hspace{1cm} (6)

The strain-induced anisotropy in the above type of representation is largely governed by the symmetry groups of internal variables $A^{(i)}$ (ONAT [1984]). A more apt description of strain-induced anisotropy would be to include certain material descriptors as variables in the functions $f$ and $g$; however, it is not contemplated in the present paper.

We now consider the special case of anisotropic or kinematic hardening through the introduction of "back-stress" $\alpha$ as a variable. We assume $\alpha$ to be symmetric, as the Cauchy stress. We note that $\alpha$ may have, in general, a non-zero trace. Let $r$ be the deviatoric part of $\alpha$. We may consider $r$ and $\text{tr } \alpha$ to be the irreducible (ONAT [1981]) internal variables. However, we now consider, for simplicity, evolution equations of the type:
\[ \sigma^* = f(\sigma, \alpha, D) \]  
\[ \alpha^* = g(\sigma, \alpha, D) \]  

where

\[ f = \Psi_1 I + \Psi_2 \sigma + \Psi_3 \sigma + \Psi_4 D + \Psi_5 (\sigma \cdot \alpha + \alpha \cdot \sigma) + \Psi_6 (\alpha \cdot D + D \cdot \alpha) + \Psi_7 (\sigma \cdot D + D \cdot \sigma) + \ldots \]  
\[ g = \theta_1 I + \theta_2 \sigma + \theta_3 \alpha + \theta_4 D + \theta_5 (\sigma \cdot \alpha + \alpha \cdot \sigma) + \theta_6 (\alpha \cdot D + D \cdot \alpha) + \theta_7 (\sigma \cdot D + D \cdot \sigma) + \ldots \]  

In (9) \( \Psi_m \) and \( \theta_m \) are functions of the invariants listed in (6). The evolution equation for \( \alpha \) may be written in the same way as in (8) and (9a). Note that \( \alpha \) is a Cauchy-type “back-stress” and is assumed to be objective; however, unlike the Cauchy-stress, it is not related to the concept of tractions on oriented surfaces in the deformed body.

We now examine and compare the various theories of plasticity such as the internal time theory (Valanis [1980]; Watanabe & Atluri [1984]), the “non-linear” hardening theory of Chaboche [1977] and Chaboche & Rousselier [1982], the multi-yield-surface theories (Mroz [1967, 1969]; Dafalias & Popov [1975, 1976]; Krieg [1975]) in the context of the general framework delineated in eqns (4) and (9).

While the discussion in eqns (4) and (9) pertains to finite deformations, the remainder of this paper is largely restricted to the case of infinitesimal deformations and infinitesimal strains. Thus, in the remainder of the paper, the left-hand sides of eqns (7) and (8) will be identified as material rates; and \( D \) will be identified as the Lagrangean (infinitesimal) strain rate, \( \dot{\epsilon} = (d\epsilon/dt) \).

IV. RATE-INDEPENDENT PLASTICITY

Endochronic (internal time) theory

Let \( S \) be the deviator of Cauchy stress (in the present infinitesimal deformation case, there is no distinction between various stress measures) and \( \alpha \) the deviator of the Cauchy-back-stress, \( \alpha \). The differential-strain tensor \( d\epsilon \) is written as: \( d\epsilon = (d\epsilon^v + d\epsilon^\nu) \equiv (d\epsilon^v + d\epsilon^\nu) + 1d\epsilon^\nu; \) thus, the plastic strains are purely distortional. Further, \( d\epsilon^\nu = d\epsilon - (dS/2\mu_o \) where \( \mu_o \) is the elastic shear modulus. The intrinsic time measure \( \xi \) is defined through:

\[ d\xi = (d\epsilon^\nu : d\epsilon^\nu)^{1/2} = (d\epsilon^\nu_d d\epsilon^\nu_d)^{1/2} . \]  

The differential intrinsic time \( dz \) is defined as:

\[ dz = \frac{d\xi}{f(\xi)} \]  

where \( f(\xi) \) is non-negative and \( f(\sigma) = 1 \). In the intrinsic-time (endochronic) theory, \( S \) is represented (Valanis [1980]) as:
\[ S = 2\mu_0 \int_{0}^{z} \rho(z - z') \frac{de^p}{dz'} \, dz' \, . \quad (12) \]

We assume the kernel \( \rho(z) \) to be of the form:

\[ \rho(z) = \rho_0 \delta(z) + \rho_1(z) \quad (13) \]

where \( \delta(z) \) is a Dirac delta function, and \( \rho_1(z) \) is a non-singular function. Use of (13) in (12) results in:

\[ S = S_v^0 \frac{de^p}{dz} + r(z) \quad (14a) \]

where

\[ r(z) = 2\mu_0 \int_{0}^{z} \rho_1(z - z') \frac{de^p}{dz'} \, dz' \quad (14b) \]

and

\[ S_v^0 = 2\mu_0 \rho_0 \quad (14c) \]

As noted in Valanis [1980] and Watanabe & Atluri [1984], the regions of material behavior may be characterized as:

(i) if \( |S - r| < S_v^0 \sigma f(\xi) \) the material is elastic, \quad (15a)

(ii) if \( |S - r| = S_v^0 \sigma f(\xi) \) and \( (S - r) : de \leq 0 \) the material is again elastic \quad (15b)

(iii) if \( |S - r| = S_v^0 \sigma f(\xi) \) and \( (S - r) : de > 0 \) then plastic deformation is permissible. \quad (15c)

Thus, in the present theory, \( f(\xi) \) signifies isotropic hardening (yield surface expansion), while \( r \) denotes kinematic hardening (yield surface translation).

It was shown by Watanabe & Atluri [1984] that the rate-form of the stress-strain relation arising out of the above sketched endochronic theory is entirely analogous in structure to that of the classical theory of plasticity and has the form:

\[ d\sigma = 2\mu_0 d\varepsilon + \lambda_o (I : de) I - \frac{2\mu_0(S - r) \langle (S - r) : de \rangle}{C(S_v^0)^2 f^2(\xi)} \Gamma \quad (16) \]

where

\[ C = 1 + \rho_1(o) + \frac{(S - r) : h^*}{S_v^0 f^2(\xi)} + \frac{S_v^0 f'}{2\mu_o} ; \quad (17) \]
and

\[ h^* = \int_0^z \frac{d\rho_1}{dz} (z - z') \frac{de^p}{dz} dz' \]  

(18)

and

\[ \Gamma = 1 \] if condition (15c) holds, i.e., when there is plastic loading,

\[ \Gamma = 0 \] if the material is elastic or undergoes elastic unloading.

Likewise, from (14b), one finds that:

\[ dr = 2\mu_0 \rho_1 (o) de^p + \frac{2\mu_0 h^*}{f(\xi)} (de^p ; de^p)^{1/2} \]  

(19)

Thus, in endochronic theory, the evolution equation for the deviatoric Cauchy-backstress is nonlinear in the plastic strain rate. We shall now discuss a specific form for the non-singular kernel \( \rho_1(z) \) and the ramifications thereof. It is convenient to assume \( \rho_1(z) \) as:

\[ \rho_1(z) = \sum_i \rho_{1i} e^{-\alpha_i z} \]  

(20)

Using (20) in (14b) and (18), we may write:

\[ r = \sum_i r^{(i)} = \sum_i 2\mu_0 \int_0^z \rho_{1i} e^{-\alpha_i z} \frac{de^p}{dz} dz \]  

(21)

\[ h^* = \sum_i h^{* (i)} = \sum_i \left( -\frac{\alpha_i}{2\mu_0} \right) r^{(i)} \]  

(22)

Now, one may treat each of the \( r^{(i)} \) as a secondary internal variable. Using (20) and (22) in (19), it is easily seen that:

\[ \rho_1(o) = \sum_i \rho_{1i} \]  

(23a)

\[ dr = \sum_i dr^{(i)} \]  

(23b)

\[ dr^{(i)} = 2\mu_0 \rho_{1i} de^p - r^{(i)} \alpha_i \frac{d\xi}{f} \] (no sum on \( i \))  

(23c)

\[ = 2\mu_0 \rho_{1i} de^p - \frac{\alpha_i (de^p ; de^p)^{1/2}}{f} r^{(i)} \] (no sum on \( i; i = 1, 2, \ldots \))  

(23d)
where
\[
\mathbf{de}^\prime = \frac{(\mathbf{S} - \mathbf{r})}{C(S^0_i)^2 f^2(\xi)} \langle (\mathbf{S} - \mathbf{r}):\mathbf{de} \rangle = \mathbf{de} - \frac{1}{2\mu_o} \, d\mathbf{S} .
\] (23e)

As noted earlier, the isotropic hardening is represented in the endochronic theory through the relation for self-similar expansion of the yield surface:
\[
S_y = S^0_y f(\xi)
\] (24)
where the non-negative \( f(\xi) \), such that \( f(0) = 1 \), may be assumed as:
\[
f(\xi) = (1 + \beta \xi) \quad \text{(linear form)}
\] (25a)
or
\[
= a + (1 - a)\exp(-\gamma \xi) \quad \text{(saturated form)} .
\] (25b)
Thus,
\[
dS_y = S^0_y \beta d\xi \quad \text{(linear)}
\] (26a)
\[
= \gamma(S^0_y - S_y) d\xi \quad \text{(saturated)} .
\] (26b)
In (26b) \( S^\infty_y \) is the saturated radius of the yield surface and is given by:
\[
S^\infty_y = S^0_y a .
\] (27)

It is perhaps worthwhile to summarize in Table 1 the presently derived rate-form of the relations in endochronic theory [with exponential kernel \( \rho(t) \)] and then contrast them with other relations presented in literature, which, at first glance, appear to be unrelated to each other. First, we comment that the above relations for the endochronic theory, namely, the evolution equations for \( \sigma \) and the deviatoric part, i.e., \( \mathbf{r} \), of the back-stress are but special cases of the general internal variable theory presented in Section III, when \( \mathbf{r} \) and the trace of \( \mathbf{a} \) are treated as irreducible internal variables.

**Other Internal Variable Theories**

In comparison, other “kinematic”-hardening plasticity relations in the literature are summarized in Table 2.

By comparing Tables 1 and 2, the similarity between the present rate-form of the endochronic theory and Chaboche’s [1977] theory is immediately apparent. However, in the endochronic theory, the coefficient \( (\alpha_i/f) \) (in the evolution equation for \( \mathbf{r} \)) is a function of the (non-negative) intrinsic time measure \( \xi \), while the coefficient \( d_i \) in Chaboche’s [1977] theory is independent of \( \xi \). Likewise, the rate of yield-surface expansion used by Chaboche [1977] is analogous to that in endochronic theory when a saturated function “\( f \)” is used. However, the expression for the yield-surface radius (eqn 29h) as used by Chaboche [1977] results in a zero value for the radius when \( \xi = 0 \); while, in the endochronic theory, at \( \xi = 0 \), the yield surface radius is \( S^0_y \). Finally, it is fair to say that the present endochronic theory is more general than the internal variable theory of Chaboche [1977], in the sense other choices of the kernel \( \rho(t) \) may be
made to model other phenomena, while the specific choice as in (20) results in an equivalence of the present theory and that of Chaboche [1977].

In closing this section, we point out that various desirable phenomenological consequences of the present endochronic modeling, in cyclic plasticity, etc. have been detailed by the authors in an earlier report (Watanabe & Atluri [1984]).

V. GEOMETRICAL REPRESENTATIONS OF VARIOUS THEORIES

The concept of multiple yield surfaces was introduced by Mroz [1967, 1969] to describe nonlinear kinematic hardening. While Mroz [1967, 1969] introduced these concepts using the total quantities \( \sigma \) and \( \alpha \) directly, we shall treat here the essential concepts of Mroz using the deviatoric tensors \( S \) and \( r \). We also note that a practically useful version of Mroz's theory—the so-called "two-surface" plasticity model—has also been proposed by Krieg [1975]. In the multi-yield-surface theories (Mroz [1967, 1969]), the translation of the \( (\ell) \)th yield surface is given by:

\[
d\mathbf{r}^{(\ell)} = d\lambda [S^{(\ell+1)} - S^{(\ell)}]
\]
where $S^{(l+1)}$ is the stress point, on the yield surface $f^{(l+1)} = 0$, at which the normal $\partial f^{(l+1)}/\partial S^{(l+1)}$ is "parallel" to the normal $\partial f^{(l)}/\partial S^{(l)}$ at the stress-point $S^{(l)}$ on the $l$th yield surface, $f^{(l)} = 0$. Thus,

\[ (S^{(l+1)} - r^{(l+1)}) = (S^{(l)} - r^{(l)}) \frac{R^{(l+1)}}{R^{(l)}} \]  

where $R^{(l+1)}$ and $R^{(l)}$ are the radii of the yield surfaces $f^{(l+1)}$ and $f^{(l)}$, respectively. (Note that $R^{(l+1)}$ and $R^{(l)}$ may change as per an isotropic hardening rule.) Use of (31) in (30) results in:

\[ \dot{r}^{(l)} = \dot{\lambda} \left[ S^{(l)} \left( \frac{R^{(l+1)}}{R^{(l)}} - 1 \right) - r^{(l+1)} - r^{(l)} \frac{R^{(l+1)}}{R^{(l)}} \right] \]  

In (32), $\dot{\lambda}$ is a constant that is determined from the consistency condition ($f = \dot{f} = 0$).

We now consider the implication of the present endochronic theory in the context of a multi-yield-surface theory. Consider the kernel $\rho(z)$ to be assumed as in (13), viz.,
Theoric~ of plasl i~ ny
and cr~ cp

(33)

Let \( \rho_1(z) \) be assumed as in (20). Let us specifically consider, for purposes of illustration, a three-term approximation:

\[
\rho_1(z) = \rho_{11}e^{-\alpha_1z} + \rho_{12}e^{-\alpha_2z} + \rho_{13}.
\]

(34)

Note that (14) represents a continuous function. For purposes of the ensuing discussion, we assume that eqn (14) is approximated by three distinct segments, each valid in a distinct range of \( z \), as:

\[
\rho_1(z) = \rho_1^*e^{-\alpha_1z} \quad \text{in segment 1}
\]

\[
= \rho_2^*e^{-\alpha_2z} \quad \text{in segment 2}
\]

\[
= \rho_3^* \quad \text{in segment 3}
\]

(35)

as shown in Fig. 1. Since \( \rho_1(z) \) is now distinct in each region of \( z \), use of (18) and (19) results in different expressions for the rate of change of \( r \), in each of the segments of \( z \). We shall henceforth denote each segment of \( z \) axis by a superscript \( \ell \); \( \ell = 1, 2, 3, \) etc. Thus, using (35) in (18) and (19), we have:

\[
\hat{r}^{(1)} = 2\mu_0\rho_1^*\hat{\kappa} - \frac{\alpha_1^*}{f}r^{(1)}\hat{\xi}.
\]

(36)
Now, in region 1 as defined in (35), \( \dot{e}^p \) is given from (14) as:

\[
\dot{e}^p = \frac{(S^{(1)} - r^{(1)})}{S_v^{(1)} f} \dot{\zeta}
\]

(37)

where, now, \( S_v^{(1)} \) is defined, for convenience, to be \( S_v^{(1)} = 2\mu_{o}, \rho_{o} \). Use of (40) in (39) results in:

\[
\dot{r}^{(1)} = \frac{\alpha^* \dot{\zeta}}{f} \left\{ \frac{\rho^*_1}{\rho_{o}(1)} S^{(1)} - r^{(1)} \left( 1 + \frac{\rho^*_1}{\rho_{o}(1)} \right) \right\}
\]

(38)

Comparing (32) and (38), the geometric interpretation of the endochronic theory with the kernel \( \rho_{o}(z) \) approximated as in (35) is evident: (i) the first yield surface \( f^{(1)} = 0 \) expands according to the criterion \( (S^{(1)} - r^{(1)}): (S^{(1)} - r^{(1)}) = (2\mu_{o}, \rho_{o} f)^2 \); (ii) the translation of \( f^{(1)} = 0 \) is according to (38); (iii) unlike the coefficient \( \lambda \) in Mroz's theory, the coefficient in endochronic theory is a function of \( \zeta \); and (iv) the second yield surface \( f^{(2)} = 0 \) expands, but does not translate until the (expanding and translating) first yield surface \( f^{(1)} = 0 \) comes into contact with \( f^{(2)} = 0 \). Comparing (32) and (38), it is further seen that the ratio of radii of the two yield surfaces is given by:

\[
\frac{R^{(2)}}{R^{(1)}} = 1 + \left( \frac{\rho^*_1}{\rho_{o}(1)} \right)
\]

(39)

such that the second yield surface, in the first segment of \( z \) as in (35), is given by:

\[
f^{(2)} = S^{(2)}: S^{(2)} = \left[ S_v^{(1)} \left( 1 + \frac{\rho^*_1}{\rho_{o}(1)} \right) f(\zeta) \right]^2 = 0
\]

(40a)

and

\[
R^{(2)} = S_v^{(2)} f = S_v^{(1)} \left( 1 + \frac{\rho^*_1}{\rho_{o}(1)} \right) f = 2\mu_{o} \left( \rho_{o} + \frac{\rho^*_1}{\alpha^*_1} \right) f
\]

(40b)

From (36) it is seen that when \( r^{(1)} \) tends to zero, we have:

\[
r^{(1)} = \frac{2\mu_{o} \rho^*_1 f}{\alpha^*_1} \frac{d e^p}{d \zeta}
\]

(41a)

or

\[
r^{(1)} = \left( \frac{2\mu_{o} \rho^*_1 f}{\alpha^*_1} \right)^2
\]

(41b)

Thus, the "orbit" of the center of the first yield surface \( f^{(1)} = 0 \) may be described by a circle as defined by (41b). After the translating and expanding surface \( f^{(1)} = 0 \) touches the surface \( f^{(2)} = 0 \) as given by (40), the surface \( f^{(2)} = 0 \) begins to translate. Now, we enter the second segment of \( z \) as defined in (35). Again, the use of (35) in (18), (19), and (14) results in:
\[ \dot{r}^{(2)} = 2 \mu_0 \rho_2^* \dot{e}^p - \frac{\alpha_2^*}{f} r^{(2)} \dot{\xi} . \]  

But, on the second yield surface, we have from (14),

\[ \dot{e}^p = \frac{S^{(2)}_{s} - r^{(2)}_{s}}{S^{(2)}_{y} f} \dot{\xi} ; \quad A^{(2)} = S^{(1)}_{y} \left( 1 + \frac{\rho_1^*}{\rho_0 \alpha^{*}_1} \right) . \]  

Thus,

\[ \dot{r}^{(2)} = \frac{\alpha_2^* \dot{\xi}}{f} \left[ \frac{2 \mu_0 \rho_2^*}{S^{(2)}_{y} \alpha^{*}_2} S^{(2)}_{s} - r^{(2)} \left( 1 + \frac{2 \mu_0 \rho_2^*}{S^{(2)}_{y} \alpha^{*}_2} \right) \right] . \]  

Thus, \( f^{(2)} = 0 \) now expands as well as translates as per (44); and \( f^{(3)} = 0 \) only expands, as discerned by comparing (32) and (44). Also, the coefficient of proportionality in the evolution law for \( \dot{r}^{(2)} \) is \( (\alpha_2^* \dot{\xi}/f) \), as opposed to being a constant, \( \lambda \), in Mroz's theory, eqn (32). The ratio \( R^{(3)} to R^{(2)} \) is given by:

\[ \frac{R^{(3)}}{R^{(2)}} = \left( 1 + \frac{2 \mu_0 \rho_2^*}{S^{(2)}_{y} \alpha^{*}_2} \right) . \]  

or

\[ R^{(3)} = f \left( S^{(2)}_{y} + \frac{2 \mu_0 \rho_2^*}{\alpha^{*}_2} \right) = f \left( 2 \mu_0 \alpha^{*}_2 \left[ \frac{\rho_0 + \rho_1^*}{\alpha^{*}_1} + \frac{\rho_2^*}{\alpha^{*}_2} \right] \right) = S^{(3)} f . \]  

Thus, until \( f^{(2)} = 0 \) touches it, the yield surface \( f^{(3)} = 0 \) is represented by:

\[ f^{(3)} = S^{(3)} \left( S^{(3)} - \left( R^{(3)} \right)^2 \right) = 0 . \]  

Analogous to (41b), the orbit of the center of \( f^{(2)} = 0 \) is governed by the "circle":

\[ \dot{r}^{(3)} : \dot{r}^{(2)} = \left( \frac{2 \mu_0 \rho_2^* f^2}{\alpha^{*}_2} \right) \]  

when \( f^{(2)} = 0 \) and \( f^{(3)} = 0 \) come into contact, we enter the third segment of \( z \) as defined in (35), and we have:

\[ \dot{r}^{(3)} = 2 \mu_0 \rho_3^* \dot{e}^p \]  

\[ = 2 \mu_0 \rho_3^* \left[ \frac{S^{(3)} - r^{(3)}}{S^{(3)} f} \right] \dot{\xi} . \]  

Now, (49a) is analogous to Prager's kinematic hardening, and thus the radius \( R^{(4)} \) of the bounding surface tends to infinity. Thus the yield surface \( f^{(3)} = 0 \) translates freely, without further constraint, in the stress-space.

The above discussion clearly shows the similarity of the endochronic theory to the multi-yield-surface plasticity theory. However, it should be recalled that this similar-
ity is due to the assumption in (35) that the kernel \( \rho_1(z) \) of the endochronic theory may be approximated in each segment of \( z \) by a single exponential term (see Fig. 1). Moreover, unlike in Mroz's theory, the coefficient of proportionality between the rate of translation \( \dot{\mathbf{r}}^{(i)} \) of \( f^{(i)} \) and the distance \( (S^{(i+1)} - S^{(i)}) \) between the stress-points \( S^{(i+1)} \) and \( S^{(i)} \) on \( f^{(i+1)} \) and \( f^{(i)} \), respectively, at which the normals have the same direction, is dependent on \( \xi \), and is different \( (a_1^1 \xi / f, a_2^1 \xi / f, \text{etc.}) \) in each segment. Inasmuch as a single continuous function, valid over \( 0 \leq z \leq \infty \), may be approximated by an infinite number of piecewise continuous functions, the endochronic theory may be considered as an infinite-number-of-yield-surfaces theory. In reality, however, there is no need to introduce the piecewise continuous approximation as in (35); instead, one may assume a uniformly valid assumption for the kernel \( \rho_1(z) \) as in (20) or (34). In this case, the geometric interpretation of the endochronic theory is that of a single yield surface that translates (kinematic) and expands (isotropic) as shown in Table 1. From a computational viewpoint, this description of anisotropic hardening (as in Table 1) is much more elegant, since one does not have to continuously monitor the event of one yield surface touching the next, etc.

From the above discussion, it is seen that the multi-yield-surface theories of Mroz, Krieg, and others may be considered as special cases of the present endochronic theory with a kernel \( \rho_1(z) \) that is continuous within \( 0 \leq z \leq \infty \) as in (20) or (34). The use of an approximation of piecewise continuous \( \rho_1(z) \) as in (35) may, however, be a useful concept for the characterization of materials, which exhibit finite "jumps" in the stress-strain curve at certain values of strain, as in the case of mild steel.

It is perhaps redundant to point out that if one uses a continuous kernel \( \rho_1(z) \) as represented by a single term, say \( \rho_1 e^{-\sigma z} \) of (34), for all ranges of \( z \), the development involving eqns (36)-(41) makes it clear that the present endochronic theory is more or less analogous to a two-surface plasticity theory, as proposed by Krieg [1975] and others. Specifically, in the present endochronic theory, (i) the equation for \( \dot{\mathbf{r}}^{(1)} \), i.e. (38) is similar to that of Krieg [1975]; (ii) the rate of expansion of \( f^{(1)} = 0 \) is somewhat different from that in Krieg [1975]; and (iii) the bounding surface only expands, while in Krieg [1975] it is assumed both to expand and translate. Apart from these differences in detail, the qualitative features of a uniaxial stress-strain curve in both models are similar.

Finally, it is interesting to observe, from Table 1, that in the present endochronic theory, when \( \dot{\mathbf{r}} \rightarrow 0 \), we have:

\[
2\mu_0 \rho_1(0) \frac{d\mathbf{e}^p}{d\xi} f = \sum_i \alpha_i \mathbf{r}^{(i)}
\]

or

\[
\left| \sum_i \alpha_i \mathbf{r}^{(i)} \right| = (2\mu_0 \rho_1(0) f)^2.
\]

Thus, if each \( \alpha_i \) in (20) is chosen to be \( \gg 1 \), \( |\mathbf{r}| \) itself is bounded when \( f \) also saturates to a limiting value. Thus, even in the case of a kernel being assumed in a general form as in (20), there exists a bounding or limiting surface in the present endochronic theory.

Dafalias and Popov [1975, 1976] also employ the concept of yield and bounding surfaces and adopt a "tangent-stiffness" approach rather than assuming a rule for trans-
lation of the yield surface \textit{a priori}. In Dafalias and Popov [1975, 1976], the uniaxial stress-strain relation is postulated as:

\[
d\sigma_{11} = E_{11}^{p} \, d\varepsilon_{11}^{p}
\]

and the tangent modulus \( E_{11}^{p} \) is expressed in terms of the stresses on the yield and bounding surfaces as

\[
E_{11}^{p} = E_{11}^{p}(\delta, \delta_{m}) = E_{o}^{p} + h \left( \frac{\delta}{\delta_{m} - \delta} \right)
\]

where \( \delta \) is the distance of the current state on the yield surface from the corresponding state on the bounding surface; \( \delta_{m} \) is the value of \( \delta \) at initiation of yielding; \( E_{o}^{p} \) is the value on the bound, and \( h \) is a positive shape parameter. Dafalias & Popov [1975, 1976] then employ the formalism of internal variables to generalize the model for general deformations.

On the other hand, in the present endochronic theory, if a continuous function is chosen for \( \rho_1(z) \) as in (34), the differential stress-strain relation in \textit{uniaxial tension}, as shown earlier by Watanabe & Atluri [1984], is:

\[
d\sigma_{11} = \frac{2\mu_{o}[1 - (1/C)]}{(2/3) + (2\mu_{o}/9K_{o})[1 - (1/C)]} \, d\varepsilon_{11}
\]

where

\[
C = 1 + \rho_1(0) + \rho_1 \frac{df}{dz} + \sqrt{\frac{3}{2}} \frac{h_{11}^{*}}{f}
\]

and \( \mu_{o} \) and \( K_{o} \) are the elastic shear and bulk modulii, respectively. For instance, if one assumes that \( f = 1 + \beta z \) and that \( \rho_1(z) = \rho_{11} e^{-n_1 z} + \rho_{13} \), then it is seen that

\[
\frac{h_{11}^{*}}{f} = \frac{1}{f} \int_{0}^{z} \frac{\partial \rho_1}{\partial z'} (z - z') \frac{d\varepsilon_{11}^{p}}{dz'} dz' = -\sqrt{\frac{2}{3}} \rho_{11} \left( \frac{n_1 - 1}{n_1} \right) (1 - f^{-n_1})
\]

where

\[
n_1 = 1 + \left[ 1 \frac{\varepsilon_{11}}{\beta} \right] \quad (54a)
\]

and

\[
C = 1 + \rho_{13} + \frac{\rho_{11}}{n_1} \left[ 1 + (n_1 - 1)f^{-n_1} \right] + \rho_1 \beta \quad (55)
\]

Since

\[
d\varepsilon_{11} = d\varepsilon_{11}^{p} + \left( \frac{d\sigma_{11}}{E_{o}} \right)
\]
it may be seen, using (56) in (52) that:
\[
\frac{d\sigma_{11}}{d\epsilon_{11}^p} = E_{11}^n = 3\mu_o(C - 1) \\
= 3\mu_o(\rho_o\beta + \rho_{13}) + \frac{3\mu_o}{n_1} \rho_{11}[1 + (n_1 - 1)(1 + \beta\xi)^{-n_1}] .
\] (57)

Thus the tangent modulus is controlled by the chosen functions \(\rho_1(z)\) and \(f(\xi)\) and is a continuous, nonlinear function of the plastic strain. By including more terms in the expansion for \(\rho_1(z)\) as in (20) and by considering a nonlinear function (saturated form, as in 25b) for \(f(\xi)\), one may model as arbitrary a function as desired, for the tangent modulus, \(E_{11}^n\).

On the other hand, if a piecewise continuous kernel, \(\rho_1(z)\) as in (35), is assumed, the piecewise-continuous tangent modulus may be derived as follows. First, when (14a) is specialized to the case of uniaxial tension, one obtains:
\[
dS_{11} = dr_{11} + 2\mu_o\rho_o \frac{d^2\epsilon_{11}^n}{d\xi^2} f d\xi + 2\mu_o\rho_o \frac{de^o}{d\xi} \frac{df}{d\xi} d\xi
\] (58)

when a segmented function \(\rho_1(z)\) as in (35) is assumed and \(f(\xi)\) is assumed to be linear as \(f = 1 + \beta\xi\), use of (58) results in the following differential stress-strain relations in each segment:
\[
d\sigma_{11}^{(1)} = \left[ 3\mu_o \frac{\rho_{11}^*}{n_1} \left( 1 + (n_1^* - 1)f^{-n_1^*} \right) + 3\mu_o\rho_o\beta \right] d\epsilon_{11}^p
\] (59)
\[
d\sigma_{11}^{(2)} = \left[ 3\mu_o \frac{\rho_{22}^*}{n_2^*} \left( 1 + (n_2^* - 1)f^{-n_2^*} \right) + 3\mu_o\beta \left( \rho_o + \frac{\rho_{11}^*}{\alpha_1^*} \right) \right] d\epsilon_{11}^p
\] (60)

and
\[
d\sigma_{11}^{(3)} = 3\mu_o\rho_{11}^* d\epsilon_{11}^n + 3\mu_o\beta \left( \rho_o + \frac{\rho_{11}^*}{\alpha_1^*} + \frac{\rho_{22}^*}{\alpha_2^*} \right) d\epsilon_{11}^p
\] (61)

where
\[
n_1^* = 1 + \frac{\alpha_1^*}{\beta} ; \quad n_2^* = 1 + \frac{\alpha_2^*}{\beta} .
\]

Thus, it is seen that the endochronic theory is entirely analogous to that of Dafalias & Popov [1975, 1976]. However, the implementation of the endochronic theory, as algorithmically described in Table 1, in computational mechanics, is much more straightforward, especially when the rate-equations for \(\dot{\sigma}\) and \(\dot{\epsilon}\), as given in Table 1, are used. This has been demonstrated by the authors' earlier work (Watanabe & Atluri [1984]) as well as ongoing efforts.
VI. CREEP

The present authors have earlier presented (Watanabe & Atluri [1984]) a simple theory for creep and creep-plasticity interaction, based on the concept of an intrinsic time measure which includes both real time and strain time as suggested earlier by Valanis [1975]. The stress-strain and stress-rate vs. strain-rate relations derived in Watanabe & Atluri [1984] are, respectively,

\[ S = S'_v \frac{d\eta}{dz} + r(z) \]  

\[ \frac{dS}{dt} = \dot{S} = 2\mu_o \left[ \dot{\varepsilon} - B \left( \frac{\|S - r\|}{S'_v f} \right)^m \frac{S - r}{\|S - r\|} \frac{1}{\sqrt{1 - \left( \frac{\|S - r\|}{S'_v f} \right)^2}} \right] \]  

where

\[ r = 2\mu_o \int^z \rho_1(z - z') \frac{d\eta}{dz'} dz' \]  

\[ dz^2 = \frac{d\xi^2}{f^2(\xi)} + \frac{dt^2}{g^2} \]  

\[ d\xi^2 = d\eta; \quad d\eta = \text{differential of inelastic (creep as well as plastic) strain} \]  

\[ g = \frac{f}{b} \left( \frac{\|S - r\|}{S'_v f} \right)^{1-m} \]  

\[ \rho(z) = \rho_o \delta(z) + \rho_1(z) ; \quad S'_v = 2\mu_o \rho_o \]  

and B and m are material constants.

It can be noted that the evolution equation (62b) for \( \dot{S} \) is a special case of the internal variable formalism given in Section 3, when the back-stress \( r \) is treated as an internal variable. We now consider \( \rho_1(z) \) to be chosen as in (20), i.e.,

\[ \rho_1(z) = \sum_i \rho_{1i} e^{-\mu z} \]  

Thus, as in (23),

\[ dr = \sum_i dr_i \]  

\[ dr_i = 2\mu_o \rho_{1i} \frac{d\eta}{dz} \quad \text{(no sum on } i) \]
The relation between $dz$ and $dt$ has been shown (Watanabe & Atluri [1984]) to be:

$$
\frac{dz}{dt} = \frac{1}{\sqrt{1 - \left( \frac{|S - r|}{S''_w f} \right)^2}} \frac{dt}{g} .
$$

(65)

Thus,

$$
\dot{r}^{(i)} = \frac{dr}{dt}^{(i)} = 2\mu_w \rho_{\text{tm}} \dot{\eta} - \frac{\alpha_i}{g} \frac{1}{\sqrt{1 - \left( \frac{|S - r|}{S''_w f} \right)^2}} r^{(i)} \quad \text{(no sum on } i; \ i = 1, 2, \ldots) \quad (66a)
$$

where

$$
\dot{\eta} = \frac{S - r}{S''_w} \frac{1}{\sqrt{1 - \left( \frac{|S - r|}{S''_w f} \right)^2}} \frac{1}{g} = \dot{\varepsilon} \frac{1}{2\mu_w} \dot{S} .
$$

(66b)

The evolution equation (66) for $\dot{r}^{(i)}$ may again be seen to be a case of the general formalism outlined in Sec. 3.

The rate equations (62b) and (66) for $\dot{S}$ (while the rate of the mean stress being as Table 1) and $\dot{r}$, respectively, constitute the present internal variable theory for creep. Now we compare this theory to some others in the literature. The classical Bailey-Orowan [1946] relation for $\dot{r}$ is:

$$
\dot{r} = C\dot{\eta} - a\dot{r}
$$

(67)

where $a$ and $C$ are material constants, and the second term on the right-hand side of (67) is called the "thermal recovery" term. Integration of (67) leads to:

$$
r = \int_0^t \int_0^t C e^{-a(t-t')} \frac{d\eta}{dt} \, dt' .
$$

(68)

which is analogous to the present expression for $r$, i.e. (62b), except that (62b) involves both the intrinsic time measure $\xi$ as well as the Newtonian time $t$.

The recent creep equation of Chaboche [1977], for instance, takes into account the strong nonlinearity in the thermal recovery term in the form:

$$
\dot{r} = C\dot{\eta} - b\dot{r}\dot{\xi} - a|\dot{r}|^m \frac{r}{|r|} .
$$

(69)

A similar effect can also be introduced in the isotropic hardening relation, as:

$$
\dot{S}_r = b(Q - S_r)\dot{\xi} - aS'^m .
$$

(70)

where $S_r$ is the radius of the yield surface. Other growth laws for back-stress and drag-stress, of the general type discussed in Section 3, have been surveyed by Walker [1980].

We now give a geometric representation to the present internal-time/Newtonian-time
theory for creep, as embodied in eqns (62a and 66). For this purpose, we suppose that
\( \rho_1(z) \) is assumed to be piecewise continuous as in (35). We will consider creep behavior under constant stress, when the applied stress is within the first yield surface \( f^{(1)} \), denoted by

\[
(S^{(1)} - r^{(1)}) : (S^{(1)} - r^{(1)}) = [S_y^{(1)} f]^2 = (2\mu_0 \rho_0 f)^2 .
\]  

(71)

The rate of translation of \( f^{(1)} \), from (64), is given by:

\[
dr^{(1)} = 2\mu_0 \rho_1 \eta - \alpha_1 r^{(1)} dz .
\]

(72)

In the present inelasticity theory, the inelastic strain increment \( d\eta \) is postulated (WATANABE & ATLURI [1984]) to be governed by the relation [see (62a)]

\[
S - r^{(1)} = S_y^{(1)} \frac{d\eta}{dz} .
\]

(73)

Using (73) in (72), one obtains:

\[
\dot{r}^{(1)} = \frac{dr^{(1)}}{dt} = \alpha_1 \frac{dz}{dt} \left[ S - \frac{\rho_1^*}{\rho_0 \alpha_1} r^{(1)} \left( 1 + \frac{\rho_1^*}{\rho_0 \alpha_1} \right) \right] .
\]

(74)

We now give a geometrical interpretation to (74) as follows. Noting that the considered stress \( S \) is within the yield surface \( f^{(1)} \), we locate the stress point \( S^{(1)} \) on \( f^{(1)} \) such that

\[
[S - r^{(1)}] = k(S^{(1)} - r^{(1)}) , \quad 0 \leq k \leq 1 .
\]

(75)

We now introduce a second surface \( f^{(2)} \), such that it may expand but not translate until a later time. The radius of \( f^{(2)} \) is chosen, as in (40), i.e.,

\[
R^{(2)} = S_y^{(2)} f = S_y^{(1)} \left[ 1 + \frac{\rho_1^*}{\rho_0 \alpha_1} \right] f = 2\mu_0 \left( \frac{\rho_1^*}{\rho_0 \alpha_1} \right) f .
\]

(76)

We locate the stress point \( S^{(2)} \) on \( f^{(2)} \) at which the normal to \( f^{(2)} \) is parallel to the normal to \( f^{(1)} \) at \( S^{(1)} \) (for \( S^{(1)} \) defined as in (75)). We define a stress \( S^* \) such that

\[
S^* = kS^{(2)} ; \quad 0 \leq k \leq 1 .
\]

(77)

Because of the aforementioned parallelism of normals, we have:

\[
\frac{S^{(2)}}{s_y^{(2)} f} = \frac{S^{(2)}}{S_y^{(1)} f \left[ 1 + \frac{\rho_1^*}{\rho_0 \alpha_1} \right]} = \frac{S^{(1)} - r^{(1)}}{S_y^{(1)} f} .
\]

(78)

In view of (75) and (77), we also have:

\[
S^* = \left( 1 + \frac{\rho_1^*}{\rho_0 \alpha_1} \right) (S - r^{(1)}) .
\]

(79)
Using (79) in (74), we have:

$$r^{(1)} = \alpha_t^* \frac{d\dot{z}}{dt} (S^* - S) \quad (80)$$

The geometrical concepts underlying eqns (74) to (80) can now be schematically represented as in Fig. 2. When $S \to S^*$, the yield surface $f^{(1)}$ ceases to translate; at this instant, the center $r^{(1)}$ of $f^{(1)}$ can be easily obtained from (79) to be:

$$r^{(1)} = \frac{(\rho_1^*/\rho_0 \alpha_1^*)}{[1 + (\rho_1^*/\rho_0 \alpha_1^*)]} S \quad (81)$$

When a saturating function $f(\xi)$ as in (25b) is used, and when (81) holds, the “steady-state” creep-strain is obtained from (62a) and (81) as:

$$\dot{\epsilon} = B \left( \frac{\|S\|}{|F_1|} \right)^m \frac{S}{\|S\|} \frac{1}{\sqrt{1 - \left( \frac{\|S\|}{|F_1|} \right)^2}} \quad (82)$$

where

$$|F_1| = 2 \rho_0 \rho_\nu \left(1 + \frac{\rho_1^*}{\rho_0 \alpha_1^*}\right) f \quad (83)$$

However, before $S \to S^*$, it may happen that the surface $f^{(1)}$ may come into contact with the surface $f^{(2)}$. In such a case, the surface $f^{(2)}$ becomes the active surface, in a manner entirely analogous to that discussed in connection with plasticity in Section 5, and hence a repetition of equations similar to eqns (74) to (80) ensues.

**VII. CLOSURE**

A general framework for inelastic constitutive relations of the rate-type has been indicated, based on concepts of internal variables. The evolution equations for the time-derivative of stress and other internal variables such as back-stress, etc., for plasticity and creep, as presented earlier by the authors (Watanabe & Atluri [1984]) and further enhanced in the present work, based on an internal-time (“endochronic”) concept, are shown to fit into the above general framework. It is shown that the present internal time theory for plasticity includes as special cases (when appropriate choices for the relevant kernels are made) the multi-surface-yield theories of Mroz, Krieg, Dafalias and Popov, and others as well as the nonlinear-hardening theories of Chaboche and others. In addition, it has been pointed out that the implementation of the present endochronic theory may be no more difficult, in a computational sense, than the widely popular linear hardening theory of Prager.

It has also been shown that the unified creep-plasticity theory presented by the authors also has an equivalent geometric representation as a multi-surface theory and includes the classical Bailey-Orowan model as a special case.

Acknowledgements—The results presented herein were obtained during the course of investigations supported by the U.S. Office of Naval Research, under Contract No. N00014-78-C-0636 with Georgia Tech. The authors thank Drs. Y. Rajapakse and A. Kushner for their encouragement. The authors express their appreciation to an associate, Ms. J. Webb, for her assistance in preparing this manuscript.
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(Received 5 October 1984; In final revised form 4 August 1985)