Explicit tangent stiffness matrix for the geometrically nonlinear analysis of laminated composite frame structures

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\section*{A B S T R A C T}

In this paper, based on Von Kármán's nonlinear theory and the classical lamination theory, a closed form expression is derived for the tangent stiffness matrix of a laminated composite beam element undergoing large deformation and rotation under mechanical and hygrothermal loads. Stretching, bending, and torsion have been considered. A co-rotational element reference frame is used as the Updated Lagrangian (UL) formulation. The model has been verified in different problems by comparison with the results of Nastran and ANSYS composite laminate tools, and the difference in the resulting large deformations is less than 5%. The major advantage of the proposed approach is that the composite structure is modeled using 1D beam elements rather than 2D shell or 3D solid elements as in the case of Nastran and ANSYS where laminates are defined over surfaces or 3D solids. The availability of an explicit expression for the tangent stiffness matrix makes the proposed model highly efficient specially when dealing with large composite space frame structures. The saving in computational time could reach 93% compared to regular FE software packages. The developed model is very useful for modeling and designing flexible composites used in new applications such as morphing aerospace structures and flexible robots.

\section*{1. Introduction}

With the appearance of new technologies and inventions in the fields of automotive design, aerospace structures, smart structures, and robotics, the design and manufacturing of laminated composite materials have seen a lot of development. Beside the well-known applications of fiber-reinforced laminated composite materials because of their various advantages, such as high specific stiffness and strength, high corrosion resistance, good thermal insulation, fatigue resistance and damping properties, new applications of composite materials emerged that necessitate the development of new effective and efficient tools and approaches in design and simulation. For example, a lot of smart material elements, such as shape memory alloy (SMA) wires or ribbons and piezoelectric patches or fibers, are embedded in polymer composite laminates to form smart composite structures with multi-functions such as sensing, actuating and load bearing [1–3]. Another example is the design of morphing aerospace structures, such as morphing wings with flexible seamless control surfaces or flexible winglets [4]. The design of such structures is challenging because of the need to have flexible, yet strong, wing skins that can morph to different shapes and still be able to stand aerodynamic loads. Composite actuators combining shape memory wires, glass fibers in a soft matrix that could morph into complex shapes utilizing coupling effects for in-plane, out-of-plane, and twisting deformations have been proposed [5–7] and applied to interesting applications such as turtle-like swimming robot [8] and morphing car spoiler [9]. Another example is robotic arms and manipulators made of flexible laminated composite end-effectors [10,11]. Flexible composites are manufactured as reinforcing fibers in flexible matrix. Such flexible composites undergo large deformations, hence for modeling these structures, geometric nonlinearity should be considered. In addition, composite beams and plates with high slenderness ratios, normally undergo large displacements and rotations even without reaching large strains and/or overcoming the material’s elastic limit [12]. Hence, it is critical to develop computational tools to efficiently and accurately predict the deformation of such composite structures subjected to any mechanical or hygrothermal loads. This will also be very useful in the design process where large number of iterations are to be done sweeping the parameters in the design space to achieve the desired goals.

Introducing geometric nonlinearity to structures is an option in almost all available commercial finite element software nowadays, such as ANSYS, ABAQUS, and MSC Nastran. Defining a composite layup
given the material properties of all plies, their fiber orientations and thicknesses can also be done in all the aforementioned software in addition to SOLIDWORKS Simulation, Autodesk Simulation Mechanical, and Nastran-in-CAD tool integrated in any CAD software such as SOLIDWORKS or Autodesk Inventor. Composite laminates are defined in these software on a surface or 3D solid to be meshed using 2D plate or shell elements or 3D solid elements, respectively. Composite laminates cannot be defined on lines to model beams or 3D frames. Hence, the computational cost of performing geometrically nonlinear static or dynamic analyses on large composite structures, such as trusses, frames or idealized structures, becomes very high. The development of a 3D frame element for the geometric nonlinear analysis of laminated composite beams will then be very effective and efficient due to the expected simplicity and low computational costs.

Several three-dimensional frame finite element formulations for the geometric nonlinear analysis of thin-walled laminated composite beams have been developed during the last two decades. Bhaskar and Librescu [13,14] developed nonlinear theory of thin-walled composite beams with closed and open sections taking into account the transverse shear deformation effect while the warping torsion component was neglected. Omidvar and Ghorbanpoor [15] and Cardoso et al. [16] proposed a three-dimensional nonlinear finite element models for thin-walled open-section composite beams with symmetric stacking sequence, including the warping effect, and based on the Updated Lagrangian (UL) formulation. Based on Von Kármán strain tensor, Vo and Lee [17,18] developed three-dimensional thin-walled laminated beam elements with open sections, and investigated the effects of fiber orientation, geometric nonlinearity, and shear deformation on the axial–flexural–torsional response. Mororó et al. [12] developed three-dimensional thin-walled laminated beam elements with closed sections using Total Lagrangian (TL) formulation allowing large displacements and moderate rotations, but without the effects of warping and transverse shear. Saravia [18] developed consistent large deformation-small strain formulation for thin-walled composite beams (TWCB) calling it a “geometrically exact TWCB formulation” suitable for modeling high aspect ratio composite beams that undergo large rigid body motions, such as wind turbine wings, satellite arms and automotive body stiffeners. Turkalj [20] presented a beam formulation for large displacement analysis of composite frames considering the flexibility of the connections.

Atluri and his co-workers extensively studied large rotations in beams, plates and shells, and attendant variational principles (see [21–24]). Explicit derivations of the tangent stiffness matrix of 3D frames including elasto-plasticity were derived [25–27] without employing either numerical or symbolic integrations. Based on a Von Kármán type nonlinear theory in rotated reference frames, Cai et al. [28] developed a simple geometrically nonlinear large-rotation beam element with arbitrary cross-section using a co-rotational reference frame for each finitely rotated beam element as the UL reference frame for the respective element, and accounting for stretching, bending and torsion. An explicit expression of a symmetric tangent stiffness matrix of the beam element in the co-rotational frame was derived and validated in multiple numerical examples of space frames undergoing large deformations.

Even in the simplest formulation of the aforementioned works that presented geometrically nonlinear composite beams [12–20], an explicit expression for the tangent stiffness matrix could not be reached. Hence in this work, a finite element formulation of laminated composite beams undergoing large deformation and rotation, in response to mechanical or hygrothermal loads, is developed based on Von Kármán nonlinear theory and the classical lamination theory. Stretching, bending, and torsion has been accounted for, and a closed form expression of the element tangent stiffness matrix has been derived, utilizing the element’s co-rotational reference frame as the UL formulation. The model has been verified in different problems by comparison with Nastran and ANSYS results. In all problems, the difference in the resulting deformations is less than 5%. The relative simplicity of the derived explicit tangent stiffness matrix is one of the major advantages of the proposed approach, which makes large deformation analyses of laminated composite beams highly efficient specially when dealing with large composite space frame structures.

2. Transformation between the global and the deformation-dependent co-rotational local frames of reference

Consider a beam element made up of fiber-reinforced composite laminate in the initial configuration and then deforms to the deformed configuration when loads are applied on it, as shown in Fig. 1 (left). The global coordinate system is denoted \( \mathbf{x}_i \), with \( \mathbf{e}_i \) as the unit vectors in the directions of the three global axes. \( \mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k \) are the coordinate syst of the initial and deformed configurations respectively, with corresponding unit vectors \( \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \), respectively as shown in Fig. 1 (left).

The unit vectors of the initial configuration are defined as follows: \( \mathbf{e}_i = (\Delta \mathbf{x}_i^1, \Delta \mathbf{x}_i^2, \Delta \mathbf{x}_i^3)/L \),

\[
\mathbf{e}_i = (\Delta \mathbf{x}_i^1, \Delta \mathbf{x}_i^2, \Delta \mathbf{x}_i^3)/L,
\]

where \( \Delta \mathbf{x}_i = \mathbf{x}_i^{(2)} - \mathbf{x}_i^{(1)} \), \( \Delta \mathbf{x}_j = \mathbf{x}_j^{(2)} - \mathbf{x}_j^{(1)} \), \( \Delta \mathbf{x}_k = \mathbf{x}_k^{(2)} - \mathbf{x}_k^{(1)} \), \( L = \sqrt{(\Delta \mathbf{x}_i^1)^2 + (\Delta \mathbf{x}_j^2)^2 + (\Delta \mathbf{x}_k^3)^2} \), and the superscript \((j)\) in \( \mathbf{x}_i^{(j)} \) indicates the node number.

\( \mathbf{e}_i \) is perpendicular to \( \mathbf{e}_j \) and \( \mathbf{e}_k \), while \( \mathbf{e}_j \) is perpendicular to \( \mathbf{e}_i \) and \( \mathbf{e}_k \):

\[
\mathbf{e}_i \times \mathbf{e}_j = (\mathbf{e}_i \times \mathbf{e}_j)/\left| \mathbf{e}_j \times \mathbf{e}_i \right| = (-\Delta \mathbf{x}_i^1, \Delta \mathbf{x}_i^2, \Delta \mathbf{x}_i^3)/Z; \quad \mathbf{e}_i = \mathbf{e}_j \times \mathbf{e}_k,
\]

where \( Z = \sqrt{(\Delta \mathbf{x}_i^1)^2 + (\Delta \mathbf{x}_j^2)^2} \).

Therefore, the relation between \( \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \) and \( \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \) could be written in matrix form as:
The unit vectors in the directions of the deformed configuration coordinates are defined as follows: $e_i$ depends on the nodal coordinates of the deformed beam element, 

$$e_i = (\Delta x, \Delta y, \Delta z)/l,$$  

where $\Delta x = x^{(2)}_i - x^{(1)}_i$; $\Delta y = y^{(2)}_i - y^{(1)}_i$; $\Delta z = z^{(2)}_i - z^{(1)}_i$; $l = l(\Delta x^2) + (\Delta y^2) + (\Delta z^2)$.

$e_i$ is perpendicular to $\overline{e_1}$ and $e_2$, while $e_i$ is perpendicular to $\overline{e_1}$ and $e_2$:

$$e_2 = (\overline{e_3} \times e_i)/|\overline{e_3} \times e_i|; \quad e_3 = e_1 \times e_2.$$  

Now, $\overline{e_1}$ and $\overline{e_3}$ can be written as:

$$\overline{e_1} = a_1 \overline{e}_2 + a_2 \overline{e}_3 + a_3 \overline{e}_1; \quad \overline{e}_3 = a_1 \overline{e}_3 + a_2 \overline{e}_2 + a_3 \overline{e}_1,$$  

where $a_1 = \overline{a}_1$, $a_2 = \overline{a}_2$, and $a_3 = \overline{a}_3$.

Using Eq. (12), $e_i$ could be expressed as:

$$e_2 = [(c_1a_1 - c_2a_3)\overline{e}_1 + (c_2a_2 - c_3a_1)\overline{e}_2 + (c_3a_3 - c_1a_2)\overline{e}_3]/l;$$

$$e_3 = b_1 \overline{e}_1 + b_2 \overline{e}_2 + b_3 \overline{e}_3,$$  

where $l' = l(\Delta x^2) + (\Delta y^2) + (\Delta z^2)$, and $b_1 = \overline{b}_1 \overline{e}_1 + \overline{b}_2 \overline{e}_2 + \overline{b}_3 \overline{e}_3$.

Finally, $e_1$ takes the form:

$$e_1 = (a_2b_3 - a_3b_2)\overline{e}_2 + (a_3b_1 - a_1b_3)\overline{e}_3 + (a_1b_2 - a_2b_1)\overline{e}_1.$$  

Therefore, the relationship between $e_1$, $e_2$, $e_3$, and $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$, is written in matrix form as:

$$\begin{bmatrix}
\overline{e}_1 \\
\overline{e}_2 \\
\overline{e}_3
\end{bmatrix} = \overline{\phi}_0
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix},$$

where $\overline{\phi}_0 = \begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
(a_2b_3 - a_3b_2) & (a_3b_1 - a_1b_3) & (a_1b_2 - a_2b_1)
\end{bmatrix}$.

$\overline{\phi}_0$ is the transformation matrix between $e_1, e_2, e_3$ and $\overline{e}_1, \overline{e}_2, \overline{e}_3$.

### 3. Von Kármán nonlinear theory for a beam with large deformation

As shown in Fig. 1 (right), the laminated composite beam is subjected to torsion $T$ about $x_3$ direction, and bending moments $M_{22}$ and $M_{33}$ about $x_1$ and $x_2$ directions, respectively. The cross-sectional area is assumed to be constant throughout the deformation of the composite beam. The displacements of the beam's centerline are denoted $u_{10}(x_1), u_{20}(x_2), u_{30}(x_3)$. The bending deformations resulting from $M_{22}$ and $M_{33}$ are $x_1(\partial u_{10}/\partial x_2)_{x_3}$ and $x_2(\partial u_{20}/\partial x_3)_{x_3}$ respectively. The total torsion of the beam about $x_3$ due to the torque $T$ is denoted $\tau(x_3)$.

With the above definitions and Fig. 1 (right), it is possible to write three equations for the total displacements of the composite beam in $x_1, x_2$, and $x_3$ directions:

$$u_1 = u_{10}(x_1) - \frac{\partial u_{10}}{\partial x_3} x_3 \frac{\partial u_{10}}{\partial x_2}; \quad u_2 = u_{20}(x_2) - x_2 \tau(x_3); \quad u_3 = u_{30}(x_3).$$  

### 3.1. Strain-displacement relations

Green-Lagrange strain-displacement relations describing the large deformations of the thin composite beam based on Von Kármán’s nonlinear theory take the following forms: Axial strain in the $x_1$ direction:

$$\varepsilon_{11} = \frac{\partial u_{10}}{\partial x_1} + \frac{1}{2} \left[ \left( \frac{\partial u_{10}}{\partial x_2} \right)^2 + \left( \frac{\partial u_{10}}{\partial x_3} \right)^2 + 2 \frac{\partial u_{10}}{\partial x_3} \frac{\partial w_{10}}{\partial x_1} - \frac{\partial u_{10}}{\partial x_2} \frac{\partial w_{10}}{\partial x_2} - \frac{\partial u_{10}}{\partial x_3} \frac{\partial w_{10}}{\partial x_3} \right] = \frac{\partial u_{10}}{\partial x_1} - \frac{\partial w_{10}}{\partial x_1} - \frac{\partial w_{10}}{\partial x_1} + \frac{\partial w_{10}}{\partial x_1}$$

where $\gamma_i = 2\varepsilon_i (i \neq j)$ are the engineering shear strains. As shown in Eq. (20), $\varepsilon_2$ and $\varepsilon_3$ do not have any linear terms and their nonlinear terms
are relatively small. Therefore, these normal strains that are perpendicular to the beam’s longitudinal direction could be neglected, and the linear and nonlinear strains can be reduced to:

\[
t^i = \begin{bmatrix}
\frac{\epsilon^{i}_{11}}{\gamma^{i}_{11}} \\
\frac{\epsilon^{i}_{12}}{\gamma^{i}_{12}} \\
\frac{\epsilon^{i}_{22}}{\gamma^{i}_{22}} \\
\end{bmatrix} = \begin{bmatrix}
u_{11} \chi_{12} \chi_{22} \chi_{13} \chi_{11} \\
0 \eta x_0 + x_1 + x_2 \chi_{13} \chi_{11} \\
0 \chi_{13} \chi_{11} \\
\end{bmatrix}, \quad \epsilon^N = \begin{bmatrix}
\epsilon^{N}_{11} \\
\epsilon^{N}_{12} \\
\epsilon^{N}_{22} \\
\end{bmatrix}
\]

(21)

3.2. Stress-strain relations

Considering each composite lamina to have linear elastic material behavior, the additional second Piola-Kirchhoff stress tensor written in matrix form could be written as [29]:

\[
\sigma = \bar{Q} \epsilon,
\]

(22)

where \( \sigma = [\sigma_{11}^1, \sigma_{12}^1, \sigma_{22}^1, \sigma_{13}^1, \sigma_{23}^1, \sigma_{33}^1]^T \), and \( \epsilon = [\epsilon_{11}, \epsilon_{22}, \gamma_{12}, \gamma_{13}, \gamma_{23}]^T \)

and

\[
\bar{Q} = \Psi^{-1} = (T^T \Psi)^{-1} = T^{-1} \Psi^{-1} \Psi^{-T}.
\]

(23)

\[
\Psi = \begin{bmatrix}
\frac{1}{k^2} & \frac{k}{k^2} & \frac{k}{k^2} & \frac{k}{k^2} & \frac{k}{k^2} & \frac{k}{k^2} \\
\frac{k}{k^2} & \frac{1}{k^2} & \frac{k}{k^2} & \frac{k}{k^2} & \frac{k}{k^2} & \frac{k}{k^2} \\
\frac{k}{k^2} & \frac{k}{k^2} & \frac{1}{k^2} & \frac{1}{k^2} & \frac{1}{k^2} & \frac{1}{k^2} \\
\frac{1}{v_{xy}} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

(24)

\( E_x, E_y, E_z \) are the three Young’s moduli, \( v_{xy}, v_{xz}, v_{yz} \) are the three Poisson’s ratios, and \( G_{xy}, G_{xz}, G_{yz} \) are the shear moduli in the local axes of the orthotropic lamina shown in Fig. 2 (right). \( c = \cos \theta \) and \( s = \sin \theta \).

As was mentioned earlier, \( \epsilon_{22}, \epsilon_{33} \) and \( \gamma_{12} \) strains are neglected, hence we do not consider the associated stresses. The stress-strain relations in a single lamina can then be written as

\[
\begin{bmatrix}
\sigma_{11}^L \\
\sigma_{12}^L \\
\sigma_{22}^L \\
\end{bmatrix} = \begin{bmatrix}
\sigma_{11}^1 \\
\sigma_{12}^1 \\
\sigma_{22}^1 \\
\end{bmatrix} + \begin{bmatrix}
\sigma_{11}^N \\
\sigma_{12}^N \\
\sigma_{22}^N \\
\end{bmatrix} = \begin{bmatrix}
\sigma_{11}^L \\
\sigma_{12}^L \\
\sigma_{22}^L \\
\end{bmatrix} = \begin{bmatrix}
\sigma_{11}^L \\
\sigma_{12}^L \\
\sigma_{22}^L \\
\end{bmatrix} = \begin{bmatrix}
\sigma_{11}^1 \\
\sigma_{12}^1 \\
\sigma_{22}^1 \\
\end{bmatrix} + \begin{bmatrix}
\sigma_{11}^N \\
\sigma_{12}^N \\
\sigma_{22}^N \\
\end{bmatrix}
\]

(24)
3.3. Hygrothermal effects

The global coefficients of thermal expansion in a single lamina are related to the local ones through the transformation matrix as follows:

\[ \alpha_{iG} = T^{-1} \alpha_i \]  

(41)

where \( \alpha_i \) are the local coefficients of thermal expansion.

The thermal stresses in a single lamina due to a temperature change \( \Delta T \) can then be written as

\[
\begin{bmatrix}
\sigma_{11}^T \\
\sigma_{22}^T \\
\sigma_{33}^T
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix} \Delta T.
\]

(42)

By neglecting \( \sigma_{12}^T, \sigma_{33}^T \) and \( \sigma_{23}^T \) as was done before, we get

\[
\begin{bmatrix}
\sigma_{11}^T \\
\sigma_{22}^T
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{bmatrix} \Delta T.
\]

(43)

The nodal axial thermal force in \( x_i \) direction can be expressed as

\[
N_{x_i}^T = \int A \sigma_{x_i}^T dA = \Delta T \int A [\overline{Q}_1 x_i + \overline{Q}_2 x_i + \overline{Q}_3 x_i] dx_i
\]

(44)

where

\[
\overline{N} = \sum_{k=1}^{n} [\overline{Q}_1 x_i + \overline{Q}_2 x_i + \overline{Q}_3 x_i] (h_k - h_{k-1})
\]

(45)

The nodal bending moments about \( x_i \) and \( x_3 \) (\( M_{x_i}^T \) and \( M_{x_3}^T \)) and the total torque about \( x_1 \) (\( T^T \)) due to temperature change are derived similarly to get

\[
M_{x_i}^T = \int A \sigma_{x_i}^T x_j dA = \Delta T \int A [\overline{Q}_1 x_i + \overline{Q}_2 x_i + \overline{Q}_3 x_i] x_j dx_j = \Delta T \overline{M}_i^T,
\]

(46)

\[
M_{x_3}^T = \int A \sigma_{x_3}^T x_j dA = \Delta T \int A [\overline{Q}_1 x_3 + \overline{Q}_2 x_3 + \overline{Q}_3 x_3] x_j dx_j = 0,
\]

(47)

\[
T^T = \int A [\sigma_{x_1}^T x_3 - \sigma_{x_3}^T x_1] dA = -\Delta T \int A [\overline{Q}_1 x_3 + \overline{Q}_2 x_3 + \overline{Q}_3 x_3] dx_3 = -\Delta T \overline{P}_T,
\]

(48)

where

\[
\overline{\sigma} = \frac{1}{2} \sum_{k=1}^{n} [\overline{Q}_1 x_1 + \overline{Q}_2 x_3 + \overline{Q}_3 x_3] (h_k - h_{k-1}),
\]

(49)

The generalized nodal forces due to temperature change can then be written in matrix form as

\[
\mathbf{s}^T = \begin{bmatrix}
S_{x_1}^T \\
S_{x_2}^T \\
S_{x_3}^T
\end{bmatrix} = \Delta T \begin{bmatrix}
\overline{N}_{x_1}^T \\
\overline{M}_{x_2}^T \\
\overline{M}_{x_3}^T
\end{bmatrix} = \Delta T \begin{bmatrix}
\overline{N} \\
\overline{\sigma}
\end{bmatrix}
\]

(50)
Following the same process, the generalized nodal forces due to moisture absorption can be derived as

$$
\mathbf{S}^{\text{Moist}} = \begin{bmatrix}
S_{11}^{\text{Moist}} \\
S_{12}^{\text{Moist}} \\
S_{13}^{\text{Moist}} \\
S_{23}^{\text{Moist}}
\end{bmatrix} = \begin{bmatrix}
\mathbf{M}_{11}^{\text{Moist}} \\
\mathbf{M}_{12}^{\text{Moist}} \\
\mathbf{M}_{13}^{\text{Moist}} \\
\mathbf{M}_{23}^{\text{Moist}}
\end{bmatrix} = b\Delta C \begin{bmatrix}
\tilde{\mathbf{F}} \\
0 \\
0 \\
-\tilde{\mathbf{X}}
\end{bmatrix}
$$

(51)

where

$$
\tilde{\mathbf{F}} = \sum_{k=1}^{n} \left[ \mathbf{Q}_{11} \mathbf{b}_{x} + \mathbf{Q}_{13} \mathbf{b}_{y} + \mathbf{Q}_{16} \mathbf{b}_{y} \right] \left( h_{k} - h_{k-1} \right) \tilde{\mathbf{W}}
$$

(52)

$$
\tilde{\mathbf{X}} = \frac{1}{2} \sum_{k=1}^{n} \left[ \mathbf{Q}_{16} \mathbf{b}_{x} + \mathbf{Q}_{13} \mathbf{b}_{y} + \mathbf{Q}_{16} \mathbf{b}_{y} \right] \left( h_{k} - h_{k-1} \right)
$$

(53)

\( \mathbf{T}^{\text{Moist}} \)

are the moisture expansion coefficients, and \( \Delta C \) is the change in the weight of moisture absorbed per unit weight of the lamina. The hygrothermal generalized nodal forces \( \mathbf{S}^{\text{Th}} \) and \( \mathbf{S}^{\text{Moist}} \) are added to the mechanical generalized nodal forces \( \mathbf{S}^{\text{Mech}} \) in Eq. (40).

4. Updated Lagrangian formulation in the co-rotational reference frame

4.1. Interpolation functions

The generalized displacement vector in a beam element with two nodes and six degrees of freedom per node can be expressed as [30]

$$
\mathbf{U} = \mathbf{N} \tilde{\mathbf{u}} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
$$

(54)

where \( \mathbf{N} \) contains the shape functions

$$
\mathbf{N}_1 = \begin{bmatrix}
\phi_1 & 0 & 0 & 0 & 0 & 0 \\
0 & N_1 & N_0 & 0 & N_2 & 0 \\
0 & 0 & N_1 & 0 & -N_2 & 0 \\
0 & 0 & 0 & \phi_1 & 0 & 0
\end{bmatrix}
$$

$$
\mathbf{N}_2 = \begin{bmatrix}
\phi_2 & 0 & 0 & 0 & 0 & 0 \\
0 & N_1 & N_0 & 0 & N_2 & 0 \\
0 & 0 & N_1 & 0 & -N_2 & 0 \\
0 & 0 & 0 & \phi_2 & 0 & 0
\end{bmatrix}
$$

(55)

\( \phi_1 = 1-x; \ \phi_2 = \xi \)

\( N_1 = 1-3\xi^2 + 2\xi^3; \ N_2 = (\xi-2\xi^2 + \xi^3)l; \ N_3 = 3\xi^2-2\xi^3; \ N_4 = (\xi^3-\xi^2)l \)

(56)

\( l \) is the length of the beam element, and \( \xi \) is the non-dimensional coordinate,

$$
\xi = \frac{x-x_1}{l}, \quad (0 < \xi < 1),
$$

and \( x_1 \) is the coordinate of first node along \( x_i \).

\( \mathbf{u} \) is the nodal degrees of freedom vector at node \( i \) in the UL co-rotational frame \( e_i \) in Fig. 1 (left):

$$
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} u_{10} \\ u_{20} \\ u_{30} \\ \tau_i^1 n_{10}^i \eta_{20}^i \end{bmatrix}, \quad [i = 1, 2]
$$

(57)

where \( \eta_{10}^i \) and \( \eta_{20}^i \) are the nodal slopes in the 1–3 and 1–2 planes respectively as shown in Fig. 1 (left).

The element generalized strains can be rewritten as

$$
\mathbf{E} = \mathbf{E}^{\text{I}} + \mathbf{E}^{\text{N}} = (\mathbf{B}^{\text{I}} + \tilde{\mathbf{B}}^{\text{N}}) \tilde{\mathbf{u}},
$$

(58)

where

$$
\tilde{\mathbf{B}}^{\text{I}} = \begin{bmatrix}
\begin{array}{cccccc}
\delta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_2 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_2
\end{array}
\end{bmatrix}
$$

(59)

$$
\tilde{\mathbf{B}}^{\text{N}} = \frac{1}{2} \mathbf{A}^{\text{HN}}
$$

$$
\delta \mathbf{E} = (\mathbf{B}^{\text{I}} + 2\tilde{\mathbf{B}}^{\text{N}}) \delta \tilde{\mathbf{u}} = (\mathbf{B}^{\text{I}} + \mathbf{B}^{\text{N}}) \delta \tilde{\mathbf{u}},
$$

(60)

where \( \delta \) indicates the variation.

4.2. Weak formulation of the beam element in the co-rotational reference frame

The stress tensor is equal to the initial Cauchy stress, \( \mathbf{\sigma}_{\text{c}} \), plus the incremental Piola-Kirchhoff stress, \( \mathbf{\sigma}_{\text{p}} \), in the UL co-rotational frame

$$
\mathbf{\sigma}_l = \mathbf{\sigma}_{\text{c}} + \mathbf{\sigma}_{\text{p}}
$$

(61)

(62)

The equilibrium static equation and boundary conditions in the composite beam can be written as

$$
\frac{\partial}{\partial x_i} \left[ \mathbf{\sigma}_l + \mathbf{\tau}_l \right] \frac{\partial \mathbf{u}}{\partial x_i} + b_j = 0,
$$

(63)

$$
(\mathbf{\sigma}_l + \mathbf{\tau}_l) \frac{\partial \mathbf{u}}{\partial x_i} n_i - f_i = 0,
$$

(64)

where \( b_j \) are the body forces per unit volume in the current reference state, and \( f_i \) are the given boundary loads.

By taking \( \delta u_j \) to be the test function, the weak form of Eqs. (63) and (64) can be expressed as

$$
\int_v \left[ \frac{\partial}{\partial x_i} \left( \mathbf{\sigma}_l + \mathbf{\tau}_l \right) \frac{\partial \mathbf{u}}{\partial x_i} + b_j \right] \delta u_j dV - \int_{S_o} \mathbf{\sigma}_l \left( \frac{\partial \mathbf{u}}{\partial x_i} \right) n_i - f_i \delta u_j dS = 0,
$$

(65)

where \( n_i \) is the outward unit normal to the boundary surface \( S_o \).

By using the Divergence theorem and integration by parts, Eq. (65) can be written as

$$
\int_v \mathbf{\sigma}_l \left( \frac{\partial \mathbf{u}}{\partial x_i} \right) \delta u_j dV + \int_v b_j \delta u_j dV + \int_{S_o} f_i \delta u_j dS = 0
$$

(66)

Using Eq. (24), the incremental Piola-Kirchhoff stress can be written as

$$
\mathbf{\sigma}_l = \mathbf{\sigma}_{\text{c}} + \frac{\partial \mathbf{\tau}_l}{\partial x_i} n_i - f_i \delta u_j dS = 0
$$

(67)
\[
\sigma^{0}_{ik} = \sigma^{L}_{ik} + \sigma^{N}_{ik} \tag{67}
\]

Therefore the first term of Eq. (66) could be developed as
\[
\sigma_{ik}(\delta_{jk} + u_{jk})\delta_{uj} = (\sigma^{0}_{ik} + \tau^{0}_{ik}u_{jk} + \sigma^{N}_{ik})\delta_{uj} + \sigma^{N}_{ik}\delta\left(\frac{1}{2}u_{jk}u_{uj}\right)
\]
\[
+ \sigma^{N}_{ik}\delta\left(\frac{1}{2}u_{jk}u_{uj}\right) \tag{68}
\]
where we used \(\delta\left(\frac{1}{2}u_{jk}u_{uj}\right) = \frac{1}{2}(u_{jk}\delta_{uj} + u_{uj}\delta_{jk}) = u_{jk}\delta_{uj} \) since \(u_{ij} = u_{ji}\).

Using Eq. (68), and the definitions \(\epsilon^{0}_{ij} = u_{ij}, \epsilon^{N}_{ij} = \frac{1}{2}u_{jk}u_{uj}\), Eq. (66) can be written as
\[
\int \left(\sigma^{0}_{ij}\delta\epsilon^{0}_{ij} + \tau^{0}_{ij}\delta\epsilon^{0}_{ij}\right)dV = \int b_{i}\delta\sigma_{ij}dV + \int f_{j}\delta u_{ij}dS
\]
\[- \int \left[\sigma^{N}_{ij}\delta\epsilon^{N}_{ij} + \tau^{0}_{ij}\delta\epsilon^{N}_{ij}\right]dV. \tag{69}
\]

The right-hand side of Eq. (69) is the “correction” term in Newton-Rapson type iterative approach. By assuming the cross sectional area of the composite beam to be constant along \(x_1\)-direction, and by using Eqs. (55)-(61), the integration terms in the above equation could be expressed as

<table>
<thead>
<tr>
<th>Mat-A</th>
<th>Mat-B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta = 0^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 0^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 0^\circ)</td>
<td>2 mm</td>
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<tr>
<td>(\theta = 0^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 10^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 30^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 0^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 30^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 0^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 45^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 0^\circ)</td>
<td>2 mm</td>
</tr>
<tr>
<td>(\theta = 45^\circ)</td>
<td>2 mm</td>
</tr>
</tbody>
</table>

Fig. 3. Six different laminates used in the simulations.
Fig. 4. Cantilever beam subjected to forces and moments in different directions at its tip.

Fig. 5. (left) \( u_2 \) displacement, (right) \( \theta_6 \) rotation vs. \( P_2 \) tip force on the laminated composite cantilever beam.

Fig. 6. (left) \( u_3 \), (right) \( u_1 \) displacements vs. \( P_3 \) tip force on the laminated composite cantilever beam.

Fig. 7. (left) \( u_3 \) displacement, (right) \( \theta_5 \) rotation vs. \( P_3 \) tip force on the laminated composite cantilever beam.
and is the current length of the beam in the current local reference frame. Again, the width of the laminated composite beam, $b$, is assumed to be constant, and the integration in Eq. (74) can be evaluated. The $12 \times 12$ linear stiffness matrix can be expressed as

$$
\mathbf{K}^L = \frac{b}{l} \begin{bmatrix}
\mathbf{K}_1^L & \mathbf{K}_2^L & \mathbf{K}_3^L
\end{bmatrix}
$$

where

$$
\mathbf{K}_1^L = \begin{bmatrix}
\hat{A} & 0 & 0 & -\hat{D} & \hat{C} & 0 \\
\frac{\nu^2}{E^L}A & 0 & 0 & \frac{\nu^2}{E^L}A & 0 & 0 \\
0 & \frac{12}{P^L} & 0 & 0 & -\frac{6}{P^L} & 0 \\
\frac{\nu^2}{E^L}A & 0 & \frac{\nu^2}{E^L}A & 0 & 0 & 0 \\
\frac{12}{P^L} & 0 & 0 & \frac{6}{P^L} & 0 & 0 \\
\frac{\nu^2}{E^L}A & 0 & \frac{\nu^2}{E^L}A & 0 & 0 & 0
\end{bmatrix}
$$

(81)

The $12 \times 12$ transformation matrix between the generalized local coordinates of the deformed configuration of the composite beam element and the global coordinates can be written in terms of $\lambda_k$ given in Eq. (9) as

$$
\mathbf{a}^k = \lambda^k \mathbf{a}^0; \quad \mathbf{c}^k = \lambda^k \mathbf{c}^0; \quad \mathbf{F}^k = \lambda^k \mathbf{F}^0
$$

(85)

Now, the stiffness matrix can be assembled, and the finite element system of equations can be expressed as:

$$
\mathbf{K} = \mathbf{F} - \mathbf{F}^{(0)}
$$

(86)

At this stage, the Newton-Raphson algorithm can be used to solve the above equation iteratively. The iterative process of the Newton-Raphson method can be written as

$$
\mathbf{K}^{(m)} \mathbf{u}^{(m)} = \mathbf{F} - \mathbf{F}^{(m)}
$$

(87)

where $m$ is the iteration number and

$$
\mathbf{F}^{(m)} = l \int_0^1 (\mathbf{B}^k)^T \mathbf{t}^{(m)} d^2
$$

(88)

The total displacements for all nodes can then be written as

$$
\mathbf{u}^{(m+1)} = \mathbf{u}^{(m)} + \mathbf{a}^{(m)}
$$

(89)

A Matlab code has been developed to implement the proposed formulation, and solve the system of equations iteratively with applied increments of the mechanical loads, until the total applied load is reached and a converged solution is obtained.
5. Numerical examples

Several numerical examples are presented in this section to demonstrate the efficiency of the proposed method. The composite laminates used in this section are made of Mat-A and Mat-B whose properties are listed in Table 1. Six different composite laminates, illustrated in Fig. 3, are used to demonstrate different cases of stacking sequence, laminae materials, and symmetry.

Nastran-in-CAD finite element tool available with SolidWorks is used in the first two examples in this section for comparison with the results of the developed Matlab code for the proposed method. This tool has the option of using a laminated composite material given the stacking sequence and the material properties and thicknesses of all laminae. Laminates should be defined on a shell, not a solid or structural elements such as beams, and loads should be applied only in the plane of the shell. It was found that applying loads perpendicular to the plane of the laminate in Nastran (2016) always give unrealistic results as compared to ANSYS. Hence, in the following examples, we compare the results of the developed Matlab code with Nastran results only when the loads are applied in the plane of the composite laminate. We also compare with ANSYS finite element tool results when loads are applied perpendicular to the plane of the laminate. It is important to note that the proposed formulation is just modeling the laminated composite structures using 1D beam elements, while Nastran requires 2D plate elements and ANSYS requires 3D Solid elements to model composite laminates.

5.1. Large deformation analysis of a cantilever laminated composite beam

Consider a cantilever beam subjected to forces $P_1, P_2, P_3$ and moments $M_1, M_2, M_3$ at its tip, in addition to temperature and moisture change on the whole beam, as shown in Fig. 4. The beam is made of a fiber-reinforced composite laminate with a rectangular cross section, as shown in Fig. 3. The length of the beam is $L = 0.5$ m, its width is $b = 0.02$ m, and the thickness of each lamina is $t = 0.002$ m. The beam is analyzed for different cases of loadings using the developed Matlab code for the proposed method as well as using ANSYS and Nastran finite element packages.

To find the solution that best balances computational capacity and accuracy, convergence study has been performed for Nastran, ANSYS and the proposed method for the cantilever beam model. The convergence study on Nastran and ANSYS began with 22 2D square shell elements and 40 3D solid cube elements along $x_1$ direction respectively. The mesh density has been increased until convergence was reached with 225 2D shell elements along $x_1$ direction for Nastran model, 234 3D solid elements along $x_1$ direction for ANSYS model and 30 1D beam element along $x_1$ direction for the developed Matlab code. Total number of 2D elements in Nastran is 2030 corresponding to 2263 nodes, total number of 3D solid elements in ANSYS is 2361 corresponding to 2576 nodes and total number of 1D elements in Matlab is 30 corresponding to 31 nodes.

The results of the different applied mechanical and hygrothermal loads on beams made of a laminated fiber-reinforced composite with a rectangular cross section as shown in Fig. 4 are presented in Figs. 5–10 using the proposed method, Nastran and ANSYS. Specifically, Fig. 5
shows $u_2$ and $u_6$ (rotation about $x_3$ axis) at the beam tip for increasing values of applied force $P_2$. Applying force $P_3$ at the tip of the composite beam, Fig. 6 shows $u_3$ and $u_1$ at the beam tip with Laminates 1 and 2 rectangular cross sections, while Fig. 7 shows $u_2$ and $u_0$ (rotation about $x_2$ axis) with Laminates 5 and 6 for increasing values of $P_3$. Fig. 8 shows the effect of temperature on $u_1$ with constant $P_1 = 10,000$ N applied force. Excellent agreement with Nastran and ANSYS can be seen in all cases. The maximum error percent is less than 5% in all cases. Fig. 9 shows the deformation of Nastran model and the proposed method with Laminate 1 cross section for the case of $P_2 = 16,000$ N, $\Delta T = 300$ K. For this specific case, Nastran takes 26 minutes and 5 seconds to find the solution, while the developed Matlab code takes only 1 minute and
As was mentioned earlier, Nastran (2016) provides unrealistic large deformations of both Matlab code and Nastran models with Laminate 1 rectangular cross section for the case of $P_3 = -10,000$ N applied using the developed Matlab code. This is 20.2% of the time required by ANSYS.

5.2. Large deformation analysis of simply supported L-shape structure

In this example, an L-shaped structure is subjected to forces $P_1, P_2, P_3$ and moments $M_1, M_2, M_3$ at the illustrated points in Fig. 11 as well as changes in temperature and moisture content on the whole structure. This structure has two equal side lengths $L_1 = L_2 = 0.25$ m, and has two fixed points at its two ends. Each side of this structure is a laminated composite beam with a rectangular cross-section of width $b = 0.02$ m. The element size of both Matlab code and Nastran models are equal to the element size of the cantilever beam example. Accordingly, in this example, 30 1D beam elements (31 nodes) are used to model the structure using the developed method, while 2029 2D elements (216 2D shell elements along $x_1$ direction) corresponding to 6573 nodes are used to model the L-shaped structure in SolidWorks with Nastran-in-CAD tool.

The displacement of the load-application point in the laminated composite L-shaped structure made of two different composite laminates for applied $P_3$ force is applied. As was mentioned earlier, Nastran (2016) provides unrealistic large deformations when the load is applied perpendicular to the plane of the laminate. Fig. 13 (right) shows the deformation of the same structure with $P_3 = -10,000$ N applied using the developed Matlab code.

5.3. Large deformation analysis of a composite 3D frame

In this example, a 3D frame structure, shown in Fig. 15, is subjected to forces $P_1, P_2, P_3$ and moments $M_1, M_2, M_3$ at the illustrated points in the figure (center point, or two end points of the upper middle member) as well as changes in temperature and moisture content on the whole structure. All sides have equal lengths $L_1 = L_2 = L_3 = L_4 = 0.25$ m, and four points are fixed as shown. Each member of the frame is a composite beam with any of the cross-sections illustrated in Fig. 3, and width $b = 0.02$ m. Four beam elements are used for each member, hence 96 beam elements (84 nodes) are used to model the 3D frame. The displacement of the load-application point of the laminated composite 3D frame structure made of two different composite laminates for applied $P_3$ and $M_1$ loads at the two ends of the upper middle member are presented in Fig. 16. Nastran tool will never yield answers for loads applied normal to the plane of the laminate. Hence, this problem cannot be solved using Nastran. Fig. 17 (left) shows the deformation of the laminated composite 3D frame structure with Laminate 3 cross section for the case of $P_3 = -10,000$ N.

The displacement and rotation of the center point of the upper middle member of the laminated composite 3D frame structure made laminates 3 and 4 for applied $P_{3.1} = P_{3.2}$ and $M_{1.1} = M_{1.2}$ loads at the two ends of the upper middle member are presented in Fig. 18. Fig. 17 (right) shows the deformation for the case of $P_{3.1} = P_{3.2} = 1,600,000$ N.
undergoing large deformation and rotation has been obtained and utilized in analyzing different structures subjected to multiple mechanical and hygrothermal loads. The proposed approach has been verified by comparison with the results of Nastran and ANSYS finite element tools that enable modeling laminated composites, and the differences in the resulting displacements and rotations are less than 5% in all examples and cases. The developed beam element is much more efficient than using composite laminate tools in FEA software because of the ability to model such composites using 1D beam elements rather than 2D plate/shell or 3D solid elements. With structures undergoing large deformations, the computational time is less than 7% the time needed for solving the problem using Nastran shell elements and less than 21% using ANSYS 3D solid elements. The developed model will be very useful in modeling and designing flexible composites which have a lot of new applications, such as morphing aerospace structures and flexible robots.

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References


