A New Endochronic Approach to Computational Elastoplasticity: Example of a Cyclically Loaded Cracked Plate

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1 Introduction

Starting from the pioneering works of Marcal and King (1967), Yamada et al. (1968), and Zienkiewicz et al. (1969) computational elastoplastic analyses have now become commonplace. These earlier approaches employed the simple isotropic hardening plasticity theory and used either the "initial strain/stress" or "tangent stiffness" approaches in the finite-element solution algorithms. To account for anisotropic hardening, the so-called linear kinematic hardening theory of Melan and Prager was widely used in the early days of the era of computational plasticity. Almost simultaneously, the search for constitutive laws that predict cyclic plastic behavior in a more realistic fashion has begun. Among such constitutive models may be mentioned: (i) the multiple-yield-surface theories of Mroz (1969), Krieg (1975), Dafalias and Popov (1976); (ii) the internal variable theories of Chaboche and Rousselier (1982), of Onat (1981), and others; and (iii) the internal time (endochronic) theories of Valanis (1980), Valanis and Fan (1983), and Watanabe and Atluri (1984a,b). A unification of concepts underlying these various theories has recently been attempted by the authors (Watanabe and Atluri, 1984b).

The endochronic theory, as developed by Valanis (1980), results in an integral relation for the stress-deviator in terms of the history of plastic strain and attempts to circumvent the introduction of a "yield-surface." Valanis and Fan (1983) recently developed a rate form of the stress/strain relation based on the theory of Valanis (1980). In this rate form (which is dimensionally homogeneous in a timelike parameter), the rate of the stress deviator is related nonlinearly to the rate of plastic-strain as well as to the history of plastic strain. Thus, Valanis and Fan (1983) used an "initial-strain" type iterative finite-element approach to study the problem of a cyclically loaded double-edge-cracked panel. Apart from the need for iterations based on a linear-elastic stiffness matrix in each increment of loading, the determination of the stress-increment at a material point corresponding to a given strain increment also involves a number of iterations.

On the other hand, using the concepts of an internal time (endochronic) theory, the authors (Watanabe and Atluri 1984a,b) have developed an alternate form of the relation between differential stress and differential strain for use in cyclic plasticity analyses. The approach of Watanabe and Atluri (1984a,b) has the following features: (i) The concept of a "yield-surface" is retained; (ii) the differential stress is related linearly to differential strain, and this relation is essentially similar in its mathematical structure to those of the classical theories of isotropic hardening or of Melan-Prager linear kinematic hardening. Thus, the present endochronic
constitutive theory is no more difficult to implement in computational algorithms than the classical theories; and (iii) the functions describing the yield-surface translation and expansion are such that the present theory (Watanabe and Atluri 1984a,b) includes, as special cases, the multiple-yield-surface theories (Mroz, 1969; Krieg, 1975; and Dafalias and Popov, 1976) and the other internal variables theories (Choboche and Rousselier, 1982; and Onat, 1981).

In this paper, we give a brief outline of the present rate form of the three-dimensional elasto-plastic constitutive law based on an internal time (endochronic) theory. We then give explicit expressions for plane stress and plane strain. We present a "tangent-stiffness" finite-element approach based on these constitutive relations. The stress increment corresponding to a given strain increment, in this approach, may be determined using a simple "radial-return" algorithm as in classical plasticity. We study the problem of a double-edge-cracked plate subject to (i) monotonic tensile loading at the upper edge parallel to the crack axis and (ii) zero-tension as well as tension-compression cyclic loading at the upper edge. In each case, we study the elastoplastic stress/strain fields at the root of the notch, for both plane-stress and plane-strain conditions. These numerical results are compared to those of Valanis and Fan (1983) for certain identical problems. These results indicate, as those of Valanis and Fan (1983), that while the external loading is zero-tension or tension-to-compression (symmetric and periodic in nature), the stress response (especially in the direction of loading) at the root of the notch does not exhibit the same symmetry and periodicity as that of the external load. However, the details of the stress response at the root of the notch are much different, in magnitude as well as periodic variation, from those of Valanis and Fan (1983). Specifically, the presently computed stress response at the notch-root, in zero-to-tension cyclic loading for both plane stress and plane strain is progressively accumulated, (ii) the hysteresis loops gradually saturate, and (iii) the peak stress at the instant of stress reversal decreases and converges to a constant value. These detailed comparisons are included in Section 5. The paper ends with a set of concluding comments in Section 6.

2 Nomenclature

Considerations in this paper are limited to small strains and infinitesimal deformations. For simplicity we employ a system of Cartesin coordinates \( x_i \), with unit basis \( e_i \). If \( A(A_{ij}, e, e) \) and \( B (B_{m, e, e, e}) \) are two second-order tensors, the notation \( A \cdot B = A_{ij} B_{ji} e_{i} e_{j} \) and \( A \cdot B = A_{ij} B_{ji} \) is employed.

3A Internal Time (Endochronic) Theory

Let \( S \) be the deviator of Cauchy stress (in the present infinitesimal deformation case, there is no distinction between various stress measures) and \( r \) the deviatoric Cauchy-backstress. The differential-strain tensor \( d\varepsilon \) is written as \( d\varepsilon = (d\varepsilon_e + d\varepsilon_p) + \varepsilon \frac{d\sigma}{d\varepsilon} \); thus, the plastic strains considered here are purely distortional. Further, \( d\varepsilon_p \equiv d\varepsilon - (d\varepsilon/2\mu_0) \) where \( \mu_0 \) is the elastic shear modulus. The intrinsic time measure \( \tau \) is defined through:

\[
\tau = (d\varepsilon^2_0 d\varepsilon_p^2)^{1/2} = (d\varepsilon_0^2 d\varepsilon_0^2)^{1/2} \tag{3.1}
\]

The differential intrinsic time \( d\tau \) is defined as:

\[
d\tau = \frac{d\tau}{f(\tau)} \tag{3.2}
\]

where \( f(\tau) \) is non-negative and \( f(0) = 1 \). In the intrinsic-time (endochronic) theory, \( S \) is represented (Valanis, 1980; Watanabe and Atluri, 1984a) as:

\[
S = 2\mu_0 \int_0^\tau \rho(z-z') \frac{d\varepsilon_p}{dz} dz' \tag{3.3}
\]

By differentiating (3.3) with respect to \( z \), one may obtain:

\[
dS = 2\mu_0 [\rho(0) d\varepsilon_p + h(z) dz] \tag{3.4a}
\]

\[
= 2\mu_0 [\rho(0) d\varepsilon_p + h(z) (d\varepsilon^2_0 d\varepsilon_p^2)^{1/2}] \tag{3.4b}
\]

\[
= 2\mu_0 \left\{ \varepsilon_0 + h(\varepsilon_0^2 \varepsilon_0^2)^{1/2} f(\tau) \right\} \tag{3.4c}
\]

where

\[
\rho(0) = \rho(z = 0); \quad h(z) = \left[ \int_0^z \frac{d\varepsilon_p}{dz} (z-z') \frac{d\varepsilon_p}{dz} dz' \right] \tag{3.5a, b}
\]

\[
2\mu_0 \rho = 2\mu_0 \left\{ 1 + \rho(0) \right\}^{-1} \tag{3.5c, d}
\]

Equations (3.3) and (3.4a) appear to negate the need for the introduction of a "yield-surface" as in classical plasticity theory. However, the differential relation for deviatoric stress, \( dS \), in equation (3.4), depends nonlinearly on the differential plastic strain \( d\varepsilon_p \) in contrast to a linear relation of the classical plasticity theory. Valanis and Fan (1983) employ (3.4b), in conjunction with an "initial-strain" type iterative finite-element algorithm, to solve certain boundary value problems of cyclic plasticity. While specific details of the convergence of iterations in each increment of solution are not indicated by Valanis and Fan (1983), it may be seen that the presence of the nonlinear term involving \( d\varepsilon_p \) on the right-hand side of (3.4), in general, results in a slow process of convergence in each increment. It should be noted that in their development, Valanis and Fan (1983) present a "tangent-stiffness" finite-element approach based on an internal time (endochronic) theory. We then give a brief outline of the present rate form of the three-dimensional elasto-plastic constitutive law based on an internal time (endochronic) theory. We then give explicit expressions for plane stress and plane strain. We present a "tangent-stiffness" finite-element approach based on these constitutive relations. The stress increment corresponding to a given strain increment, in this approach, may be determined using a simple "radial-return" algorithm as in classical plasticity. We study the problem of a double-edge-cracked plate subject to (i) monotonic tensile loading at the upper edge parallel to the crack axis and (ii) zero-tension as well as tension-compression cyclic loading at the upper edge. In each case, we study the elastoplastic stress/strain fields at the root of the notch, for both plane-stress and plane-strain conditions. These numerical results are compared to those of Valanis and Fan (1983) for certain identical problems. These results indicate, as those of Valanis and Fan (1983), that while the external loading is zero-tension or tension-to-compression (symmetric and periodic in nature), the stress response (especially in the direction of loading) at the root of the notch does not exhibit the same symmetry and periodicity as that of the external load. However, the details of the stress response at the root of the notch are much different, in magnitude as well as periodic variation, from those of Valanis and Fan (1983). Specifically, the presently computed stress response at the notch-root, in zero-to-tension cyclic loading for both plane stress and plane strain is progressively accumulated, (ii) the hysteresis loops gradually saturate, and (iii) the peak stress at the instant of stress reversal decreases and converges to a constant value. These detailed comparisons are included in Section 5. The paper ends with a set of concluding comments in Section 6.

In the following, we present a sketch of an alternate development (Watanabe and Atluri, 1984a) of the differential stress/strain relation of the internal time (endochronic) theory of plasticity where: (i) the notion of a "yield-surface" is retained by assuming the kernel \( \rho(z) \) to have a singularity at \( z = 0 \) and (ii) the differential stress \( dS \) depends linearly on \( d\varepsilon_p \) so that expedient finite-element algorithms depending on the concepts of "tangent stiffness" or "initial stress," as elaborately developed for classical plasticity, may be employed. In that sense, the present differential stress/strain relations for endochronic theory are not different in mathematical structure from those of classical plasticity (and hence their computational implementation is no different from those of the classical theory). However, through deliberate and specific choices of the functions \( \rho(z) \) and \( f(\tau) \), the present theory can account for several phenomenological features of cyclic plasticity that the classical plasticity theory is incapable of.

Toward the objective stated in the foregoing, assume the kernel \( \rho(z) \) to be of the form:

\[
\rho(z) = \rho_0 \delta(z) + \rho_1(z) \tag{3.6}
\]

where \( \delta(z) \) is a Dirac delta function, and \( \rho_1(z) \) is a non-singular function. Use of (3.6) in (3.3) results in:

\[
S = S_0 + \rho_0 \frac{d\varepsilon}{dz} + r(z) \tag{3.7a}
\]

where

\[
r(z) = 2\mu_0 \int_0^z \rho_1(z-z') \frac{d\varepsilon_p}{dz'} dz' \tag{3.7b}
\]

and

\[
S_0 = 2\mu_0 \rho_0 \tag{3.7c}
\]

As noted by Valanis (1980) and Watanabe and Atluri (1984a), the regions of material behavior may be characterized as:

\[
\frac{dS}{dz} = 2\mu_0 \rho_0 \tag{3.7d}
\]
where $oPu_i$ 

(i) if $\mathbf{S} = \mathbf{I} < S^2f(\mathbf{r})$ the material is elastic, \hspace{1cm} (3.8a) 

(ii) if $\mathbf{S} - \mathbf{I} = S^2f(\mathbf{r})$ and 

$(S-r):de \leq 0$ the material is again elastic \hspace{1cm} (3.8b) 

(iii) if $\mathbf{S} - \mathbf{I} = S^2f(\mathbf{r})$ and $(S-r):de > 0$ 

then plastic deformation is permissible \hspace{1cm} (3.8c) 

Thus, in the present theory, $f(\mathbf{r})$ signifies isotropic hardening (yield surface expansion), while $r$ denotes kinematic hardening (yield surface translation).

The present authors (Watanabe and Atluri, 1984a) have shown that the rate form of the stress-strain relation arising out of the preceding sketched endochronic theory is entirely analogous in structure to that of the classical theory of plasticity and has the form:

$$
\frac{da}{dt} = 2\mu_0 \left[ \frac{2\mu_0}{C(S^2_\eta) - \rho^2} \right] \cdot \frac{\mathbf{r}}{(S-r):de} \\
= 2\mu_0 \left[ \frac{2\mu_0}{C(S^2_\eta) - \rho^2} \right] \cdot \frac{\mathbf{r}}{(S-r):de}
$$

or

$$
de^\rho = \frac{(S-r) - (S-r):de}{C(S^2_\eta) - \rho^2} \cdot \mathbf{r}
$$

where

$$
C = 1 + \rho_i(0) + \frac{(S-r):\mathbf{h}^*}{S^2_\rho(S^2_\eta)} + \frac{S^2_\rho}{2\mu_0} ;
$$

and

$$
\mathbf{h}^* = \int_0^z \frac{dp_i}{dz} \, (z-z') \cdot \frac{de^\rho}{dz'} \, dz'.
$$

and $\Gamma = 1$ if condition (3.8c) holds, i.e., when there is plastic loading; 

$\Gamma = 0$ if the material is elastic or undergoes elastic unloading.

Likewise, from (3.7b), one finds that:

$$
dx = 2\mu_0 \rho_i(0) \, de^\rho + 2\mu_0 \mathbf{r}^* \cdot de^\rho \cdot \mathbf{r}^{(1/2)}
$$

Thus, in endochronic theory, the evolution equation for the deviatoric Cauchy-back stress is nonlinear in the plastic strain rate (this may be labeled as nonlinear kinematic hardening).

We now discuss a specific form for the nonsingular kernel $\rho_i(z)$ and the ramifications thereof. It is convenient to assume $\rho_i(z)$ as:

$$
\rho_i(z) = \sum_i \rho_i e^{-\alpha_i z}
$$

Using (3.13) in (3.7b) and (3.11), we may write

$$
\mathbf{r} = \sum_i \mathbf{r}^{(i)} = \sum_i 2\mu_0 \int_0^z \frac{dp_i}{dz} \, (z-z') \cdot \frac{de^\rho}{dz'} \, dz'
$$

$$
\mathbf{h}^* = \sum_i \mathbf{h}^{(i)} = \sum_i \left( \frac{\alpha_i}{2\mu_0} \right) \mathbf{r}^{(i)}
$$

Now, one may treat each of the $\mathbf{r}^{(i)}$ as a secondary internal variable. Using (3.13) and (3.15) in (3.12), it is easily seen that:

$$
\rho_i(0) = \sum_i \rho_i
$$

$$
\frac{dr}{dt} = \sum_i \mathbf{r}^{(i)}
$$

$$
\frac{dr^{(i)}}{dt} = 2\mu_0 \rho_i \, de^\rho - \frac{\rho_i}{f} \, \frac{dr^{(i)}}{dt}
$$

$$
= 2\mu_0 \rho_i \, de^\rho - \frac{\rho_i}{f} \, \frac{dr^{(i)}}{dt}
$$

where

$$
\rho_i = \frac{2\mu_0}{C(S^2_\eta) - \rho^2} \cdot \mathbf{r}
$$

As noted earlier, the isotropic hardening is represented in the endochronic theory through the relation for self-similar expansion of the yield surface:

$$
S_y = S_y^0 f(\mathbf{r})
$$

where the non-negative $f(\mathbf{r})$, such that $f(0) = 1$, may be assumed as:

$$
f(\mathbf{r}) = 1 + \beta \gamma
$$

or

$$
\rho = a + (1-a)\exp(-\gamma f)
$$

Thus,

$$
dS_y = S_y^0 \beta \, df
$$

$$
\gamma = S_y^0 - S_y
$$

Table 1 Summary of the present internal-time theory of plasticity

<table>
<thead>
<tr>
<th>Endochronic theory:</th>
<th>[ ] denotes a derivative of ( ) with respect to Newtonian-time or a Newtonian-time-like external parameter such as external load</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{dS}{dt} = 2\mu_0 \phi - \frac{2\mu_0}{C(S^2_\eta) - \rho^2} \cdot (S-r) ]</td>
<td>[ \phi = \frac{(S-r):de}{C(S^2_\eta) - \rho^2} ]</td>
</tr>
<tr>
<td>[ f = (1 + \beta \gamma) ] (linear)</td>
<td>[ f = (1 + \beta \gamma) ] (linear)</td>
</tr>
<tr>
<td>[ \gamma = \frac{S_y^0}{S_y} ] (saturated)</td>
<td>[ \gamma = \frac{S_y^0}{S_y} ] (saturated)</td>
</tr>
</tbody>
</table>

It is perhaps worthwhile to summarize the presently derived rate form of the relations in endochronic theory [with exponential kernel $\rho_i(z)$] and then contrast them with other relations presented in literature, which, at first glance, may appear to be unrelated to each other.

3B Other Theories of Plasticity

In comparison to the presently developed internal time (endochronic) theory, other "isotropic hardening" and "kinematic hardening" plasticity theories in literature are summarized in the following:
Table 2 Summary of other theories of plasticity

<table>
<thead>
<tr>
<th>Theory</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prandtl-Reuss isotropic hardening</td>
<td>[ \sigma = \frac{2 \mu_0 \varepsilon}{(C + 2 \mu_0)^{1/2}} ]</td>
</tr>
<tr>
<td>Linear kinematic hardening (Prager, 1956)</td>
<td>[ \dot{\varepsilon} = \frac{6 \mu_0^2}{(C + 2 \mu_0)^{1/2}} ]</td>
</tr>
<tr>
<td>Mroz-Shrivastava-Dubey (1976)</td>
<td>[ \dot{\varepsilon} = \frac{6 \mu_0^2}{(C + 2 \mu_0)^{1/2}} ]</td>
</tr>
<tr>
<td>Chaboche and Rousselier (1982)</td>
<td>[ \dot{\varepsilon} = \frac{6 \mu_0^2}{(C + 2 \mu_0)^{1/2}} ]</td>
</tr>
</tbody>
</table>

It may be easily seen that the present endochronic theory encompasses, as special cases, all the earlier classical theories of plasticity (Table 2). Detailed discussions of this comparison, as well as the interesting aspect of how the multiphase-surface theories of Mroz (1969), Krieg (1975), and Dafalias and Popov (1976) may also be considered as special cases of the present theory, are presented by the authors (Watanabe and Atluri, 1984b).

### 4 Plane Problems of Plasticity

#### 4.1 Plane Strain

Here \( d\varepsilon_2 = d\varepsilon_1 = d\varepsilon_{12} = 0 \). Defining \( \gamma_1 = 2\varepsilon_{12} \), it is easy to derive the relevant plane-strain relations from Table 1 as:

\[
\begin{align*}
\dot{\varepsilon}_{11} & = (\lambda_0 + 2\mu_0) \varepsilon_{11} + \frac{2 \mu_0}{C(6 \mu_0)^{1/2}} \varepsilon_{11} \varepsilon_{11} + \frac{2 \mu_0}{C(6 \mu_0)^{1/2}} \varepsilon_{112}^2, \\
\dot{\varepsilon}_{12} & = 0, \\
\dot{\varepsilon}_{22} & = \frac{2 \mu_0}{C(6 \mu_0)^{1/2}} \varepsilon_{11}^2, \\
\dot{\varepsilon}_{112} & = \frac{2 \mu_0}{C(6 \mu_0)^{1/2}} \varepsilon_{11} \varepsilon_{12}.
\end{align*}
\]

### 5.1 Plane Stress

As usual, the derivation of the differential stress-strain relation in this case is somewhat tedious. First, note that \( \dot{\varepsilon}_{11} = \dot{\varepsilon}_{11} = \dot{\varepsilon}_{22} = 0 \). To obtain the stress/strain relations, note from Table 1 that

\[
\begin{align*}
\dot{\sigma}_{11} & = \frac{2 \mu_0}{C(6 \mu_0)^{1/2}} \varepsilon_{11}^2, \\
\dot{\sigma}_{12} & = 0, \\
\dot{\sigma}_{22} & = \frac{2 \mu_0}{C(6 \mu_0)^{1/2}} \varepsilon_{11} \varepsilon_{12}, \\
\dot{\sigma}_{112} & = \frac{2 \mu_0}{C(6 \mu_0)^{1/2}} \varepsilon_{11} \varepsilon_{12}.
\end{align*}
\]

Thus

\[
\dot{\varepsilon}_{11} = \dot{\varepsilon}_{11} = \dot{\varepsilon}_{22} = 0
\]

Using (4.13) in (4.9) and taking the trace of both sides of the resulting equation with \( \sigma_{11} - \sigma_{22} \) yields:

\[
\begin{align*}
\dot{\sigma}_{11} & = \frac{1}{M} \left( \sigma_{11} - \sigma_{22} \right) \dot{\varepsilon}_{11} + \frac{\nu_0}{E_0} \left( \sigma_{11} - \sigma_{22} \right) \dot{\varepsilon}_{112}, \\
\dot{\sigma}_{12} & = 0, \\
\dot{\sigma}_{22} & = \frac{1}{M} \left( \sigma_{11} - \sigma_{22} \right) \dot{\varepsilon}_{11} + \frac{\nu_0}{E_0} \left( \sigma_{11} - \sigma_{22} \right) \dot{\varepsilon}_{112}.
\end{align*}
\]
where \( S_i \), \( Safi \) are as previously defined.

The relations (4.16) may be inverted to find:

\[
\begin{align*}
(\delta e_{11}) & = \frac{E_0}{Q} \\
(\delta e_{22}) & = \frac{E_0}{Q} \\
(\delta e_{12}) & = \frac{1 - 2\nu_0}{E_0 C_0}
\end{align*}
\]

\[
\begin{bmatrix}
\delta e_{11} \\
\delta e_{22} \\
\delta e_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} \\
S_{22} \phi_{22} \\
\phi_{12}
\end{bmatrix} +
\begin{bmatrix}
S_{12} \\
S_{21} \phi_{22} \\
\phi_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
s_{11} \\
S_{22} \phi_{22} \\
\phi_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} \\
S_{22} \phi_{22} \\
\phi_{12}
\end{bmatrix} +
\begin{bmatrix}
S_{12} \\
S_{21} \phi_{22} \\
\phi_{12}
\end{bmatrix}
\]

where \( S_{11} = S_{22} = S_{12} = \frac{1}{2} \theta \), \( S_{21} = S_{22} = \frac{1}{2} \theta \), \( \phi_{12} = \frac{1}{2} \phi \).

5 Finite Element Analysis and Numerical Results

5.1 Finite Element Analysis. In so far as the presently developed theory results in a linear relation between the differential stress \( da_{ij} \) and the differential strain \( de_{ij} \), and since the differential plastic strain \( de^p \) is related to \( de \) as in (3.9b), the implementation of the present endochronic theory in a finite-element algorithm is no different from that of the classical theory of isotropic hardening or Prager-hardening plasticity theory (Marcal and King, 1967; Yamada et al., 1968; Zienkiewicz et al., 1969; and Krieg and Key, 1976). For this reason, all details of finite-element algorithms are omitted here. However, we note briefly that one may use the elastic-plastic "tangent stress/strain" relations derived in equation (3.9a) (the three-dimensional case) or in Section 4 for plane stress/strain to obtain "tangent stiffness" matrices for each finite element. Alternatively, one may use an initial strain/stress approach. Irrespective of the type of finite-element approach (tangent-stiffness/initial strain/stress) that is used, the integration of the constitutive relation, i.e., to determine \( da_{ij} \) for a given \( de_{ij} \), may proceed in any number of ways—for instance, the popular "radial return" method (Krieg and Key, 1976). In contrast, the type of differential stress/strain relations derived by Valanis and Fan (1983) mandates the use of an initial-strain method, requiring a number of iterations to account for plastic pseudoforces arising out of the nonlinear part in the \( ds \) versus \( de \) relation [see (3.4c)]. Furthermore, in the approach of Valanis and Fan (1983), the stress integration for a given relation [see (3.4c)].

5.2 Numerical Results. All the results presented here pertain to the behavior of a double-edge-cracked square plate...
subjected to monotonic or cyclic (zero-to-tension as well as tension-compression) loading. The external load is assumed to be uniformly distributed at the upper and lower edges of the plate. Because of the symmetry of loading and geometry, a quarter of the plate is modeled by finite elements, as shown in Fig. 1, with 42 elements and 284 degrees of freedom. For future reference, we designate the Gauss integration point closest to the crack-tip as point “A” and the point at the center of the upper edge of the modeled region as point “B”.

5.2.1 Monotonic Loading. The material constants used correspond to SUS 304 stainless steel (see Watanabe and Atluri, 1984a): \( E_0 = 153.8 \text{ GPa} \); \( \nu_0 = 0.3 \); \( \sigma_0 = 20.0 \text{ MPa} \). The kernel \( p_1(z) \) is approximated as: \( p_1(z) = p_{11}\exp(-\alpha_1 z) + p_{12} \), with \( p_{11} = 9.298 \times 10^{-2} \); \( \alpha_1 = 314 \); \( p_{12} = 3.511 \times 10^{-4} \). The isotropic hardening function, i.e., \( f(t) \), is assumed to be \( f = 1 + 5.0t \).

Figure 2 shows the load \( (P) \) versus deflection \( (\delta) \) curve, where \( \delta \) refers to the deflection at \( B \) in the direction of the loading. It is seen that the internal time theory results in a slightly stiffer behavior of the specimen as compared to the classical isotropic hardening theory. For both the cases of plane stress and plane strain, Fig. 3 shows the solutions for displacement \( u_2 \) and stress \( \sigma_{22} \) at Gauss points closest to the upper edge of the specimen and at points closest to the crack-axis, respectively, at applied stress levels of 98.1 MPa (i.e., 10 kgf/mm\(^2\), for plane stress) and 107.9 MPa (i.e., 11 kgf/mm\(^2\), for plane strain).

From Figs. 2 and 3, it is seen that for monotonic loading the present endochronic theory agrees fairly well with the classical isotropic hardening theory.

5.2.2 Zero-to-Tension Cyclic Loading. First, for comparison purposes, we solve the same problem as reported in Valanis and Fan’s (1983) work. The material is OFHC copper, with \( E_0 = 133 \text{ GPa} \); \( \nu_0 = 0.3 \); \( \sigma_0 = 20.0 \text{ MPa} \). [Recall \( \sigma_0 = 2\mu_0 \sigma_0 = \sqrt{2/3} \sigma_0^2 \).] The kernel \( p_1(z) \) is approximated, presently, as:

\[
p_1(z) = p_{11}\exp(-\alpha_1 z) + p_{12} \exp(-\alpha_2 z) + p_{13}
\]

with \( p_{11} = 5.979 \); \( p_{12} = 4.886 \times 10^{-1} \); \( p_{13} = 4.746 \times 10^{-2} \); \( \alpha_1 = 41389 \); \( \alpha_2 = 4497 \). The function \( f(t) \) is assumed as \( f = a + (1 - a)\exp(-\gamma t) \) with \( a = 1.01 \), and \( \gamma = 500 \), to be valid in the range \( 0 < t < 0.3 \text{ percent} \). These assumptions for \( p(z) \) and \( f(t) \) are different from those in Valanis and Fan (1983), which are: \( f(t) = 1 + 0.53(t)^{0.72} \) and \( p(z) = \rho_0 \exp(-\alpha_2 z) \) (see Valanis and Fan, 1983, for specific values for \( \rho_0 \) and \( \alpha_2 \)).

Figure 4 shows the uniaxial stress/strain curve as modeled in the foregoing, along with the comparison curve of Valanis and Fan (1983) and the experimentally determined curve. Figure 5 shows the hysteresis loops of \( (\sigma_{22} \text{ versus } \epsilon_{22}) \) response at notch-root (point A in Fig. 1) in a DEC copper (OFHC) sheet subject to zero-to-tension external cyclic loading (plane stress).
present calculations. Note that, in general, there may be stress singularities near the crack tip, which are not modeled either in the present study or in that of Valanis and Fan (1983). Apart from this, the reasons for a better estimation of stress at point A in the present series of computations, as contrasted to those of Valanis and Fan (1983) are believed to be: (i) the use of eight-node parametric elements in the present, as opposed to constant strain triangles by Valanis and Fan (1983); (ii) the use of a better finite-element algorithm in the present; and (iii) the use of a better algorithm, in the present, for the determination of stresses from the computed strains at each material point. From Fig. 5, it is seen that ratcheting takes place and plastic strain accumulates progressively at point A.

However, the present results shown in Fig. 5 for zero-to-tension cyclic loading have the following features that are different from the comparison results presented in Fig. 4 of Valanis and Fan (1983): (i) the present hysteresis loops for $\sigma_{r22}$ versus $\epsilon_{r22}$ at the root of the notch, point A, saturate gradually, due to the use of a saturating function $f(\xi)$ as in the present, and (ii) the amplitude of peak stress at point A, at the instant of stress reversal in the zero-to-tension cyclic loading, decreases gradually and converges to a steady-state value.

As a second example, we consider the zero-to-tension cyclic loading of a double-edge-cracked SUS 304 stainless steel panel with: $E_0 = 153.8$ GPa, $\nu = 0.3$, $\sigma_0 = 103.8$ MPa; $\Delta \epsilon_1 = 1.502 \times 10^{-4}$; $\Delta \epsilon_2 = 9.298 \times 10^{-4}$; $\rho_{\lambda} = 3.511 \times 10^{-4}$; $\rho_\delta = 0.072$; $\sigma_3 = 314$; $\epsilon = 1.2$; $\gamma = 25$ (note that $\rho_\lambda(z)$ and $f(\xi)$, for SUS 304 steel, are assumed to be of the same form as for OFHC copper described earlier).

Figures 6(a) and 6(b) show the $\sigma_{r22}$ versus $\epsilon_{r22}$ loops at the root of the notch (point A) of the 304 steel plate subjected to (zero-to-tension) cyclic loading under plane stress (with applied $\sigma_{\max} = 73.5$ MPa or 7.5 kgf/mm$^2$) and plane strain ($\sigma_{\max} = 98$ MPa or 10 kgf/mm$^2$) conditions, respectively. Qualitatively, the stress response is the same as in OFHC copper. Figure 7 illustrates the plastic zones at the instants of reversal of loading at various cycles of zero-tension loading under plane strain. It may be seen that the elastic region increases in size progressively, after cyclic tensile loading.

5.2.3 Tension-Compression Cyclic Loading. Here again the considered material is SUS 304 stainless steel, and the problem is that of a double-edge-cracked panel with dimensions as shown in Fig. 1.

The applied loading is uniform at the upper edge of the panel, and varies cyclically from tension to compression, with amplitude of ± 73.5 MPa (± 7.5 kgf/mm$^2$). Figure 8 shows the overall load versus deflection (at point B) curve for this tension-compression cyclic loading, under plane-stress conditions. The corresponding behavior of $\sigma_{r22}$ versus $\epsilon_{r22}$ at point A is shown in Fig. 9. From Figs. 8 and 9, it is seen that the hysteresis loops saturate gradually in the present tension-compression cyclic loading, instead of the progressive ratcheting response observed under zero-to-tension cyclic loading (see Fig. 6(a)).

6 Concluding Comments

In all the examples, dealing with either monotonic or cyclic loading, discussed in the foregoing, the increments of external loading are taken to be about 1 MPa. The computational time...
for the present endochronic theory, per increment, was about 3 sec on a CYBER855 and is about the same as for classical isotropic hardening plasticity theory. Considering the facts that: (i) the constitutive relations (in differential form) of the present endochronic theory are exactly similar in mathematical structure to those of classical plasticity theory, (ii) the computational implementation of the present endochronic theory is no more difficult and no more expensive than a classical isotropic hardening theory, and that (iii) the present theory can predict the phenomenological behavior of materials under cyclic loading in a much better fashion, as demonstrated in several examples in this paper, the present constitutive theory may be a viable option for implementation in general purpose computational codes for analyzing structures, especially those susceptible to crack-growth, under cyclic loading.

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