On My Inheritance

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The Goal

One of the most important duties of a man is to *preserve, improve and pass on to the next generation his heritage* – the best ideas and institutions left to him by his elders.

For a long time I thought that this can be done by simply raising my children, teaching and doing research.

But lately I started to realize that this strategy is failing, and that more explicit articulation of this heritage is needed.

That is what I attempt to do in this talk.

Leningrad School of Logic

Around 1965 I started to attend seminars of *Mathematical Logic and the Theory of Algorithms research group* at Leningrad Branch of Steklov Mathematical Institute.

The group was led by Nikolai Aleksandrovich Shanin, who also taught a course in Mathematical Logic at Leningrad University.

This course and the influence of V. Lifschitz who started to attend this seminar when he was in high school are primarily responsible for the change of my interest from topology to logic.

Nikolai Aleksandrovich Shanin

Members of this group concentrated on

- Development of the version of Constructive Logic and Mathematics pioneered by Andrei Andreivich Markov,
- Automated Theorem Proving,
- Decidability and Complexity.

To really appreciate the existence of this group in the USSR at that time, one needs to realize that in 1954 Logic and Cybernetics were still viewed as “bourgeois pseudo-science” and denounced by the authorities.

Constructive mathematics was born from the crisis in foundations of mathematics and attempts to answer fundamental questions such as

- What is the nature of mathematical objects and mathematical proofs?
- What does it mean for a mathematical object to exist?
- How can we establish properties of such objects?

Learning answers to these and similar questions greatly influenced my thinking and taught me a number of important lessons.

In this talk I'll try to share some of them.

Lesson 1: “Think Slowly”

I always remember the promise Shanin made in his first lecture: *Throughout this course, I will teach you to think slowly.*

It took me a lot of time to understand what this means, but I am very grateful for the lesson.

I saw great scientists seeing a mystery and depth in seemingly simple things, and realized how important, fruitful and satisfying it is to take such a mystery seriously.

It taught me the value of *clarity* and *precision* of language and thought.

Lesson 2: Philosophy Matters

I have also learned the huge role philosophy and intuition play in mathematical work.

The prevailing philosophy of mathematics in the USSR in the 60s was a confused form of Platonism.

Mathematical objects were highly abstract (often axiomatically defined) entities existing in "heavens" – the ideal objective platonic world. Proofs were means for establishing the true properties of these objects.

This view was advocated by the Bourbaki group whose highly rigorous textbooks covered important parts of mathematics based on set theory, but excluded algorithms, combinatorics, logic, etc.

Constructive mathematics had a different perspective.

Objects, though still objective entities, were results of human activity. Humans were not separate from the world, they were part of it, and were looking for truth in close, accessible places.

For instance, to prove the existence of a mathematical object satisfying some property, it was necessary to *construct* such an object by good, reliable means.

Moreover, to prove a statement

$$\forall X \exists Y \; p(X, Y)$$

one needed to find a construction F such that

$$\forall X \; p(X, F(X)).$$

Recursive Constructive Mathematics (Russian School)

Schools of constructivism differ in the precise meaning of the terms *object* and *construction*.

Russian school viewed "objects" as *words* or *symbols* generated by some collection of rules, and "constructions" as algorithms understood, via the Church Thesis, as *partial recursive functions*.

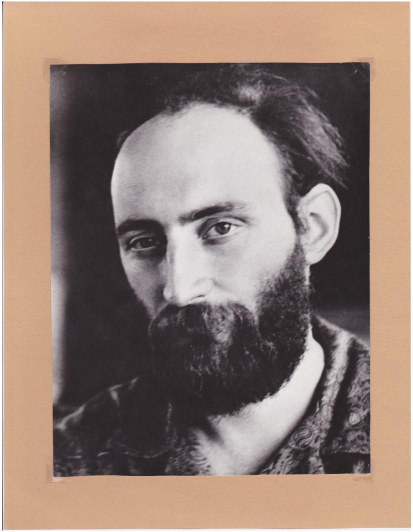
The relation to CS is obvious, and it is small wonder that Vladimir, Vladik Kreinovich, myself and some other people "raised" in this tradition eventually found their homes in CS departments.

Strong belief in foundational advantages of the constructive approach over the traditional one and its importance for computation led to an ambitious project of *replacing* the substantial parts of classical mathematics by their constructive analogues.

Classical real numbers were replaced by computable ones in the sense of Turing.

Constructive counterparts were developed for notions from calculus, functional analysis, probability theory, etc., together with constructive versions of important theorems from these fields.

The vision of the group was expressed in its hymn written by Sergey Maslov:



The day will come when every single freshman will comprehend the meaning of a "construct", the world of algorithms and of recursion will take the place of alephs.

This has not yet happened in math, but the work continues.

It is clear now that a widely held belief that constructivism imposes such restrictions as to render the development of serious mathematics impossible, is false: large parts of deep modern mathematics have been produced by purely constructive methods (Stanford Encyclopedia of Philosophy).

Lesson 3: Do What is Important

Establish your own long term goals, make sure they are important, and have courage to pursue them, ignoring current fads and opinions.

At times it seemed that, in social terms, all this work did not not lead anywhere. But the impression was wrong.
Even though math students do not spend much time on algorithms and recursion, a substantial part of the dream is realized in Computer Science Departments.

(And I still believe that some day, constructive analyses will replace classical one in our curriculum, but that day is yet to come.)

Logic

In the 60s substantial contributions to logic and automatic theorem proving had already being made by Shanin, Davydov, Maslov, Mints, Orevkov.

This was followed by work on complexity by Slisenko, Matiyasevich, and others.

Here are two examples relevant to LP:

1. The *Inverse Method* of proof, similar to Robinson's resolution, was developed at about the same time by Sergey Maslov.

2. Work on discovery of Glivenko classes – theories for which classical and constructive provability coincide.

In Pure Prolog, given a program P, we answer a query $\mathsf{q(X)}$ by using classical logic to prove $\exists \mathsf{X}\mathsf{q(X)}$.

But even though inference is classical, we still manage to extract X from the proof. Why?

Here is the answer provided by Vladimir Orevkov (without any reference to Prolog): If P consists of Horn clauses, then for every classical proof of $\exists \mathsf{X}\mathsf{q(X)}$ there is a constructive proof of the same statement.

More recent examples of the use of Glivenko classes can be found in work by V. Lifschitz and by P. Tarau.

Shanin liked to repeat that our goal as scientists is to search for *the truth*.

Our perceptions of this *one and only truth* may differ, but it is there and we can get closer to it.

This was in sharp contrast with the Soviet view of multiple truths being functions of class interests.

It is also different from the post-modernist view according to which there is no such thing as *the truth* (except, of course, the one and only indisputable truth that *"there is no truth"*).

My teachers' interests were not limited to their narrow fields. They taught me that

- **We are all participants in the long conversation which goes across centuries, countries, and scientific fields, which constitutes the world's culture and that**
- **the more aware you are about this conversation, the better is your life and your work.**

A remarkable expression of this view was ***Maslov's seminar*** **which included many Leningrad logicians.**

There were talks given by linguists, historians, philosophers, biologists, priests, poets, etc.

These talks and discussions created *the atmosphere of learning and possibility of growth* which, I believe, would have been otherwise impossible in the USSR.

It helped us to free ourselves from the shackles of official ideology and group thinking.

It allowed to practice *free thought and free speech*, opened our minds and made us happy.

It is also interesting that we were usually rather ruthless in our criticism of ideas (though not people) but no one was ever offended.

Lesson 5: Do not confuse Tolerance with Indifference

Without knowing it at the time, I had learned another lesson which proved especially useful for me in the USA: *Tolerance and desire not to offend are important, but thought should be met with thought - not with tolerance.*

Not doing this and leaving an honest argument without a response is actually a sign of indifference and disrespect, of not taking a person's thoughts seriously.

In my years in the US I attempted to pass on these and other lessons to my students. I know that I was not very successful, but hope that I have not completely failed either.

Maslov's "Theory of Deductive Systems"

This is just another example of the breadth of interests of my teachers.

Among other things in this book, Maslov studied two types of cognitive mechanisms: *horizontal* and *vertical*, loosely associated with the right and left hemispheres of the brain.

The first deals with reasoning in a fixed deductive system. The second is concerned with moving from one system to another.

The book contains many interesting mathematical observations related to this topic but I'd like to mention a non-mathematical one.

Maslov studied “in great detail the history of Russia and a number of countries in Western Europe from 16th to 18th centuries” and established an amazing correlation between so called *horizontal* and *vertical* modes of cognition and artistic and social styles of the society.

Roughly speaking, prevalence of the horizontal mode of cognition correlated with classical architecture and less authoritarian forms of government.

The vertical mode correlated with a baroque, romantic style and increased authoritarianism.

The art of living together in peace seemed to be dependent on maintaining the balance between the classical and romantic views of the world.

THANK YOU VERY MUCH FOR LISTENING!