ALGEBRA PRELIM – AUGUST 2026

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element and all ring homomorphisms are assumed to preserve the 1-elements.

GROUPS

Problem 1: Let G be a group of order pqr, where p, q and r are primes with $p < q < r$. Show that G has only one Sylow r-subgroup.

Problem 2: Let N be a normal subgroup of a group G. Assume that N has trivial center and that every automorphism on N is inner. Show that $G \cong N \times H$ for a subgroup H of G.

Problem 3: A subgroup M of a group G is *maximal* if $M \neq G$ and the only subgroups of G which contain M are M and G.

(a) Define the quasi-dihedral group of order 16, QD_{16}, by

$$QD_{16} := \langle r, s \mid r^8 = s^2 = e; srs^{-1} = r^3 \rangle.$$

Find three maximal subgroups of QD_{16}.

(b) Show that if a finite group G has only one maximal subgroup, then G is cyclic.

(c) Show that if a maximal subgroup M of G is normal, then the index of M in G is finite and prime.

RINGS

Problem 4: Show that

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$

is a Euclidean Domain with respect to the norm

$$N(a + b\sqrt{2}) = \left|a^2 - 2b^2\right|$$

and find the prime factorization of $11 + 6\sqrt{2}$.

Problem 5: Show that the ideal in the polynomial ring $\mathbb{Z}[x]$ generated by 2 and $x^4 + x + 1$ is prime.

Problem 6: Let R be the group ring $\mathbb{Z}S_3$.

(a) Find an element of R which is a zero-divisor.

(b) Find an element in the center of R which is not in $\mathbb{Z}$.

(c) The trivial units of R are $\pm S_3$. Find an element of R which is a non-trivial unit. (Hint: R contains non-trivial units of order two.)

MODULES

Problem 7: Let F be a field. The abelian group $H = \mathrm{Hom}_F(F[x], F[x])$ has two F-vector space structures. One that comes from the first copy of $F[x]$:

$$(f \cdot \varphi)(p) = \varphi(fp) \qquad \text{for } f \in F \text{ and } p \in F[x]$$

and one that comes from the second copy:

$$(f \times \phi)(p) = f\phi(p) \qquad \text{for } f \in F \text{ and } p \in F[x].$$

(a) Show explicitly that these two structures coincide.

(b) The group H also has two obvious $F[x]$-module structures; define them and show that they do not coincide.

Problem 8: Let R be an integral domain, and let I and J be ideals in R. Show that $I \cong J$ as R-modules if and only if there exists non-zero elements x and y in R with $xI = yJ$.

Problem 9: Let R be a commutative ring. An R-module M is called *simple* if the only submodules of M are $\{0\}$ and M. Show that every simple R-module is isomorphic to a quotient R/J where J is a maximal ideal in R.

FIELDS

Problem 10: Let

$$f(x) = (x-4)(x-2)x(x+2)(x+4)(x^2+4)+2$$

Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$. Let M be the splitting field for $f(x)$ over $\mathbb{Q}$. Determine $\mathrm{Gal}(M/\mathbb{Q})$.

Problem 11: Let $p(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$, and let M be its splitting field (over $\mathbb{Q}$). Assume that $p(x) \mid p(x^2)$. Show that $\mathrm{Gal}(M/\mathbb{Q})$ is abelian.

Problem 12: Let F be a field in prime characteristic p, and assume that F is not perfect, i.e., that not every element in F is a p^{th} power. Show that there exist algebraic extensions of F of arbitrarily large degree.