Complex Variables
Preliminary Exam
August 2026

Directions: Do all of the following ten problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. **(a)** Prove that any analytic function f on the domain $D = \mathbb{C}\setminus\{z : \Im(z) = 0, \Re(z) \leq 0\}$ has a primitive on D.
   (b) Give an example of a domain $G \subset D$ and an analytic function g on G which does not have a primitive on G.
2. Determine for which $k \in \mathbb{N}$ there exists an analytic function f on the disk $D_2 := \{z : |z| < 2\}$ such that
$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^k}$$
   for all positive integers n.
3. Find all entire functions $f(z)$ such that $\Re(f(z)) \neq \Im(f(z))$ for each $z \in \mathbb{C}$.
4. Let f be an analytic function on an open set containing the closed unit disk $\overline{\mathbb{D}}$. Prove that if $z \in \mathbb{D}$, then
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{\sin(w-z)}\,dw,$$
   where γ traverses the unit circle once in the positive direction.
5. Let
$$f(z) = \frac{e^{\frac{1}{z+1}}}{e^z+1}.$$
   Locate and classify all the singularities of f (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential). Calculate the residues of f at its poles.
6. Use the Residue Calculus to evaluate the integrals
$$(a)\quad \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} \qquad (b)\quad \int_0^\infty \frac{x^2\,dx}{x^4+x^2+1}.$$
7. **(a)** State any version of Rouché's Theorem.
   (b) Determine how many poles, counting multiplicity, the rational function
$$R(z) = \frac{1-z^2}{z^5-6z^4+z^3+2z+2}$$
   has in the unit disk $\{z : |z| \leq 2\}$.
8. Find a conformal mapping $w = f(z)$ from the upper half-plane $\mathbb{H} := \{z : \Im(z) > 0\}$ onto the exterior of the ellipse $L := \{w : |w-1|+|w+1| = 4\}$ such that $f(-1+i) = \infty$.