**Mathematical Finance Preliminary Examination**

May 2023

**Instruction:** Please solve 3 out of the 4 problems provided below and specify in the designated box which 3 problems you want to be graded. Make sure to provide clear and detailed solutions.

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**Problem 1.** Consider a stock $\mathcal{S}$ with one-share price process $S_t, t = 0, \dots, N$. The present time is $t = 0$, and $S_0 > 0$. The riskless bank account $\mathcal{B}$ has with value $\beta_t = (1+r)^t, t = 0, \dots, N$, where $r$ is the riskless short rate. The price process $S_t, t = 0, \dots, N$ follows $N$-period binomial pricing tree

$$S_{t+1} = \begin{cases} S_t u, & \text{with probability} \quad p, \\ S_t d, & \text{with probability} \quad q = 1-p, \end{cases}$$

where $p \in (0,1)$, and $0 < d < 1 + r < u$. Thus, the arithmetic returns,

$$R_{t+1} = \frac{S_{t+1} - S_t}{S_t} = \begin{cases} u, & \text{with probability} \quad p, \\ d, & \text{with probability} \quad q = 1-p, \end{cases} \quad t = 0, \dots, N-1, \;\; R_0 = 0,$$

are assumed independent identically distributed random variables, determining the stochastic basis $(\Omega, \mathbb{F} = \{\mathcal{F}_t = \sigma(R_0, \dots, R_t), t = 0, \dots, N\}, \mathbb{P})$,

$$\omega = (\omega_1, \dots, \omega_N) \in \Omega, \qquad \omega_i = \begin{cases} 1 \text{ (up)}, \\ 0 \text{ (down)}, \end{cases}$$

filtration $\mathbb{F}$, and natural probability measure $\mathbb{P}$. The risk-neutral measure $\widetilde{\mathbb{P}}$ with Radon-Nikodym derivative $Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$, is determined by the risk-neutral probabilities

$$\tilde{p} = \frac{1 + r - d}{u - d}, \qquad \tilde{q} = 1 - \tilde{p} = \frac{u - 1 - r}{u - d}.$$

An investor with initial wealth of $W_0 > 0$ at time $t = 0$, invests in $\mathcal{S}$ and $\mathcal{B}$ in a self-financing portfolio $P_t, \; t = 0, \dots, N, \; P_0 = W_0$. At $t = 0$, the investor buys $\Delta_0$ shares and deposits the rest of his wealth in $\mathcal{B}$. The self-financing portfolio dynamics, $P_t, \; t = 0, \dots, N$, which represents the investor wealth process $W_t, \; t = 0, \dots, N$, is given by

$$P_{t+1} = W_{t+1} = \Delta_t S_{t+1} + (1 + r)(P_t - \Delta_t S_t), \;\; t = 0, \dots, N-1, \;\; P_0 = W_0.$$

The investor's goal is to find an $\mathbb{F}$- adapted sequence of stock allocations $\Delta_0, \dots, \Delta_{N-1}$ that maximizes $\mathbb{E} \ln(W_N)$.

Show that the optimal allocations $\Delta_0^{(opt)}, \dots, \Delta_{N-1}^{(opt)}$ maximizing $\mathbb{E}U(W_N)$, with $U(x) = \ln(x)$, $x > 0$, is determined by the optimal portfolio

$$W_{t+1}^{(opt)} = P_{t+1}^{(opt)} = \Delta_t^{(opt)} S_{t+1} + (1+r)\left(P_t^{(opt)} - \Delta_t^{(opt)} S_t\right), t = 0, \dots, N-1,$$

where:

$P_t^{(opt)} = \frac{W_0}{\zeta_t},\ t = 0,1, \dots, N;$

$\zeta_t,\ t = 0, 1, \dots, N$ is the state price density process $\zeta_t = Z_t(1+r)^{-t}, t = 0, \dots, N$; and $Z_t = \mathbb{E}_n(Z) = \mathbb{E}^{(\mathbb{P})}(Z|\mathcal{F}_t)$ is the Radon-Nikodym derivative process.

**Problem 2.** Consider the binomial pricing model, $(\Omega, \mathbb{F} = \{\mathcal{F}_n, n = 0, \dots, N\}, \mathbb{P})$,

$$\omega = (\omega_1, \dots, \omega_N) \in \Omega,\ \omega_I = \begin{cases} 1\ (\text{up}) \\ 0\ (\text{down}) \end{cases},$$

filtration $\mathbb{F}$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, natural probability measure $\mathbb{P}$, and equivalent risk-neutral measure $\widetilde{\mathbb{P}}$. Let $S_n,\ n = 0, \dots, N,\ S_0 > 0$ be the price of a risky asset (stock) with price dynamics determined by the pricing tree

$$S_{t+1} = \begin{cases} S_t u, & \text{with probability} \quad p, \\ S_t d, & \text{with probability} \quad q = 1 - p, \end{cases}$$

where $p \in (0,1)$ and $0 < d < u$. Define the interest rate process $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_{N-1}$ which is $\mathbb{F}$-adapted such that $d < \mathcal{R}_i < u$, $\mathbb{P}$- almost surely, $i = 0, \dots, N-1$. Define the discount process,

$$\mathcal{D}_n = \frac{1}{(1+\mathcal{R}_0)(1+\mathcal{R}_1)\dots(1+\mathcal{R}_{n-1})}, \qquad n = 1, \dots, N, \qquad \mathcal{D}_0 = 1.$$

The price at time $n,\ n = 0, \dots, m < N$ of a zero-coupon bond maturing at time $m$ is $\mathcal{B}_{n,m} = \widetilde{\mathbb{E}}_n\left(\frac{\mathcal{D}_m}{\mathcal{D}_n}\right)$, where $\widetilde{\mathbb{E}}_n(\cdot) = \mathbb{E}^{\widetilde{\mathbb{P}}}(\cdot\,|\mathcal{F}_n)$ is the conditional expectation with respect to $\widetilde{\mathbb{P}}$. For $n = 0, \dots, m$, the forward (delivery) price is $For_{n,m} = \frac{S_n}{\mathcal{B}_{n,m}}$ and the futures price is $Fut_{n,m} = \widetilde{\mathbb{E}}_n(S_m)$.

$(i)$ Suppose at each time $n = 0, \dots, N-1$, a trader takes a long position in the forward contract with maturity $m, n < m \le N$, and sells the contract at $n+1$. Show that this strategy generates the cash amount $\left(S_{n+1} - S_n \frac{\mathcal{B}_{n+1,m}}{\mathcal{B}_{n,m}}\right)$ at $n+1$.

$(ii)$ Assume the interest rate is constant, $\mathcal{R}_i = r,\ i = 0, \dots, N-1$. At each time $n,\ 0 \le n < m < N-1$, the trader takes a long position of $(1+r)^{m-n-1}$ forward contracts with maturity $m$, and sells these contracts at time $n+1$. Show that the resulting cash flow is the same as the difference in the futures prices $Fut_{n+1,m} - Fut_{n,m},\ 0 \le n \le m < N-1$.

**Problem 3.** Let $B(t),\ t \in [0,T],$ be a standard Brownian motion generating a stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$, with canonical filtration $\mathbb{F} = \{\mathcal{F}_t = \sigma(B_u, 0 \le u \le t),\ t \in [0,T]\}$. Let $t_{k,n} = \frac{kT}{n}$, $k = 0, \dots, n$. For $\alpha \in (0,1]$ set $\tau_{k,n}^{(\alpha)} = \alpha t_{k,n} + (1-\alpha) t_{k+1,n},\ k = 0, \dots, n-1$. Define the Stratonovich $\alpha$-variation of $B_t, t \in [0,T]$ as

$$\mathbb{S}_T^{(\alpha)} = \lim_{n \uparrow \infty} \mathbb{S}_{T,n}^{(\alpha)}, \quad \mathbb{S}_{T,n}^{(\alpha)} = \sum_{k=0}^{n-1} \left( B\left(\tau_{k,n}^{(\alpha)}\right) - B\left(t_{k,n}\right) \right)^2.$$

$(i)$ Show that $\mathbb{E}\left(\mathbb{S}_T^{(\alpha)}\right) = (1-\alpha)T$ and that the variance of $\mathbb{S}_{T,n}^{(\alpha)}$ converges to zero as $n \uparrow \infty$.

$(ii)$ Define the Stratonovich $\alpha$-integral of $B(t), t \in [0,T]$ as

$$\int_0^T B(t) \overset{(\alpha)}{\circ} dB(t) = \lim_{n \uparrow \infty} \sum_{k=0}^{n-1} B\left(\tau_{k,n}^{(\alpha)}\right) \left( B\left(t_{k+1,n}\right) - B\left(t_{k,n}\right) \right).$$

Show that

$$\int_0^T B(t) \overset{(\alpha)}{\circ} dB(t) = \frac{1}{2} B(T)^2 + \left( \frac{1}{2} - \alpha \right) T.$$

**Problem 4.** Let $B(t),\ t \ge 0,$ be a standard Brownian motion generating a stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$,with canonical filtration $\mathbb{F} = \{\mathcal{F}_t = \sigma(B(u), 0 \le u \le t),\ t \ge 0\}$. Let $S_t,\ t \ge 0,\ S_0 > 0$, be the price process of a risky asset (stock) with continuous diffusion dynamics determined by the SDE,

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB(t), t \ge 0,$$

where $\mu_t,\ t \ge 0$, and $\sigma_t,\ t \ge 0$, are $\mathbb{F}$- adapted processes satisfying the usual regularity conditions guaranteeing that the SDE for $S_t,\ t \ge 0$, has a unique strong solution. Let $\beta_t,\ t \ge 0$, be the value of a riskless bank account

$$\beta_t = e^{\int_0^t r_s ds}, \qquad t \ge 0,$$

where $r_t > 0,\ t \ge 0$, is the instantaneous riskless (short) rate, which is $\mathbb{F}$-adapted and $\mathbb{P}\left(\max\left\{r_t + \frac{1}{r_t}; t \ge 0\right\} < \infty\right) = 1$. Define the market price of risk,

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}$$

and the state price density process

$$\zeta_t = \exp\left\{-\int_0^t \theta_s dB_s - \int_0^t \left(r_s + \frac{1}{2}\theta_s^2\right) ds\right\}, \qquad t \geq 0$$

($i$) Show that $\frac{d\zeta_t}{\zeta_t} = -\theta_t dB(t) - r_t dt.$

($ii$) Let $P_t,\ t \geq 0$, be the value of an investor's self-financing portfolio in the stock and in the riskless bank account when he uses a portfolio process $\Delta(t)$. Assume $P_t$ satisfies the dynamics

$$dP_t = \Delta(t)dS_t + r_t(P_t - \Delta(t)S_t)dt.$$

Show that $\zeta_t P_t,\ t \geq 0$ is an $\mathbb{F}$-martingale.

($iii$) Let $T$ be the investment terminal time of an investor with initial capital (wealth) $W_0$ at $t = 0$. The investor applies the self-financing strategy $P_t,\ t \geq 0$ in $(ii)$with the goal of achieving a terminal wealth $W_T = P_T$. Show that his initial capital should be $W_0 = P_0 = \mathbb{E}[\zeta_t W_t]$.