Mathematical Finance Preliminary Examination

August 2026

Instruction: Please solve three (3) out of the four (4) problems provided below and specify in the designated boxes which three problems you want to be graded. Make sure to provide clear, rigorous, and detailed solutions. Justify each step carefully.

Problem 1. (One-Period Binomial Pricing and Hedging)

A stock is currently priced at $S_0 = 80$. At time $T = \frac{1}{4}$ year, the stock price will be either $S_u = 92$ or $S_d = 72$. The continuously compounded risk-free rate is $r = 4\%$ per annum. Consider a European call option with strike $K = 84$.

(i) Compute the option payoff in each terminal state.

(ii) Determine the replicating portfolio (Δ, B), where Δ is the number of shares held and B is the amount invested in the money market account at time 0.

(iii) Compute the no-arbitrage price of the option.

(iv) Verify your answer using the risk-neutral probability.

Problem 2. (Two-Period Binomial Model)

Consider a two-period binomial model with stock price process S_n, $n = 0,1,2$, where

$$S_0 = 100, u = 1.2, d = 0.9,$$

and the risk-free gross return per period is $1 + r = 1.05$.

Let $X = (K - S_2)^+$ be the payoff of a European put option with strike $K = 100$.

(i) Draw the stock-price tree and compute the option payoff at each terminal node.

(ii) Compute the risk-neutral probability.

(iii) Compute the option value at time $n = 1$ at each node.

(iv) Compute the time $n = 0$ value of the option.

Problem 3. (Itô Formula for a Nonlinear Transform)

Let S_t satisfy the geometric Brownian motion SDE

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and W_t is a standard Brownian motion. Let

$$Y_t = S_t^\alpha,$$

where $\alpha \in \mathbb{R}$.

(i) Apply Itô's formula to derive the SDE satisfied by Y_t.

(ii) Specialize your answer to the case $\alpha = -1$.

Problem 4. Derivation of the Black–Scholes PDE

Let $V(t, S)$ denote the price of a European derivative written on a stock whose price satisfies

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

The bank account satisfies

$$dB_t = r B_t \, dt.$$

Assume $V \in C^{1,2}$. Derive the Black–Scholes PDE satisfied by $V(t, S)$.