August 2026 Numerical Analysis Prelim Exam

1. [**20 points**] Interpolation.

(a) [10 points] Let $f(x)$ be a given function with the following data values given:

$$x : 0, 2, 4, 5$$
$$f(x) : 1, 3, 2, 9$$

Use Newton's divided difference interpolation to construct the interpolating polynomial P(x). Then use the polynomial to approximate f(3).

(b) [10 points] Let
$$q(x) = 2x^4 + x^3 - x^2 + 5.$$
Suppose the values of $q(x)$ at $x = 0, 1, 2, 3, 4, 5$ are denoted by

$$f_0 = q(0), \quad f_1 = q(1), \quad f_2 = q(2), \quad f_3 = q(3), \quad f_4 = q(4), \quad f_5 = q(5).$$

Construct a polynomial $p(x)$ of degree at most 5 such that

$$p(1) = f_1, \quad p(2) = f_2, \quad p(3) = f_3, \quad p(4) = f_4, \quad p(5) = f_5, \quad p(6) = 10.$$

without explicitly calculating the values $f_0, f_1, \ldots, f_5$.

2. [**20 points**] Numerical Integration.

(a) [10 points] Write down explicitly the interpolating polynomial $p(x)$ that agrees with $f(x)$ at the three points a, b and $(a+b)/2$. Next, integrate this polynomial $p(x)$ from a to b to obtain Simpson's rule.

(b) [5 points] Divide $[a, b]$ into N subintervals of the same length $h = (b-a)/N$. Apply Simpson's rule on each subinterval to derive the composite Simpson's rule.

(c) [5 points] How small would h need to be to guarantee an error no larger than 10^{-6} in the composite Simpson's rule for $f(x) = x^4$ on $[0, 1]$? Hint: the error in Simpson's rule obtained in (a) is

$$E = \left| \int_a^b (f(x) - p(x))\, dx \right| \le \frac{M}{2880}(b-a)^5.$$

3. [**10 points**] Write down the forward Euler method and the trapezoidal rule for solving the ODE

$$y' = f(t, y), \quad y(0) = y_0.$$

(a) [5 points] Find the stability region for both methods respectively.

(b) [5 points] Write our the numerical schemes corresponding to forward Euler method and the trapezoidal rule to solve the ODEs

$$\begin{cases} u' = -10u + 9v, \\ v' = 10u - 11v, \\ u(0) = \alpha, \quad v(0) = \beta, \end{cases}$$

What is the largest time step that can be used in the two methods?

4. [**10 points**] Consider the following numerical scheme

$$y_{n+2} + (\mu - 1)y_{n+1} - \mu y_n = \frac{h}{4}\left((\mu + 3)f_{n+2} + (3\mu + 1)f_n\right),$$

where $f_n = f(t_n, y_n)$. Show that the method is consistent with the ODE $y' = f(t, y)$, and that the order of consistency is 2 if $\mu \neq -1$ and 3 if $\mu = -1$.

5. [**10 points**] Consider the linear advection equation

$$\partial_t U(x, t) + a\partial_x U(x, t) = 0,$$

with periodic boundary condition on $[0, 1]$, where a is a constant. Let $h = 1/N$, $x_j = jh$, and $t_n = n\Delta t$. Consider the semi-discrete centered difference approximation

$$\frac{du_j}{dt} = -\frac{a}{2h}\left(u_{j+1} - u_{j-1}\right),$$

together with a Runge Kutta method.

(a) [5 points]Apply the classical fourth-order Runge Kutta method to derive the fully discrete scheme. Perform the stability analysis, determine the amplification factor of the scheme.

(b) [5 points] Derive a CFL condition involving Δt, h, and a under which the method is stable.

6. [**15 points**] Consider the heat equation

$$\partial_t U = \nu\Delta U \quad \text{in} \quad D := (0, 1) \times (0, 1),$$

with initial condition

$$U(x, y, 0) = U^0(x, y)$$

and homogeneous Dirichlet boundary condition

$$U(x, y, t) = 0 \quad \text{for} \quad (x, y) \in C := \partial D.$$

Here $\nu > 0$ is a constant.

Fix a positive integer N, let $h = 1/N$, and define the grid points

$$x_i = ih, \qquad y_j = jh.$$

Let $u_{i,j}^n$ denote the numerical approximation to $U(x_i, y_j, t_n)$, where $t_n = n\Delta t$. Derive the forward Euler finite difference scheme for this problem using a centered finite difference approximation for the Laplacian. Then answer the following questions:

(a) [5 points] Write down the full difference scheme for the interior grid points, including the boundary and initial conditions.

(b) [5 points] Derive the local truncation error of the scheme, assuming that the exact solution is sufficiently smooth.

(c) [5 points] Derive a sufficient CFL condition on Δt, h, and ν under which the scheme is stable.

7. [**15 points**] Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite. Define a sequence of matrices $\{A_k\}$ as follows:

$$A_0 = A.$$

For each $k = 0, 1, 2, \ldots$, compute the Cholesky factorization

$$A_k = L_k L_k^\top,$$

where L_k is lower triangular with positive diagonal entries. Then define the next matrix by

$$A_{k+1} = L_k^\top L_k.$$

In other words, each step first factors A_k by Cholesky decomposition and then reverses the order of the two Cholesky factors to obtain A_{k+1}.

(a) [5 points] Show that A_k is symmetric and positive definite, so that the iteration is well defined, and similar to A.

(b) [10 points] Show that if

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a \geq c,$$

has eigenvalues $\lambda_1 \geq \lambda_2 > 0$, then A_k converges to $\mathrm{diag}(\lambda_1, \lambda_2)$.