

ODE Preliminary Exam

August 2025

Instructions Please read the instructions carefully.

- You are not allowed to use any electronic devices, print or online media.
- There are two parts in the Question paper. Attempt all problems from Part A, and one from Part B.
- Please start answering a new question on a new page.
- In case you attempted more than one question in Part B, please specify, on the top page of your answer sheet, which question we should grade. If you failed to specify, we will grade the first attempted answer for Part B, and delete the last.

Part A Answer all questions in this section.

1. Consider a linear, time-varying ODE in $\mathbb{R}^d$:

$$\frac{d}{dt}x = A(t)x. \quad (1)$$

- (a) [5] Prove that the solution to (1) with initial conditions $x(t_0) = x_0$ can be written as

$$x(t) = \Phi_A(t_0, t)x_0,$$

where $\Phi_A(t_0, t)$ is a matrix valued function of the start time t_0 and present time t .

- (b) [10] Now suppose that $A(t)$ is T -periodic. Prove that

$$\Phi_A(0, t) = \Phi_A(T, t + T), \quad \forall t \in \mathbb{R}.$$

2. [15] Investigate the stability of the origin for the following ODE, using the center manifold theorem

$$\begin{aligned} \frac{d}{dt}x_1 &= -x_1^2 \\ \frac{d}{dt}x_2 &= x_1^2 - x_2 \end{aligned} \quad (2)$$

Provide the full statement of any theorem that you use. You do not need to prove the theorem.

3. Consider the ODE

$$\begin{aligned} \frac{d}{dt}x_1 &= 2x_1 + x_1^2 \\ \frac{d}{dt}x_2 &= -x_2 \end{aligned} \quad (3)$$

- (a) [3] Describe the stability properties of the fixed point at the origin.
- (b) [2] Define the global and local stable sets of the fixed point.
- (c) [3] Express the global stable set in terms of the local stable manifold.

(d) [12] Use the iterative method with up to 3 iterations, to estimate the stable manifold.

4. Consider the following ODE in $\mathbb{R}^2$ expressed in polar coordinates

$$\frac{d}{dt}r = 4r^2 - 2r^3 + \mu r \quad (4)$$

Here μ is a parameter that can vary.

- (a) [4] Find all fixed points and periodic cycles for various parameter values, and describe their stability,
- (b) [8] Identify and characterize all the bifurcations taking place.
- (c) [3] Sketch a bifurcation diagram.

Part B Answer any one from the following section :

5. Consider the ODE

$$\begin{aligned} \frac{d}{dt}x_1 &= -x_2 - \mu x_2^2 x_1 \\ \frac{d}{dt}x_2 &= x_1 \end{aligned} \quad (5)$$

in which μ is a parameter.

- (a) [4] Classify the stability of all fixed points, for various values of μ .
- (b) [4] Explain with reason whether a Hop bifurcation occurs, and where.
- (c) [12] Determine the stability of the fixed point for $\mu > 0$.

6. Consider the following ODE in $\mathbb{R}^2$ expressed in polar form :

$$\begin{aligned} \frac{d}{dt}r &= -2r^3 + r(1.1 + \cos \theta) \\ \frac{d}{dt}\theta &= 1 \end{aligned} \quad (6)$$

- (a) [5] Find all fixed points and periodic cycles for the system. You should also eliminate the possibility of fixed points and periodic cycles other than the ones you list.
- (b) [3] Characterize the stability of the origin
- (c) [4] Prove that all trajectories of this ODE remain bounded.
- (d) [8] Use the Poincare Bendixson theorem and your previous observations to characterize the stability of all the periodic cycles that you have found.