ODE Preliminary Exam

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Instructions. Please read the instructions carefully.

- You are not allowed to use any electronic devices, print or online media.
- There are two parts in the Question paper. Attempt all problems from Part A, and one from Part B.
- Please start answering a new question on a new page.
- In case you attempted more than one question in Part B, please specify, on the top page of your answer sheet, which question we should grade. If you failed to specify, we will grade the first attempted answer for Part B, and delete the last.

Part A. Answer all questions in this section.

1. Consider a general ODE in $\mathbb{R}^d$:

$$\frac{d}{dt}x = V(x). \tag{1}$$

Suppose that this ODE has global solutions.

1.1 [3] Define the ω-limit set of a point $x \in \mathbb{R}^d$.

1.2 [5] Prove that the orbit of x converges to $\omega(x)$.

1.3 [1+2+2] Explain with reason which of the following are invariant sets

(i) Fixed points

(ii) Omega-limit sets

(iii) Orbits.

2. [15] Investigate the stability of the origin for the following ODE, using the center manifold theorem

$$\begin{aligned} \frac{d}{dt}x_1 &= -x_1x_2 \\ \frac{d}{dt}x_2 &= -x_2 + x_1^2 \end{aligned} \tag{2}$$

Provide the full statement of any theorem that you use. You do not need to prove the theorem.

3. Consider a general parametric IVP in $\mathbb{R}^d$

$$\frac{d}{dt}x(t) = V(x(t), \mu), x(0) = x_0. \tag{3}$$

with μ being a parameter picked from $\mathbb{R}$.

3.1 [5] State in precise terms what it means for solutions to be *locally* continuous with respect to x_0 and with respect to μ.

3.2 [5] State without proof the result on local continuity of solutions to ODEs with respect to initial conditions. Frame the statement in terms of a general non-parametric ODE, and use a variable other than "x" to denote the state-variable.

3.3 [5] Use this theorem to prove that the solutions to (3) depends continuously on μ

4. Consider the ODE

$$\begin{aligned} \frac{d}{dt}x_1 &= -x_1 + x_2^2 \\ \frac{d}{dt}x_2 &= 2x_2 + x_1x_2 \end{aligned} \tag{4}$$

4.1 [3] Describe the stability properties of the fixed point at the origin.

4.2 [2+2] Define the global and local, stable and unstable sets of the fixed point.

4.3 [5] Describe some sufficient conditions under which the local stable set is a manifold. Are these conditions met for this problem ?

4.4 [3] Express the global stable set in terms of the local stable manifold.

Part B. Answer any one from the following set :

5. Consider a general time-invariant linear ODE

$$\frac{d}{dt}x = V(x) = Ax \tag{5}$$

(i) [2] What does the ODE (5) turn into under a linear change of variables $y = Px$ for some invertible matrix P ?

We assume henceforth that the origin is a hyperbolic fixed point.

(ii) [6] Prove that the stable and unstable sets $W^s(\vec{0})$ and $W^u(\vec{0})$ of the origin are linear subspaces.

(iii) [3] Explain the Jordan canonical decomposition for any square matrix.

(iv) [3] Prove that there is a choice of P from (ii) under which the right hand side of the transformed ODE is a matrix in Jordan canonical form.

$$\frac{d}{dt}y = Bx, \quad B \text{ in Jordan canonical form} \tag{6}$$

(v) [3] How are the stable and unstable manifolds of the origin for the ODE (6) related the same for ODE (5) ?

(vi) [3] Use this to deduce that the stable and unstable spaces are complimentary.

6. Consider the following ODE in $\mathbb{R}^2$ expressed in polar form :

$$\begin{aligned} \frac{d}{dt}r &= \mu r + r^3 - r^5 \\ \frac{d}{dt}\theta &= 1 \end{aligned} \tag{7}$$

(i) [5] Find all fixed points and periodic cycles for the system. You should also eliminate the possibility of fixed points and periodic cycles other than the ones you list.

(ii) [3] Characterize the stability of the origin

(iii) [4] Prove that all trajectories of this ODE remain bounded.

(iv) [8] Describe all bifurcations taking place.