ODE Preliminary Exam

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Instructions. Please read the instructions carefully.

- You are not allowed to use any electronic devices, print or online media.
- There are two parts in the Question paper. Attempt all problems from Part A, and one from Part B.
- Please start answering a new question on a new page.
- In case you attempted more than one question in Part B, please specify, on the top page of your answer sheet, which question we should grade. If you failed to specify, we will grade the first attempted answer for Part B, and delete the last.

Part A. Answer all questions in this section.

1. [15] Explain rigorously why the following third order differential equation in scalar variable y :

$$y''' - 3y'' + \sin(t)y' + ty = \cos(2t) \tag{1}$$

has unique solutions defined over the entire time interval $(-\infty, \infty)$. If you wish to invoke any theorem from the theory of first-order differential equations, you need to state it without proof.

2. Let A be a fixed $n \times n$ real matrix. Consider the autonomous matrix differential equation:

$$\frac{dX}{dt} = AX - XA \tag{2}$$

(i) [3] Argue whether this ODE is linear.

(ii) [5] Fix a matrix $Y_0 \in \mathbb{R}^{n\times n}$. Describe the initial value problem satisfied by this function

$$Y(t) = e^{tA} Y_0 e^{-tA}.$$

(iii) [7] Prove that any solution curve $X(t)$ of (2) with initial condition Y_0 has the same spectrum throughout the curve.

3. [15] Investigate the stability of the origin for the following ODE, using the center manifold theorem

$$\begin{aligned} \frac{d}{dt}x_1 &= x_1^2 \\ \frac{d}{dt}x_2 &= x_1^2 - x_2 + x_1 x_2 \end{aligned} \tag{3}$$

Provide the full statement of any theorem that you use. You do not need to prove the theorem.

4. Consider the ODE

$$\begin{aligned}\frac{d}{dt}x_1 &= -x_1 \\ \frac{d}{dt}x_2 &= x_2 + x_1^2 + x_1 x_2\end{aligned} \tag{4}$$

4.1 [3] Describe the stability properties of the fixed point at the origin.

4.2 [2] Define the global and local stable sets of the fixed point.

4.3 [3] State without proof the local stable manifold theorem.

4.4 [7] Use the iterative method with up to 3 iterations, to estimate the stable manifold.

Part B. Answer any one from the following section :

5. Consider the following ODE in $\mathbb{R}$ modelling a certain population dynamics.

$$\frac{dx}{dt} = r + \mu x - x^2. \tag{5}$$

Here μ is the bifurcation parameter that can be varied independently of t.

5.1 [8] Find all fixed points for various parameter values, and describe their stability.

5.2 [8] Identify and characterize all the bifurcations taking place.

5.3 [4] Sketch a bifurcation diagram.

6. Consider the linear, time-varying ODE in $\mathbb{R}^d$:

$$\frac{d}{dt}x = A(t)x. \tag{6}$$

(i) [6] Prove that for any two time instants t_0, t there is a matrix $\Phi(t_0, t)$ such that for any vector x_0, $\Phi(t_0, t)x_0$ satisfies the ODE (6) with initial conditions $x(0) = x_0$.

(ii) [2] Explain how the variable x in the ODE (6) be interpreted to represent matrices instead of vectors.

(iii) [4] Suppose that the solution curves corresponding to two initial conditions X_0 and Y_0 are respectively $X(t)$ and $Y(t)$. Prove that

$$X(t)X_0^{-1} = Y(t)Y_0^{-1}, \quad \forall t \in \mathbb{R}.$$