Preliminary Exam on PDE, August 2026.

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Problem #1. Maximum and minimum principles:

i. Let Ω be a bounded, smooth, simply connected set with boundary $\partial\Omega$. Let $u(\boldsymbol{x})$ be a smooth solution of $\Delta u = u^3 - u$ in Ω, with $u(\boldsymbol{x}) = 0$ on $\partial\Omega$. Show that $|u(\boldsymbol{x})| \leq 1$ in Ω.

ii. Give a direct proof that if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is harmonic within a bounded open set Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Hint: define $u_\epsilon(\boldsymbol{x}) = u(\boldsymbol{x}) + \epsilon\frac{|\boldsymbol{x}|^2}{2d}$ for $\epsilon > 0$ and show that u_ϵ cannot attain its maximum at an interior point of $\overline{\Omega}$.

Problem #2. Consider the following first-order <u>semilinear</u> problem:

$$\begin{cases} uu_x + u_y = 1 \\ \quad u(x,0) = h(x) \end{cases}$$

Then

i) Find the expression of the characteristic surface $[x(r,s), y(r,s), u(r,s)] \subset \mathbb{R}^3$ in terms of general data $h(s)$. Here s parametrizes the data while r parametrizes the tangent direction of the characteristic.

ii) For the specific case of $h(s) = s$: eliminate the parameters r and s and write down the specific formula for $u = u(x,y)$

Note: this PDE is NOT a conservation law! Therefore, you cannot assume that characteristics are straight lines. You may use either semilinear theory or fully-nonlinear theory of characteristics to solve this problem. However, you are encouraged to use semilinear theory since its easier.

Problem #3. Fix $\boldsymbol{x}_0 \in \mathbb{R}^d$ and $t_F > 0$. Consider the space-time cone with apex $(\boldsymbol{x}_0, t) \in \mathbb{R}^d \times \mathbb{R}^+$, that is:

$$C(\boldsymbol{x}_0, t_0) = \big\{(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}^+ \,\big|\, |\boldsymbol{x} - \boldsymbol{x}_0| \leq t_0 - t\big\}$$

Let $u(\boldsymbol{x}, t)$ be a solution of the wave equation $\partial_t u - \Delta u = 0$. Prove that if

$$u(\boldsymbol{x}, 0) = 0 \text{ and } \partial_t u(\boldsymbol{x}, 0) = 0 \text{ for all } \boldsymbol{x} \in B(\boldsymbol{x}_0, t_0)$$

then $u \equiv 0$ in the space-time cone $C(\boldsymbol{x}_0, t_0)$.

Hint: Start by defining an energy $E(t)$ at each cross-section $B(\boldsymbol{x}, t_0 - t)$ of the space-time cone $C(\boldsymbol{x}_0, t_0)$. You will have to assume that $u(x,t)$ is a strong solution of the wave equation, that is $u(x,t) \in \mathcal{C}^2$ and $\partial_t^2 u - \Delta u = 0$ holds in a strong sense.

Problem #4. Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open set with a $\mathcal{C}^1$ boundary $\partial\Omega$. Assume that $1 \le p \le \infty$. Let $\overline{u}$ be the mean value of u in Ω that is

$$\overline{u} := \fint_{\Omega} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

Then there exists a constant $c = c(d, p, \Omega)$ such that

$$\|u - \overline{u}\|_{L^p(\Omega)} \le c(d, p, \Omega)\|\nabla u\|_{L^p(\Omega)}$$

Hint: proceed by contradiction and use compactness.

Problem #5. Let u_i for $i = 1, 2$ be the solution of

$$\begin{cases} -\mathrm{div}\,(a_i \nabla u_i) = f \text{ in } \Omega \\ \qquad\qquad\quad u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ is bounded, $f \in L^2(\Omega)$ and the coefficients satisfy $a_i \in \mathcal{C}(\overline{\Omega})$ and

$$0 < \alpha_0 \le a_i(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega.$$

Prove the stability bound

$$\|\nabla(u_1 - u_2)\|_{L^2(\Omega)} \le \frac{const}{\alpha_0^2}\,\|a_1 - a_2\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)}.$$

Problem #6. Let Lu be defined as

$$Lu = -\mathrm{div}\,\big(A(\boldsymbol{x}, t)\nabla u(\boldsymbol{x}, t)\big) + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \nabla u(\boldsymbol{x}, t) + c(\boldsymbol{x}, t)u(\boldsymbol{x}, t)$$

with smooth coefficients $A(\boldsymbol{x}, t) \in \mathbb{R}^{d\times d}$, $\boldsymbol{v}(\boldsymbol{x}, t) \in \mathbb{R}^d$, and $c(\boldsymbol{x}, t) \in \mathbb{R}$. In particular, we assume that $A(\boldsymbol{x}, t)$ satisfies:

$$\beta|\boldsymbol{z}|_{\ell^2(\mathbb{R}^d)} \le \boldsymbol{z}^\top A(\boldsymbol{x}, t)\boldsymbol{z} \le \gamma|\boldsymbol{z}|_{\ell^2(\mathbb{R}^d)} \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^d\,,\ \boldsymbol{x} \in \Omega\,,\ t \in [0, T]$$

Consider the hyperbolic problem:

$$\partial_t^2 u + Lu = f$$

Prove that there exists a constant c, depending on Ω, T and the coefficients of the operator L, such that the following estimate holds:

$$\begin{aligned} &\max_{t\in[0,T]} \big(\|\partial_t \boldsymbol{u}(t)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}(t)\|_{L^2(\Omega)}\big) + \|\partial_t^2 \boldsymbol{u}(t)\|_{L^2(0,T,H^{-1}(\Omega))} \\ &\quad \le c\big(\|\boldsymbol{f}(t)\|_{L^2(0,T,L^2(\Omega))} + \|g\|_{H^1(\Omega)} + \|h\|_{L^2(\Omega)}\big) \end{aligned}$$

Note: the diffusion matrix $A(\boldsymbol{x}, t)$ is time dependent!
