Probability and Statistics Preliminary Examination: August 2026

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); i.i.d. (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbf{E}(X)$: expectation of random variable X.
  - $\mathbf{Var}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  (a) $Bin(n,p)$: $\mathbf{E}(X) = np$, $\mathbf{Var}(X) = np(1-p)$, and pmf and mgf given respectively by

  $$f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}I(x \in \{0,1,\cdots,n\}), \qquad M(t) = (1-p+pe^t)^n$$

  (b) $Pois(\lambda)$: $\mathbf{E}(X) = \lambda$, $\mathbf{Var}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}I(x \in \{0,1,\cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  (c) $Geom(p)$: $\mathbf{E}(X) = 1/p$, $\mathbf{Var}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1}I(x \in \{1,2,\cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  (d) $NB(r,p)$: $\mathbf{E}(X) = r(1-p)/p$, $\mathbf{Var}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

  $$f(x) = \binom{r+x-1}{x}p^r(1-p)^x I(x \in \{0,1,\cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  (e) $Gamma(\alpha, \beta)$: $\mathbf{E}(X) = \alpha\beta$, $\mathbf{Var}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

  $$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

  (f) $Exp(\beta)$: Gamma$(1, \beta)$.

(g) $Beta(\alpha, \beta)$: $\mathbf{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbf{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1)$$

(h) Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i)f(x_j)[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-1-i}[1-F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

1. Suppose that $X_1, \ldots, X_n$ are i.i.d. from $Unif(0, \theta)$ for $\theta > 0$. Consider the hypotheses $H_0 : \theta \geq 10$ V.S. $H_1 : \theta < 10$. We reject H_0 if $X_{(n)} < 6$.

   (a) Calculate the power function of the test.

   (b) Calculate the size of the test.

2. Suppose the joint pdf of X and Y is $f(x, y) = x + y$ for $0 < x < 1, 0 < y < 1$.

   (a) Calculate $\mathbf{P}(X > Y^3)$.

   (b) Calculate $\mathbf{E}(X|Y)$.

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. from $Exp(\beta)$ with $\beta > 0$. Let $X_{(n)}$ be the sample maximum. Find the limiting distribution of

$$\frac{X_{(n)} - \beta \log(n)}{\beta} \xrightarrow{D} ?$$

   (**Comment:** The pdf of $\exp(\theta)$ is $\theta^{-1} e^{-x/\theta}$.)

4. Let $\theta \in \mathbb{R}$ be the parameter. Suppose that $X_1, \ldots, X_n$ are i.i.d. observations. For each X_i, it is generated according to the following rule:

$$X_i = \begin{cases} Exp(\theta) & \text{with probability } 0.3, \\ Exp(1) & \text{with probability } 0.7. \end{cases}$$

   (a) Find the pdf of X_i.

   (b) Let $\widehat{\theta}$ be the MLE. Find the expression of $v(\theta)$ such that $\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{D} N(0, v(\theta))$. You can leave the expression of $v(\theta)$ as an integral.

   (**Comment:** The pdf of $\exp(\theta)$ is $\theta^{-1} e^{-x/\theta}$.)

5. Suppose $X_1, \ldots, X_n$ are i.i.d. $Unif(0, \theta)$ with $\theta > 0$. Find the UMVUE of θ^r, where $r > 0$ is a known constant.