Real Analysis Preliminary Examination

May, 2022

DIRECTIONS: Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

PROBLEMS:

1. Let μ^* be an outer measure on X and $\{E_n : n \in \mathbb{N}\} \subset \mathcal{P}(X)$ be a sequence of subsets of X such that $\lim_{n\to\infty} \mu^*(E_n) = 0$. Find $\mu^*\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} E_n\right)$ and show that $\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} E_n$ is μ^*-measurable.

2. Let $\mathcal{M}$ be the countable/co-countable σ-algebra over $\mathbb{R}$ (i.e. the σ-algebra of all countable subsets of $\mathbb{R}$ and their complements), and let μ be the measure on $\mathcal{M}$ defined by

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is countable,} \\ 1, & \text{if } E^c \text{ is countable.} \end{cases}$$

Prove that for any $(\mathbb{R}, \mathcal{M}, \mu)$-measurable $\mathbb{R}$-valued function f,

$$S = \left\{\alpha \in \mathbb{R} : \mu\left(f^{-1}((-\infty, \alpha))\right) = 0\right\}$$

is nonempty and bounded above, and $f(x) = \sup S$ μ-a.e.

3. Let μ^* be an outer measure on X and $E \subset X$ such that $\mu^*(E) < \infty$. Prove that E is μ^*-measurable if and only if there is a μ^* measurable set $A \subset E$ such that

$$\mu^*(E) = \mu^*(A).$$

(*Hint: For the sufficiency, apply the Carathéodory condition of A on E*)

4. Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be an enumeration of all the rational numbers in $\mathbb{R}$. Define

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{[r_n, \infty)}(x).$$

Prove that F is strictly increasing and right continuous, and the Lebesgue-Stieltjes measure μ_F generated by F has the property that

$$\mu_F(\mathbb{Q}) = 1 \qquad \text{and} \qquad \mu_F(\mathbb{Q}^c) = 0.$$

5. Let $\alpha > 0$. Compute (with justification) $\displaystyle\lim_{n\to\infty} \int_0^{\infty} \frac{e^{n(\alpha - x)} \sin x}{1 + e^{n(\alpha - x)}}\, dx.$

6. Let F, G be increasing functions from $\mathbb{R}$ to $\mathbb{R}$, F be right continuous, G be continuous, and let μ_F and μ_G be the Lebesgue-Stieltjes measures generated by F and G, respectively. Prove that for any continuous function $f : \mathbb{R} \to \mathbb{R}$, the graph

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}$$

is Borel measurable, and $\mu_F(x) \times \mu_G(y)(\Gamma(f)) = 0$.

7. Let ν be a signed measure on $(X, \mathcal{M})$. Prove that for any $E \in \mathcal{M}$,

$$\nu^+(E) = \sup\{\nu(A) : A \subset E, A \in \mathcal{M}\} \quad \text{and} \quad -\nu^-(E) = \inf\{\nu(A) : A \subset E, A \in \mathcal{M}\}.$$

8. Let X be a normed vector space over $\mathbb{R}$. Prove that for any $x \neq 0$ in X, there is a bounded linear functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

9. Let $f \in L^\infty(X, \mathcal{M}, \mu)$. Prove that $|f| \leq \|f\|_\infty$ a.e., and for all $0 < \epsilon < \|f\|_\infty$

$$\mu\left(\left\{x \in X : \|f\|_\infty - \epsilon < |f(x)| \leq \|f\|_\infty\right\}\right) > 0.$$