Real Analysis Preliminary Examination

August 2026

DIRECTIONS

Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded.

If you do not do this, then problems 1-7 will be graded. Strive for clear, detailed, and legible solutions.

On this exam, $(X, \mathcal{M}, \mu)$ denotes a measure space and m, m_n denote Lebesgue measure on $\mathbb{R}, \mathbb{R}^n$, respectively.

PROBLEMS

1. Let μ^* be an outer measure on X. Prove that if the sequence $(E_k) \subset X$ satisfies $\sum_{k=1}^{\infty} \mu^*(E_k) < \infty$ then $\mu^*\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n\right) = 0$.

2. Suppose (f_n) is a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$.
   (a) Prove that $g(x) = \sup_n f_n(x)$ is measurable.
   (b) If $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$, prove that f is measurable.

3. Let $A \subset \mathbb{R}$. Prove that if A is not Lebesgue measurable, then there exists an $\epsilon > 0$ such that if B is measurable and $A \subset B$, then $m^*(B \setminus A) \geq \epsilon$.
   Hint: Proceed by contradiction.

4. Let $a, b \in \mathbb{R}$ such that $a > b > 1$. Evaluate
$$\lim_{n\to\infty} \int_0^{\infty} \frac{n|\cos x|}{1 + n^a x^b}\, \mathrm{d}x,$$
   and prove your answer. *Hint:* Let $y = nx$.

5. Let X be a Banach space and $S \subset X$ such that $\sup_{x\in S} f(x) < \infty$ for all $f \in X^*$. Prove that $\sup_{x\in S} \|x\| < \infty$.

6. Let $D \in \mathcal{M}$ and $f : D \longrightarrow [0, \infty)$ be measurable. Let $E \subset X \times \mathbb{R}$ such that the cross-sections satisfy
$$E_x = \emptyset, \qquad \text{for } x \in D^c,$$
$$E_x = [0, f(x)), \qquad \text{for } x \in D.$$
Prove that $E \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ and $(\mu \times m)(E) = \displaystyle\int_D f(x)\,\mathrm{d}\mu$.

Hint: Approximate by simple functions.

7. If ν is a complex measure and λ is a positive measure (both on $(X, \mathcal{M})$), prove that $\nu \ll \lambda$ if and only if $|\nu| \ll \lambda$.

8. Prove that if $f : [a, b] \longrightarrow \mathbb{R}$ is absolutely continuous, then f is of bounded variation.

9. Suppose $\mu(X) < \infty$, $\quad f, f_n \in L^3(X)$, $\quad f_n \longrightarrow f$ μ-a.e., and there exists $M > 0$ such that $|f_n| \leq M$ for all $n \in \mathbb{N}$. Let $g \in L^{3/2}(X)$. Prove that
$$\lim_{n\to\infty} \int f_n g\,\mathrm{d}\mu = \int fg\,\mathrm{d}\mu.$$