TOPOLOGY PRELIMINARY EXAM
AUGUST 2026

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem.

There are eight questions on the exam. You will be graded on your seven best answers.

(1) Give an example of two topological spaces X and Y that are homotopy equivalent but not homeomorphic. Make sure to provide a proof of your assertion.

(2) Consider the real line $X := \mathbb{R}$ and the subspace given by the open interval $A := (-1, 1)$. Recall that the quotient X/A is defined by the equivalence relation

$$x_1 \sim x_2 \equiv [x_1 = x_2 \vee (x_1 \in A \wedge x_2 \in A)].$$

Prove that X/A is *not* Hausdorff.

(3) Determine whether the following spaces are homeomorphic. Make sure that you provide a proof:
  (a) The real line $\mathbb{R}$ and the plane $\mathbb{R}^2$.
  (b) The open interval $(0, 1)$ and the closed interval $[0, 1]$.
  (c) The torus $T = S^1 \times S^1$ and the cylinder $C = S^1 \times [0, 1]$.

(4) Consider the n sphere S^n for $n \geq 2$ defined by

$$S^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

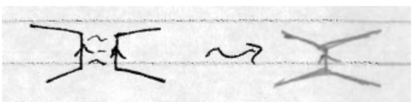
Let $G = \{e, g\}$ be the cyclic group of order two, and consider the action of G on S^n that permutes antipodal points:

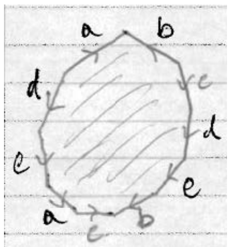
$$g \cdot \mathbf{x} := -\mathbf{x}.$$

Compute the fundamental group of the quotient S^n/G.

(5) Give a complete proof of the Tietze Extension Theorem.

(6) Find the genus of the following surface. Recall that arrows are identified tip-to-tip.





(7) Let $\rho : \widetilde{X} \to X$ be a covering space mapping a basepoint $\tilde{x}_0 \in \widetilde{X}$ to a basepoint $x_0 \in X$. Prove that any path $\gamma : [0, 1] \to X$ with initial point x_0 has a lift $\tilde{\gamma} : [0, 1] \to \widetilde{X}$ starting at $\tilde{x}_0$,

$$\rho \circ \tilde{\gamma} = \gamma.$$

(8) Recall that the sphere is defined by

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Compute the singular homology groups of the space

$$W := \{(x, y, z) \in S^2 : x = 0 \text{ or } y = 0\}.$$