ALGEBRA PRELIM – Aug 2019

Work two problems from each of the four sections, i.e. eight in total. Clearly indicate which two problems from each section are to be graded; otherwise, problems 1, 2, 4, 5, 7, 8, 10, and 11 will be graded. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element, and ring homomorphisms preserve 1.

GROUPS

Problem 1: A *maximal* subgroup of a group G is a proper subgroup H such that there are no subgroups K with $H \subsetneq K \subsetneq G$.

(a) Prove that if M is a maximal subgroup of G that is not normal in G, then the number of non-identity elements of G that are contained in conjugates of M is at most

$$(|M|-1)\,[G:M].$$

(b) Let G be a simple group in which every proper subgroup is abelian. If M and N are distinct maximal subgroups of G prove that $M \cap N = 1$.

Problem 2: Let p be prime and $G = S_p$ be the symmetric group on a set of p elements. Prove that if a subgroup $H \leq G$ has order p, then there is an isomorphism

$$\mathrm{N}_G(H)/\mathrm{C}_G(H) \cong \mathrm{Aut}(H)\,.$$

Problem 3: Prove that there are no simple groups of order 144.

RINGS

Problem 4: Let $\mathrm{M}_2(\mathbb{Z})$ be the ring of 2×2-matrices over the integers. Set

$$J = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{and} \quad R = \{A \in \mathrm{M}_2(\mathbb{Z}) \mid AJ = JA\}\,.$$

1. Show that R is a subring of $\mathrm{M}_2(\mathbb{Z})$.
2. Show that R is commutative.

Problem 5: Let R be a commutative ring and I and J be non-zero ideals in R. Show that if I and J are comaximal (i.e. $I + J = R$), then $IJ = I \cap J$ holds. Show also that the converse holds if R is a Principal Ideal Domain.

Problem 6: Let R be a ring and C be the left ideal in R generated by the set

$$\{xy - yx \mid x, y \in R\} ;$$

it is known as the *commutator ideal.*

(a) Prove that C is also a right ideal in R.

(b) Prove that the quotient ring R/C is commutative.

MODULES

Problem 7: Let $\varphi\colon R \to S$ be a ring homomorphism and M be an R-module.

(a) Show that $\mathrm{Hom}_R(S, M)$ is an S-module via the action given by

$$(s \cdot \varphi)(m) = \varphi(ms) .$$

(b) Show that if M is an injective R-module, then $\mathrm{Hom}_R(S, M)$ is an injective S-module.

Problem 8: Show that the matrix $A = \left(\begin{smallmatrix} 0 & -5 \\ 1 & -2 \end{smallmatrix}\right)$ satisfies $A^2 + 2A + 5I = 0$. Use this to prove that there is a real $n \times n$ matrix with minimal polynomial $t^2 + 2t + 5$ if and only if n is even.

Problem 9: Let R be a commutative ring and M an R-module that satisfies the *ascending chain condition;* that is, every ascending chain of submodules

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq N_{i+1} \subseteq \cdots$$

is eventually stationary. ($N_n = N_{n+1}$ holds for all n sufficiently large.)

(a) Show that M is generated by a finite number of elements.

(b) Show that every surjective R-module homomorphism $\varphi\colon M \to M$ is an isomorphism. [Hint: consider iterations of φ.]

FIELDS

Problem 10: Let M be the splitting field over $\mathbb{Q}$ of the polynomial

$$x^3 - 3x + 3\,.$$

What is the Galois group for $M/\mathbb{Q}$?

Problem 11: Let F be a field and $K \subseteq F$ a subfield that contains all squares in F; that is

$$\{\alpha^2 \mid \alpha \in F\} \subseteq K\,.$$

(a) Show that if F does not have characteristic 2, then $K = F$ holds.

(b) Show that if F is finite of characteristic 2, then $K = F$ holds.

Problem 12: Consider the polynomial

$$f(x) = x^4 + ax^3 - 6x^2 - ax + 1\,.$$

(a) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$ for all integer values of a, except $a = 0$ and $a = \pm 3$.

(b) Show that if α is a root of $f(x)$, then $(1+\alpha)/(1-\alpha)$ is a root as well.

(c) The discriminant of $f(x)$ (i.e. the product of the squares $(\alpha_i - \alpha_j)^2$ for distinct roots α_i and α_j) is $4(a^2+16)^3$. Use this information to determine the Galois group over $\mathbb{Q}$ for the splitting field of $f(x)$ for any integer value of a, except $a = 0$ and $a = \pm 3$.