# ALGEBRA PRELIM – MAY 2026

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element and all ring homomorphisms are assumed to preserve the 1-elements.

## GROUPS

**Problem 1:** Determine, up to isomorphism, all groups of order $20449 = 11^2 \cdot 13^2$.

**Problem 2:** Let $G$ be a group of order 24, and assume that $G$ has no elements of order 6. Show that $G \cong S_4$.

**Problem 3:** Let $p$ be an odd prime number.

- **(a)** Determine all subgroups of $D_{2p}$, the dihedral group of order $2p$.
- **(b)** Determine the conjugacy classes of the subgroups of $D_{2p}$. (Recall, two subgroups $H$ and $K$ of a group $G$ are conjugate if $H = gKg^{-1}$ for some $g \in G$.)

## RINGS

**Problem 4:** Let $R$ and $S$ be integral domains with $R \subseteq S$, and assume that $S$ is finitely generated as an $R$-module. Show: If $I$ is a proper ideal in $R$, then the ideal in $S$ generated by $I$ is also a proper ideal.

**Problem 5:** Determine the group of units in $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ and describe its structure.

**Problem 6:** Let $Q_8$ denote the group of quaternions. Determine whether or not the group ring $\mathbb{R}Q_8$ is isomorphic to $\mathbb{H}$, the ring of real quaternions.

## MODULES

**Problem 7:** Let $R$ be a commutative ring, and let $F$ be a free module of rank $n$. Show that a generating set for $F$ with $n$ elements is a basis.

**Problem 8:** Let $F$ be a field, and let $M_n(F)$ be the ring of $n \times n$ matrices over $F$. Show that two finitely generated left $M_n(F)$-modules are isomorphic if and only if they have the same dimension as $F$-vector spaces.

**Problem 9:** Let $R$ be a ring, and let $M$ and $N$ be $R$-modules.

- **(a)** Show that $\mathrm{Hom}_R(M, N)$ is a left $\mathrm{Hom}_R(N, N)$-module.
- **(b)** Show that $\mathrm{Hom}_R(M, N)$ is a right $\mathrm{Hom}_R(M, M)$-module.
- **(c)** Show that $\mathrm{Hom}_R(M, N)$ is a $\mathrm{Hom}_R(N, N)$-$\mathrm{Hom}_R(M, M)$-bimodule.

## FIELDS

**Problem 10:** Let $F$ be a field in prime characteristic $p$. A polynomial of the form

$$x^{p^n} + a_{n-1}x^{p^{n-1}} + \cdots + a_1x^p + a_0x \in F[x]$$

is called a *vectorial* polynomial. Show that a vectorial polynomial is separable if $a_0 \neq 0$, and that, in this case, the Galois group of its splitting field over $F$ can be considered as a subgroup of $\mathrm{GL}(n, p)$, the group of invertible $n \times n$ matrices over $\mathbb{Z}/p\mathbb{Z}$.

**Problem 11:** Let $p(x) = x^4 + ax^2 + b \in \mathbb{Q}[x]$ be irreducible with exactly two real roots. Determine $\mathrm{Gal}(M/\mathbb{Q})$, where $M$ is the splitting field for $p(x)$ over $\mathbb{Q}$.

**Problem 12:** Let $\mathbb{F}_5$ be the field with five elements. Determine the ideals of the ring $\mathbb{F}_5[x]/(x^2 + 1)$.