Algebra Preliminary Examination 1996

There are three sections, each containing four problems. Do three problems from each section.

I. Groups

1. Let G be a nonabelian group with commutator subgroup G'. Prove that G/G' cannot be the union of an ascending chain $\langle g_1 \rangle \subseteq \langle g_2 \rangle \subseteq \cdots$ of cyclic subgroups.

2.
   a. State Sylow's Theorems.
   b. Show that there is no simple group of order 48.
   c. Let H be a subgroup of a finite group G, and let p be a prime divisor of the order of H. Let P and Q be distinct Sylow p-subgroups of H and let P^* and Q^* be Sylow p-subgroups of G containing P and Q, respectively. Prove that $P^* \neq Q^*$.

3. Prove that every group with more than two elements has more than one automorphism.

4.
   a. Let p be prime and let a finite p-group act on a finite set X. Let X^G be the set of fixed points of the action. Prove that $|X^G| \equiv |X| \pmod{p}$.
   b. Prove that the center of a finite p-group contains nonidentity elements.

II. Rings and modules

1. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} | a, b \in \mathbb{Z}\}$. Prove that R is not a principal ideal ring.

2. Let R be a ring with identity. Prove that every ideal of the ring $\mathrm{M}_n(R)$ of $n \times n$ matrices has the form $\mathrm{M}_n(I)$, for some ideal I of R. Show that this is not the case for left ideals.

3. Show that if R is a ring whose additive group is isomorphic to $\mathbb{Q}/\mathbb{Z}$, then $rs = 0$ for all r and s in R.

4. Show that if $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$ is an exact sequence of left R-modules, then for any left R-module C,

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{f_*} \mathrm{Hom}_R(C, N) \xrightarrow{g_*} \mathrm{Hom}_R(C, L)$$

is exact. Give an example to show that g_* need not be surjective.

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III. Fields and linear algebra.

1. Show that every vector space over a division ring has a basis.

2. Let $f(X) = X^4 - 12X^2 + 35 \in \mathbb{Q}[X]$. Let K be a splitting field of f over $\mathbb{Q}$. Find
   a) The Galois group of K over $\mathbb{Q}$;
   b) the intermediate fields of the extension $K/\mathbb{Q}$;
   c) a primitive element for K as an extension of $\mathbb{Q}$.

3. Given that every abelian group G is the Galois group of a Galois extension $K/\mathbb{Q}$, show that in fact, G is the Galois group of infinitely many such extensions. (Hint: Use direct products of copies of G).

4. Let J_n be the $n \times n$ complex matrix

$$\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

Find the rational and Jordan canonical forms of J_n.