Algebra Preliminary Examination

May 1997

There are three sections, each containing four problems. Do THREE problems from each section. **Clearly indicate the problems you wish to have graded.**

I. Groups

1. Suppose G is a finite p-group having a unique subgroup of index p. Show that G is cyclic. [Use induction and look at $G/Z(G)$.]

2. Show that there is no simple group of order 36.

3. If p is a prime, $|G| = p^3$ and G is not abelian, show that $G' = Z(G)$.

4. Suppose G is a finite group, $H \triangleleft G$, and P is a Sylow p-subgroup of H. Let $N = N_G(P)$. Show that $G = NH$.

II. Rings and Modules

1. Let $R = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$. Show that R is not a principal ideal ring.

2. Let R be a simple ring with identity. Show that $M_n(R)$ is a simple ring.

3. Show that if $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence of left R-modules, then for any left R-module N,
$$0 \to Hom_R(C, N) \xrightarrow{g_*} Hom_R(B, N) \xrightarrow{f_*} Hom_R(A, N)$$
is exact.

4. Suppose R is a commutative ring, I and J are ideals in R, P is a prime ideal in R, and $I \cap J \subseteq P$. Show that $I \subseteq P$ or $J \subseteq P$.

III. Fields and Vector Spaces

1. Find the rational and Jordan canonical forms of the complex matrix
$$\begin{pmatrix} -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & -2 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let K be a field and $\alpha \in K$, and let m and n be relatively prime positive integers. Prove that $X^{mn} - \alpha$ is irreducible in $K[X]$ if and only if $X^m - \alpha$ and $X^n - \alpha$ are irreducible in $K[X]$.

3. Determine all the subgroups of the Galois group and all the intermediate fields (over $\mathbb{Q}$) of the polynomial $f(X) = X^4 - 7X^2 + 10 \in \mathbb{Q}[X]$.

4. If A is an $n \times n$ matrix over a field F, show that A is similar (over F) to its transpose A^t.