There are three sections, each containing four problems. Do three problems from each section.

I. Groups

1. Show that no group of order 192 is simple.

2 .
   a. Let G be a finite group and let H be a proper subgroup of G. Show that there is an element of G that is not in any conjugate of H.
   b. Show that the result in part a is not generally true for infinite groups. (Hint: Let G be the group of invertible 2×2 complex matrices, and H the subgroup consisting of the upper triangular ones. Use the fact that every complex matrix has an eigenvector and that change of basis amounts to a conjugation).

3.
   a. Let G be the group of positive reals under multiplication, and let H be the group of all reals under addition. Show that G and H are isomorphic.
   b. Now let $\tilde{G}$ be the group of positive rationals under multiplication, and $\tilde{H}$ the group of all rationals under addition. Prove that $\tilde{G}$ is a free abelian group (Hint: unique factorization), and hence that there are infinitely many homomorphisms from $\tilde{G}$ to $\tilde{H}$. Show that there is only one homomorphism from $\tilde{H}$ to $\tilde{G}$.

4. Let G be a finite group of odd order, and let x be a nonidentity element of G. Show that x and x^{-1} are not conjugate in G.

II. Rings and modules

1. Let R be an integral domain, and suppose that every descending chain

$$I_1 \supseteq I_2 \supseteq I_3 \ldots$$

eventually becomes constant: $I_k = I_{k+1} \ldots$. Prove that R is a field.

2. A module over a ring is *simple* if its only submodules are itself and (0).
   a. Prove that if S is a simple left R-module, then its ring of R-endomorphisms is a division ring.
   b. Let R be the ring of 2×2 upper triangular matrices over a field F. The 2-dimensional column vectors over F form an R-module M via matrix multiplication. Show that the R-endomorphism ring of M is isomorphic to F, but M is not a simple R-module.

3. Let R be an integral domain. A *minimal prime ideal* of R is a nonzero prime ideal that contains no other prime ideal except (0). Prove that if R is a unique factorization domain, then every minimal prime ideal of R is principal.

4. Let M be a nonzero left module over a ring R. Let N be the submodule

$$\{(0, m) | m \in M\}$$

of $M \times M$. For any R-endomorphism σ of M, let

$$M^\sigma = \{(m, \sigma(m)) | m \in M\}$$

(the graph of σ).

a. Show that M^σ is a submodule of $M \times M$, and is isomorphic to M.

b. For a submodule K of $M \times M$, show that $M \times M$ is the direct sum of K and N if and only if $K = M^\sigma$ for some σ.

III. Fields and linear algebra.

1. Let K be the splitting field of $f(X) = X^3 - 2$ over $\mathbf{Q}$. Determine the Galois group of K. Find all the intermediate fields, and determine which of them are Galois over $\mathbf{Q}$.

2. Let K be a field of characteristic $p > 0$. For a polynomial $f(X) = \sum_{k=0}^{n} a_k X^k$ over K, define $f'(X) = \sum_{k=1}^{n} k a_k X^{k-1}$. Prove that $f'(X)$ is the zero polynomial if and only if $f(X) = g(X^p)$ for some polynomial $g(X)$.

3. Find the rational and Jordan canonical forms of the complex matrix

$$\begin{pmatrix} 2 & 1 & -6 & -6 \\ 0 & 2 & 0 & 0 \\ -3 & -1 & 5 & 6 \\ 3 & 1 & -6 & -7 \end{pmatrix}.$$

4. Let $T : V \to V$ be a linear transformation of a finite-dimensional vector space over a field K. Notice that

$$\ker(T) \subseteq \ker(T^2) \subseteq \ldots,$$

and

$$\operatorname{Im}(T) \supseteq \operatorname{Im}(T^2) \supseteq \ldots.$$

Both eventually become constant. (Why?) Let

$$\ker(T^k) = \ker(T^{k+1}) = \ldots = V_1,$$

and

$$\operatorname{Im}(T^m) = \operatorname{Im}(T^{m+1}) = \ldots = V_2.$$

Show that

a. $T(V_i) \subseteq V_i$, for $i = 1, 2$.

b. $T|_{V_1}$ is nilpotent.

c. T maps V_2 isomorphically onto V_2.

d. $V = V_1 \oplus V_2$.