Preliminary Exam: Algebra
May 1999

Work any 3 problems from each part.

Part I. - Groups

1. Suppose there is only one group (up to isomorphism) of order n. Show that $(n, \phi(n)) = 1$ where $\phi(n)$ = Euler phi function.

2. Suppose G is finite, p is the smallest prime dividing $|G|$, $H \leq G$, and $[G : H] = p$. Show $H \lhd G$.

3. If a finite group is nilpotent, show that each of its Sylow subgroups is normal.

4. Let G be a group, $Aut\ G$ the group of automorphisms of G.

For $a \in G$, define $f_a : G \to G$ by $f_a(x) = axa^{-1}$ for all $x \in G$, and Inn $G = \{f_a : a \in G\}$.

- **a.** Show that $f_a \in Aut\ G$.
- **b.** Show that $Inn\ G \lhd Aut\ G$ and $Inn\ G \cong G/C(G)$ where $C(G)$ is the center of G.

Part II - Rings and Modules

1. Let R be a commutative ring with the property that every prime ideal is maximal. Let $R[x]$ be the polynomial ring over R.

Show that if P is a prime ideal of R, then $P[x]$ is a prime ideal of $R[x]$.

Also, show that no prime ideal of $R[x]$ is properly contained in $P[x]$.

2. A commutative domain R is called a Dedekind ring if all ideals I of R are projective as R-modules.

Assume R is a Dedekind ring. Let A and B be R-modules such that $A = B \oplus R$ as R-modules. Let M be a submodule of A.

Show that there is an exact sequence

$$0 \to M \cap B \to M \to J \to 0$$

where J is an ideal of R.

3. Let R be a commutative ring with identity. If I and J are ideals, define

$$(I : J) = \{r \in R : rJ \subseteq I\}.$$

Show that $(I : J)$ is an ideal.

4. Prove that if R is a simple ring with identity, then $M_n(R)$ is also a simple ring.

Part III - Fields and linear algebra

1. Suppose $K \subseteq L$ is a Galois field extension with $[L : K] = 20$.

Show that there exists an irreducible polynomial $f(x) \in K[x]$ of degree 5 such that L contains a root of f.

2. Find the rational and Jordan canonical forms of the complex matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{pmatrix}.$$

3. Show that the only automorphism of the field of real numbers is the identity.

4. Prove that any vector space over an arbitrary field has a basis (that is, it is a free module).