Algebra Preliminary Examination

May 2000

There are three sections, each containing four problems. Do THREE problems from each section. **Clearly indicate the problems you wish to have graded.**

I. Groups

1. Let N_i be a normal subgroup of G_i for $i = 1, 2$. Prove that

$$(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2).$$

2. Let G be a group of order p^n, where p is a prime and n is a positive integer. Prove that G has a normal subgroup of every order p^k such that $1 \leq k \leq n$.

3. Prove that every group of order 56 is solvable.

4. Let G be a finite simple group and let p be a prime divisor of $|G|$. Prove that every element of G can be written as a product of elements of order p.

II. Rings and Modules

1. Let K be a field. Prove that $K[x, y]$ is not a principal ideal domain.

2. If $R_1, ..., R_n$ are rings with identity and if I is an ideal in $R_1 \times \cdots \times R_n$, then show that $I = A_1 \times \cdots \times A_n$ where each A_i is an ideal in R_i. (Possible hint: Given I, let $A_k = \pi_k(I)$, where $\pi_k : R_1 \times \cdots \times R_n \to R_k$ is the canonical epimorphism.)

3. Let M be a left R-module. Prove that the following are equivalent:

   i) For each ascending sequence $S_1 \subset S_2 \subset S_3 \subset \cdots \subset M$ of submodules S_i of M, there exists an integer m such that $S_m = S_{m+1} = S_{m+2} = \ldots$

   ii) Every submodule of M is finitely generated.

4. Let R be a commutative ring with identity. A subset S of R is called a multiplicative set if it contains 1, is closed under multiplication, and does not contain the zero element.

   a. Prove that an ideal I of R is prime if and only if there is a multiplicative set S such that I is maximal among ideals disjoint from S.

   b. Prove that the set of all nilpotent elements of R equals the intersection of all the prime ideals of R. (Hint: If s is not nilpotent, $\{1, s, s^2, \ldots\}$ is a multiplicative set.)

III. Fields and Vector Spaces

1. Find the rational canonical form and the Jordan canonical form of the complex matrix

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 2i & 2i & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

2. Consider the field extension $K \subset K(x)$. Let $\beta \in K(x) - K$. Prove that β is transcendental over K and that $[K(x) : K(\beta)] < \infty$.

3. Let F be the splitting field of $x^3 - 2$ over the field $\mathbb{Q}$ of rational numbers. Describe the Galois group of F over $\mathbb{Q}$ and determine all the intermediate fields.

4. Let K_1 and K_2 be Galois extensions of $K = K_1 \cap K_2$ with Galois groups G_1 and G_2. Let $K_3 = K_1K_2$ be the composite field (which is generated by $K_1 \cup K_2$).

   i) Prove that K_3 is a Galois extension of K.

   ii) Prove that the Galois group G_3 of K_3 over K is isomorphic to $G_1 \times G_2$.