Algebra Preliminary Examination
August 2002

Work any eight problems. Clearly indicate which eight are to be graded.

1. Let p be a prime. Show that there is no simple group of order $8p$.

2. a) Show that any finitely generated group G has a maximal proper subgroup.

   b) Show that the additive group $\mathbb{Q}$ of rational numbers is not finitely generated. (Hint: $\mathbb{Q}$ is a divisible group).

3. Let R be a commutative local ring with maximal ideal J, and suppose that every element of J is nilpotent. Show that if S is a multiplicative set in R, then $S^{-1}R$ is either isomorphic to R or is the zero ring.

4. Let R be an artinian ring. Show that if $r \in R$ has left inverse a, then a is also a right inverse for r.

5. Let R be an algebra over a field K and let
$$E : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
be a short exact sequence of left R-modules. Suppose that $\dim_K B < \infty$ and that $B \cong A \oplus C$ as R-modules. Prove that E splits, i.e. there is an R-homomorphism $\lambda : C \to B$ with $g\lambda = 1_C$. (Hint: apply the functor $\mathrm{Hom}_R(C, -)$ to E and count dimensions).

6. Let R be a ring in which all left ideals are projective as left R-modules (such a ring is called a *left hereditary* ring). Prove that if I and J are two left ideals of R, then $I \oplus J \cong (I + J) \oplus (I \cap J)$ as left R-modules.

7. Let F/K be a field extension and let A and B be two $n \times n$ matrices over K. Prove that A and B are similar over F if and only if they are similar over K.

8. Let $T : V \to V$ be K-linear, where V is a finite-dimensional vector space over a field K. Prove that there is a vector $v_0 \in V$ such that for $f(X) \in K[X]$, $f(T) = 0$ if and only if $f(T)(v_0) = 0$.

9. Let K be an infinite field. Prove that if F/K is a finite separable field extension, then there is an element u of F such that $F = K(u)$.

10. A (very difficult) theorem states that if G is a finite solvable group, then there is a finite Galois extension of the rational field $\mathbb{Q}$ with Galois group G. Assuming this, prove the corollary that there are infinitely many non-isomorphic Galois extensions of the rational field $\mathbb{Q}$ with Galois group G. (Hint: a direct product of finitely many solvable groups is solvable).