Algebra Preliminary Examination
May 2002

Work any eight problems. Clearly indicate which eight are to be graded.

1. Let P be a p-Sylow subgroup of a finite group G, and let H be a p-subgroup of the normalizer $N_G(P)$. Prove that $H \subseteq P$.

2. Prove that, if $n \geq 3$, the center of the symmetric group S_n is $\{e\}$.

3. Let p be a non-zero non-unit of an integral domain R. Prove that p is irreducible if and only if (p) is maximal in the set of proper principal ideals of R.

4. Let R be a ring and suppose that an element $r \in R$ has a left inverse a_0, but does not have a right inverse. Prove that r has infinitely many left inverses and that the left ideal $J = \{s \in R \,|\, s\,r = 0\}$ of R is infinite. (Hint: Let A be the set of all left inverses of r. Show that $F(a) = r\,a - 1 + a_0$ defines a function from A to itself, and examine its basic set-theoretic properties.)

5. Let φ be an endomorphism of a Noetherian left module M over a ring R. Prove that $\mathrm{Im}(\varphi^n) \cap \mathrm{Ker}(\varphi^n) = 0$ for sufficiently large n. Deduce that if φ is surjective, then it is an automorphism of M.

6. Let $\varphi : M \to N$ be a homomorphism of left modules over a ring R and suppose that M has finite length n. Let ρ be the length of $\mathrm{Im}(\varphi)$ and let ν be the length of $\mathrm{Ker}(\varphi)$. Show that $\rho\,\nu \leq \left[\dfrac{n^2}{4}\right]$, where $[\ ]$ is the greatest integer function.

7. Let φ be an endomorphism of a finite-dimensional vector space V over a field K and suppose that $\varphi^2 = \varphi$. Show
   a) $\mathrm{Im}(\varphi) \cap \mathrm{Ker}(\varphi) = 0$
   b) $V = \mathrm{Im}(\varphi) \oplus \mathrm{Ker}(\varphi)$
   c) There is a K-basis of V with respect to which φ has a diagonal matrix with all diagonal entries equal to either 0 or 1.

8. Let V be a non-zero vector space over an infinite field K. Show that V is not the union of any finite set of proper subspaces.

9. Let K be a splitting field of $F(x) = x^4 - 10x^2 + 1$ over the rational field $\mathbb{Q}$. *Hint!* Before you do parts a) and b) show that there are integers m and n such that $\sqrt{5 \pm 2\sqrt{6}} = \sqrt{m} \pm \sqrt{n}$
   a) Describe K and determine the Galois group of K over $\mathbb{Q}$.
   b) Determine the intermediate fields of the extension $K/\mathbb{Q}$.

10. Let F/K be a finite Galois extension with Galois group G. Prove that there is an intermediate field L such that
    a) L is Galois over K.
    b) The Galois group of L over K is abelian.
    c) If L' is an intermediate field with the properties described in a) and b), then $L' \subseteq L$.
    Describe the Galois group of L/K in terms of G.