ALGEBRA PRELIM
August, 2004

Work any three problems from each of the two parts below:

I. General Algebra: Groups, Rings and Linear Algebra

1. Let G be a group of order 70. Show that G has a subgroup of index 2.

2. Let G be a finite group. Let H be a subgroup of G, and let N be a normal subgroup of G. Assume that $|H|$ and $[G : N]$ have greatest common divisor 1. Show that $H \subseteq N$.

3. Let S be a commutative ring and $R \subseteq S$ a subring. Let $X_1, \ldots, X_n$ be independent indeterminates over the ring S. Show that for every ideal I in the polynomial ring $R[X_1, \ldots, X_n]$ that

$$S \otimes_R (R[X_1, \ldots, X_n]/I) \cong S[X_1, \ldots, X_n]/IS[X_1, \ldots, X_n]$$

as S-algebras.

4. Prove that a nonzero prime ideal in a principal ideal domain is maximal.

5. Let A be an $n \times n$ matrix over $\mathbb{C}$. Suppose that $\lambda \in \mathbb{C}$ is the only eigenvalue of A. Prove that $(\lambda I_n - A)^n = 0$ (as usual, I_n denotes the identity matrix of order n).

II. Fields and Galois Theory

1. Let E be a finite field with n elements and let F be the subfield of E generated by the multiplicative identity $1 \in E$. Show that

   a. F is isomorphic to $\mathbb{Z}_p = \mathbb{Z}/(p)$ for some prime $p \in \mathbb{Z}$.

   b. $n = p^r$ for some $r > 0$, where p is as in part **a**.

   c. For all $a \in E^\times$, $a^{n-1} = 1$, where $E^\times$ denotes the group of units of E.

   d. E is the splitting field of $f(x) = x^n - x$ over F.

2. Let F be a field and let D be an integral domain which is a finite dimensional F-algebra. Prove that D is a field.

3. Let
$$\eta = \frac{2+i}{\sqrt{5}} \in \mathbb{C}.$$
Show that η is algebraic over $\mathbb{Q}$, and that η is on the complex unit circle, but that η is *not* an n^{th} root of unity for any n. (*Hint.* Consider η^2.)

4. Let $p(X) = X^4 + aX^2 - 1 \in \mathbb{Q}[X]$, and assume that $p(X)$ is irreducible in $\mathbb{Q}[X]$. Show that the Galois group of the splitting field is isomorphic to the dihedral group D_4. (D_4 is the group of all rotations and reflections of a square.)

5. Let $L/\mathbb{Q}$ be the splitting field of a polynomial $p(X) \in \mathbb{Q}[X]$, and let $M/\mathbb{Q}$ be the splitting field of a polynomial $q(X) \in \mathbb{Q}[X]$. Assume L and M to be subfields of $\mathbb{C}$. Furthermore, assume that $\mathrm{Gal}(L/\mathbb{Q})$ is cyclic of order 4, and $\mathrm{Gal}(M/\mathbb{Q})$ is cyclic of order 6, and that $L \cap M$ is a quadratic extension of $\mathbb{Q}$. Show that the splitting field for $p(X)q(X)$ over $\mathbb{Q}$ has degree 12.