Algebra Prelim

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Work any eight of the following problems:

1. Show that $\mathbb{Z}[\sqrt{5}]$ is not a unique factorization domain by producing an element with two different factorizations in irreducible factors. You must justify why the factors are irreducible.

2. Let R be a commutative ring and let G be the group of R-module automorphisms of R. Show that G is isomorphic to the group $R^\times$ of units of the ring R.

3. Let k be a field and let R be a finite dimensional k-algebra (note that R is not necessarily commutative). Assume that R contains no zero divisors. Prove that R is a division algebra.

4. Let I and J be ideals in a ring R such that $R/J \cong R/I$ as R- modules. Prove that $I = J$.

5. Show that an infinite simple group cannot have a proper simple subgroup of finite index. (*Hint.* Consider G acting as symmetries of the set X of left cosets of H in G.)

6. Let G be a group of order 1225. Prove that G is Abelian.

7. Let G be a group of order 24, and assume that G contains no element of order 6. Prove that $G \simeq S_4$.

8. The polynomial $f(X) = X^3 + X^2 - 2X - 1$ is irreducible over $\mathbb{Q}$. (This is given, and need not be proved.) Let L be its splitting field over $\mathbb{Q}$. Show: If θ is a roo t of $f(X)$, then so is $\theta^2 - 2$. (*Hint.* Consider the polynomial $f(X^2 - 2)$.) Use this to describe the Galois group for $L/\mathbb{Q}$.

9. Let $c \in \mathbb{Q}$, and assume that $a = 1 + c^2$ is not a square in $\mathbb{Q}$. Prove that $\mathbb{Q}(\sqrt{a + \sqrt{a}})/\mathbb{Q}$ is a Galois extension with cyclic Galois group of order 4.

10. Let p be a prime, and let $\mathbb{F}_p$ be the finite field with p elements. Also, let $\mathrm{GL}(n, p)$ be the multiplicative group of invertible $n \times n$ matrices over $\mathbb{F}_p$. Prove that there is an element in $\mathrm{GL}(n, p)$ of order $p^n - 1$. (*Hint.* There is a multiplicative element of order $p^n - 1$ in $\mathbb{F}_{p^n}$.)