ALGEBRA PRELIM
AUGUST 2005

Work any eight problems. Clearly mark the eight problems you want graded.

1. Let G be a group of order 6235. Prove that G is cyclic. (*Hint.* Find all the prime divisors of 6235 knowing that 43 is one of them.)

2. Let S_n be the symmetric group on n symbols, $n \geq 5$, and let H be a subgroup of S_n of index n. Prove that $H \cong S_{n-1}$. (*Hint.* S_n has no normal subgroups other than S_n, A_n and 1, where 1 denotes the trivial group.)

3. Provide a description, with proof, of the elements which are units in the ring $\overline{\mathbb{Z}}$ of algebraic integers.

4. Let R be a commutative ring, $R[X]$ the ring of polynomials over R, I an ideal of $R[X]$. Prove that if I contains a monic polynomial, then $R[X]/I$ is a finitely generated R-module.

5. Let R be a commutative ring. Recall that an element $x \in R$ is called nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$. Show that the set of all nilpotent elements of R is an ideal in R. Give an example of a noncommutative ring M with two nilpotent elements x and y such that their sum $x + y$ is not a nilpotent element of M.

6. Let V be a vector space of finite dimension. If φ is any linear transformation from V to V prove that there is an integer m such that the intersection of the image of φ^m and the kernel of φ^m is $\{0\}$.

7. Suppose that I is a principal ideal in the integral domain R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements (i.e., $rm = 0$ with $0 \neq r \in R$ and $m \in I \otimes_R I$ implies $m = 0$).

8. Let $p(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree 5, and assume that its Galois group is cyclic. Prove that all its roots are real.

9. Let $p(X) = X^4 + aX^2 + 1 \in \mathbb{Q}[X]$, and assume that $p(X)$ is irreducible over $\mathbb{Q}$. Let $\theta \in \mathbb{C}$ be a root. Show that $-\theta$ and $1/\theta$ are also roots, and use this to find the Galois group.

10. Let K be a field of prime characteristic p, and let $a \in K$ be an element that is not a p^{th} power in K. Prove that the polynomial $X^{p^n} - a$ is irreducible for all natural numbers n.