Algebra Preliminary Examination

May 2006

There are four sections. Do the indicated number of problems from each section. **Clearly mark the problems you wish to have graded.**

Groups

Do **three** of the following four problems.

(1) Let G be a finite group of order pq where p, q are prime numbers such that $p < q$ and p does not divide $q - 1$. Show that G is abelian.

(2) A partition of $n \in \mathbb{N}$ is a sequence of natural numbers $1 \le i_1 \le i_2 \le \cdots \le i_r$ such that $i_1 + \cdots + i_r = n$. Let p be a prime number. Show that the number of nonisomorphic abelian groups of order p^n is equal to the number of partitions of n.

(3) Let G be a finite group of order p^n where p is a prime number. Show that the center of G is nontrivial.

(4) Show that there is no simple group of order 300.

Rings

Do **one** of the following two problems.

(1) Consider the canonical projection

$$\varphi : \mathbb{Q}[x, y, z] \longrightarrow \mathbb{Q}[x, y, z]/(xy - z^2).$$

Consider the ideal $I = (x, z) \subseteq \mathbb{Q}[x, y, z]$. Show that $\varphi(I) \subseteq \mathbb{Q}[x, y, z]/(xy - z^2)$ is prime.

(2) Let $\mathbb{F}$ be an arbitrary field. Show that $\mathbb{F}[x, y]/(y^2 - x)$ and $\mathbb{F}[x, y]/(y^2 - x^2)$ are not isomorphic as rings.

Modules

Do **two** of the following three problems.

(1) Let R be a commutative ring with unity. Show that $\mathrm{Hom}_R(R, R)$ and R are isomorphic as rings.

(2) Show that $\mathbb{Z}[\mathbf{i}] \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{C}$ are isomorphic as rings.

(3) Show that the direct sum of two modules M and N is projective if and only if both M and N are projective.

Fields

Do **two** of the following three problems.

(1) Let p be a prime number. Prove that $\mathbb{Q}$, resp. $\mathbb{Z}/p$, is the prime field in characteristic 0, resp. $p > 0$.

(2) Let $\mathbb{F}(u)/\mathbb{F}$ be a simple field extension such that $|\mathbb{F}(u) : \mathbb{F}|$ is odd. Show that $\mathbb{F}(u) = \mathbb{F}(u^2)$.

(3) Let $\mathbb{F}$ be a field of characteristic not equal to two. Consider the field extension $\mathbb{F}(u_1, u_2)/\mathbb{F}$. Assume that $u_1^2, u_2^2 \in \mathbb{F}$, but $u_1, u_2, u_1 u_2 \notin \mathbb{F}$. Show that $\mathbb{F}(u_1, u_2)/\mathbb{F}$ is Galois with the Klein-Four group as Galois group .

Algebra Preliminary Examination

August 2006

There are four sections. Do the indicated number of problems from each section. **Clearly mark the problems you wish to have graded.**

Groups

Do **three** of the following four problems.

(1) Let G be a finite group. Assume that every Sylow subgroup of G is normal in G. Show that G is the direct product of its Sylow subgroups.

(2) Let G be a nonabelian group of order 21. How many Sylow subgroups are there?

(3) Let G be a group of odd order. Let $g \in G$ be a nonidentity element. Show that g and g^{-1} are not conjugate in G.

(4) Let G be a finite abelian group. Define $\nu_k(G) \in \mathbb{N}_0$ to be the number of elements of order k in G. Show that two finite abelian groups G and H are isomorphic if and only if $\nu_k(G) = \nu_k(H)$ for all k.

Rings

Do **one** of the following two problems.

(1) Consider the ring $\mathbb{Z}[x,y]/(x^2,y^2,2)$. Show that the square of every element is either zero or one. What is the characteristic of this ring?

(2) Let R be a commutative ring with unity. Let $I, J \subseteq R$ be two ideals in R. Show that

$$\varphi : R/(I \cap J) \longrightarrow R/I \times R/J, \; r \longmapsto (r+I, r+J)$$

defines a ring monomorphism. Show that φ is an isomorphism if $I + J = R$.

Modules

Do **two** of the following three problems.

(1) Let $I \subseteq R$ be a left ideal in a ring. Let $n \in \mathbb{N}$. Show that

$$R^n/IR^n \cong \underbrace{R/IR \times \cdots \times R/IR}_{n}$$

as R-modules, where the product on the right has n factors.

(2) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are **not** isomorphic as left $\mathbb{R}$-modules.

(3) Show that every R-module is projective if and only if every R-module is injective for a ring R.

Fields

Do **two** of the following three problems.

(1) Let $\mathbb{F}$ be a finite field of characteristic p. Show that $\mathbb{F}$ has exactly p^n elements for some $n \in \mathbb{N}$.

(2) Show that no finite field is algebraically closed.

(3) Let $\mathbb{K}$ be a field of characteristic not equal to two. Assume that the field extension $\mathbb{F}/\mathbb{K}$ is Galois with the Klein-Four group as Galois group. Show that $\mathbb{F} \cong \mathbb{K}(u_1, u_2)$ such that $u_1^2, u_2^2 \in \mathbb{K}$, but $u_1, u_2, u_1u_2 \notin \mathbb{K}$.