ALGEBRA PRELIM—AUGUST 2007

Solve eight of the twelve problems below. If you provide solutions (full or partial) to more that eight problems, clearly mark which eight should be graded.

Group theory.

(1) Let G be a finite Abelian group. A *group character* of G is a homomorphism χ from G to the multiplicative group of non-zero complex numbers $\mathbb{C}^*$. Let $\widehat{G}$ denote the set of group characters of G.

(i) Show that $\widehat{G}$ is a group with the operation of point-wise multiplication, i.e., $\chi_1\chi_2$ is the map sending g to $\chi_1(g)\chi_2(g)$.
(ii) Prove that if G is cyclic, then $G \simeq \widehat{G}$.

(2) Prove that a group of order 255 is Abelian. Then use this to determine how many groups there are of order 255 (up to isomorphism).

(3) Let G be a finite group, and let N be a normal subgroup of G. Let p be a prime, and let P denote a p-Sylow subgroup of N. Assume P is normal in N. Show that P is normal in G.

(4) An automorphism φ on a group G is called *inner*, if it has the form $\varphi(g) = aga^{-1}$ for some fixed $a \in G$. Show that all automorphisms on the symmetric group S_4 are inner.

Ring theory and modules. All rings are assumed to be commutative, and to have an identity element 1.

(5) Let F be a field, and let $F[x, 1/x]$ be the ring of *Laurent polynomials* over F, i.e., polynomials in x and $1/x = x^{-1}$. Show that $F[x, 1/x]$ is a principal ideal domain.

(6) Let R be a ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R with $\mathfrak{a}+\mathfrak{b} = R$ and $\mathfrak{a}\cap\mathfrak{b} = \{0\}$. Show that there exists an element $e \in R$ with $\mathfrak{a} = Re$, $\mathfrak{b} = R(1-e)$, and $e^2 = e$.

(7) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

(8) Let R be a ring. The *annihilator* of an R-module M is the set

$$\operatorname{Ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$

(i) Show that $\operatorname{Ann}(M)$ is an ideal in R.
(ii) Assume that $\operatorname{Ann}(M) + \operatorname{Ann}(N) = R$ for two modules M and N. Show that $M \otimes_R N = 0$.

Cont. on p. 2

Fields and Galois theory.

(9) Let F be a field and let $E = F(x)$ be a purely transcendental extension of F. Let $u \in E \setminus F$. Prove that $[F(u) : F] = \infty$.

(10) Let $p(x) = x^4 + ax^2 + 1 \in \mathbb{Q}[x]$ and assume that $p(x)$ is irreducible over $\mathbb{Q}$. Let θ be a root of $p(x)$. Show that $\mathbb{Q}(\theta)$ is the splitting field of $p(x)$, and that both $\left(\theta + \frac{1}{\theta}\right)^2$ and $\left(\theta - \frac{1}{\theta}\right)^2$ are in $\mathbb{Q}$. Compute the Galois group of $\mathbb{Q}(\theta)/\mathbb{Q}$.

(11) Prove that an algebraically closed field must be infinite.

(12) Let E/F be a Galois extension of fields with $[E : F] = 2$, where F has characteristic $\neq 2$. Prove that there exists an $\alpha \in E \setminus F$ with $\alpha^2 \in F$.